1. These 100 people have a total of $100 \times 22$ cookies. The total number of cookies for $100 + N$ people is $(100 + N) \times 20$. Therefore, we must have

$$100 \times 22 = (100 + N) \times 20 \Rightarrow 110 = 100 + N \Rightarrow N = 10.$$  

The answer is a.

2. A point is purple if it is on the line $y = x + 6$ and on the parabola $y = x^2$. This yields the equation $x^2 = x + 6$, which can be written as $x^2 - x - 6 = 0$, and can be factored as $(x - 3)(x + 2) = 0 \Rightarrow x = 3, -2$. The answer is c.

3. By the distance formula, the length of the path of terrapin is $\sqrt{8^2 + 15^2} = 17$ and the length of rabbit’s path is $8 + 15 = 23$. Thus, the rabbit travels $23 - 17 = 6$ units farther. The answer is d.

4. Since there are six pairs of gloves, it is possible to select six non-matching gloves. Once we choose seven gloves, we would have to have a pair of matching gloves. The answer is d.

5. Since the plane travels at 600 mph and the GPS watch shows their speed at 602 mph, their speed must be 2 mph. Therefore, the length of the plane is $\frac{2}{60} \times 5280 = 176$ feet. The answer is d.

6. The largest two-digit prime numbers are 97, 89, 83, 79. The answer is b.

7. $8(\log 20 + \log 50) = 8 \log(20 \times 50) = 8 \log(1000) = 8 \times 3 = 24$. The answer is c.

8. Suppose $p < q$ and $pq = a^2 - 1$ for some positive integer $a$. Then, $pq = (a - 1)(a + 1)$. Since $pq \geq 6$, we must have $a \geq 3$. Therefore, both $a - 1$ and $a + 1$ are more than 1, and since $p, q$ are both prime, we must have $p = a - 1$ and $q = a + 1$. This yields $q - p = 2$. Furthermore, if $q - p = 2$, then such an integer $a$ exists. Among the given primes, there are four pairs of primes that differ by 2. (They are $(3, 5)$, $(5, 7)$, $(11, 13)$, and $(17, 19)$). Since there are eight primes, we can select two of them in $\binom{8}{2} = 28$ ways. Therefore, the probability of choosing a pair $p, q$ for which $pq$ is one less than a perfect square is $4/28 = 1/7$. The answer is a.

9. The given assumption yields the following system of equations:

$$\begin{align*}
81 + 9b + 3c + d &= 0 \\
16 + 4b + 2c + d &= 0 \\
1 + b + c + d &= 0
\end{align*}$$

Subtracting the last equation from the other two yields

$$\begin{align*}
80 + 8b + 2c &= 0 \\
15 + 3b + c &= 0
\end{align*}$$

Subtracting twice the second equation from the first one we obtain $50 + 2b = 0$, or $b = -25$. Substituting into $15 + 3b + c = 0$ we obtain $c = 60$ and hence $d = -36$. When $a = 0$, we obtain
the equation $-25x^2 + 60x - 36 = 0$. This can be factored as $-(5x - 6)^2 = 0$. Thus, $x = 6/5$. The answer is e.

10. This year is $3 + 2b + b^2$. For this year to be in the 1900s we need to have $1900 \leq 3 + 2b + b^2 < 2000$. This can be simplified to $1898 \leq (b + 1)^2 < 1998$. Therefore, $b = 43$ and the year is 1938. The answer is b.

11. **First solution.** The point $(\cos(x), \sin(x))$ is on the unit circle. In order for it to also satisfy $\sin(x) + \cos(x) = 0.1$, it must lie on the line $x + y = 0.1$. This line and the unit circle intersect at two points. Therefore, the answer is c.

**Second solution.** Dividing both sides by $\sqrt{2}$ we obtain

$$\frac{\sin(x)}{\sqrt{2}} + \frac{\cos(x)}{\sqrt{2}} = \frac{0.1}{\sqrt{2}} \Rightarrow \cos\left(\frac{\pi}{4}\right)\sin(x) + \sin\left(\frac{\pi}{4}\right)\cos(x) = \frac{0.1}{\sqrt{2}} \Rightarrow \sin(x + \pi/4) = \frac{0.1}{\sqrt{2}}$$

There are two solutions to this equation, one in the first quadrant and one in the second quadrant. The answer if c.

12. Call the triangle $ABC$, and let $H$ be the foot of perpendicular from $O$ to one side $AB$ of the triangle. since the circumcenter and incenter of $ABC$ coincide, $ABC$ must be equilateral. By assumption $r = |OH|$ and $R = |OA|$. Since $AO$ is the angle bisector of $\angle BAC$, we have $\angle HAO = 30^\circ$. Thus, $\frac{|OA|}{|OH|} = \csc(30^\circ) = 2$. The answer is a.

13. Setting $a = |AB| = |AC|$, since the triangle $ABC$ is isosceles, $|BC| = a\sqrt{2}$. Using the Angle Bisector Theorem, we obtain

$$\frac{|BD|}{|AD|} = \frac{a\sqrt{2}}{a} \Rightarrow \frac{|BD|}{a - |BD|} = \sqrt{2}$$

$$\Rightarrow |BD| = a\sqrt{2} - |BD|\sqrt{2}$$

$$\Rightarrow |BD| = \frac{a\sqrt{2}}{1 + \sqrt{2}} = a\sqrt{2}(\sqrt{2} - 1) = 2a - a\sqrt{2}$$

$$\Rightarrow |BD| = |AB| + |AC| - |BC|.$$

The answer is b.

14. Rearranging the terms and using difference of squares we obtain the following:

$$1 + 4 + \sum_{k=1}^{505}(-(4k - 1)^2 + (4k + 1)^2) + \sum_{k=1}^{505}(-(4k)^2 + (4k + 2)^2)$$

$$= 5 + \sum_{k=1}^{505}2(8k) + \sum_{k=1}^{505}2(8k + 2) = 5 + (\sum_{k=1}^{505}32k) + 4 \times 505$$

$$= 2025 + 32 \frac{505 \times 506}{2} = 4090505$$

The answer is e.
15. Suppose there are \( r \) red balls and \( 8 - r \) green balls. The chance of selecting 1 red and 2 green balls is,

\[
\frac{r \binom{8-r}{2}}{\binom{8}{3}} = \frac{r(8-r)(7-r)}{112}
\]

Therefore, in order to maximize this, we need to maximize \( r(8-r)(7-r) \). After evaluating this expression for \( r = 1, 2, 3, 4, 5, 6 \) we see that \( r = 2 \) and \( r = 3 \) both yield the maximum value of 60. The answer is c.

16. There are four possibilities for the first move. By symmetry we will find the number of frog’s trajectories that start with the move \((2, 2) \rightarrow (3, 2)\) and multiply the answer by 4. We will count these based on the number of moves after the first move. For that we will use \(R, L, U, D\) for right, left, up, and down, respectively.

**One move:** The only possibility is “R”. So there is only 1 possible move.

**Two moves:** The possibilities are UR, UU, DR, DD. So, there are four possible moves.

**Three moves:** The possibilities are ULU, UDR, DLD, DUR, LLLL, LRR, LUU, LDD. So, there are eight possibilities.

Therefore, the number of possibilities is \(4 \times (1 + 4 + 8) = 52\). The answer is e.

17. Subtracting the equations we obtain \(ax + b - bx - a = 0\), which yields \((x - 1)(a - b) = 0\), and hence \(x = 1\) or \(a = b\). If \(x = 1\) is a common root of the equations, we must have \(1 + a + b = 0\), or \(a = -b - 1\). We could have \(-20 \leq b \leq 19\). So we obtain 40 different pairs of \((a, b)\).

When \(a = b\), we need the discriminant of \(x^2 + ax + a = 0\) to be nonnegative. Which implies \(a^2 - 4a \geq 0\). Therefore, \(a \leq 0\) or \(a \geq 4\). This yields 21 + 17 = 38 pairs of \((a, b)\). Adding these up, we conclude there are 78 pairs of \((a, b)\). The answer is e.

18. For simplicity, let \(d(k)\) be the number of positive divisors of \(k\). Let \(p\) be a prime factor of \(n\) and write \(n = p^\alpha m\), where \(m\) is an integer, relatively prime to \(p\). Each factor \(p^\beta\) with \(0 \leq \beta \leq \alpha\) appears in exactly \(d(m)\) positive divisors of \(n\). Therefore, the exponent of \(p\) in the product of the positive divisors of \(p\) is,

\[
\sum_{\beta=0}^{\alpha} \beta d(m) = \frac{\alpha(\alpha + 1)d(m)}{2}.
\]

By assumption, we would need this to be \(3\alpha\). This implies \((\alpha + 1)d(m) = 6\). This yields three possibilities for \(n\).

**Case I.** \(n = p^5\). In that case, \(n = 32\) is the only integer in the given range.

**Case II.** \(n = pq^2\) for distinct primes \(p, q\). This yields the following answers,

\[n = 2 \cdot 3^2, 2 \cdot 5^2, 2 \cdot 7^2, 3 \cdot 2^2, 3 \cdot 5^2, 5 \cdot 2^2, 5 \cdot 3^2, 7 \cdot 2^2, 7 \cdot 3^2, 11 \cdot 2^2, 11 \cdot 3^2, 13 \cdot 2^2, 17 \cdot 2^2, 19 \cdot 2^2, 23 \cdot 2^2.\]

There are 16 possibilities. So, the answer is e.
19. In order for a point \( P \) inside the triangle to have the given property, the distance from \( P \) to \( AB \) must be 1/3 the distance from \( C \) to \( AB \). Similarly for sides \( AC \) and \( AB \). Therefore, there is a unique point inside the triangle \( ABC \) with the given properties. If a point \( P \) outside triangle \( ABC \) and inside the \( \angle BAC \) has this property, then \([ABC] = [PAB] + [PAC] - [PBC] \) Therefore, \([PAB] = [PBC] = [PAC] = [ABC] \). This implies, point \( P \) lies on the line parallel to \( AB \) and through \( C \). Similarly, \( P \) lies on the line passing through \( B \) and parallel to \( AC \). This yields a unique point \( P \). Three other points can similarly be found. The area of the convex region containing all these points is four times \([ABC] \). By Heron’s formula

\[
[ABC] = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 7 \cdot 3 \cdot 4 = 84.
\]

Therefore, the total area is 336. The answer is \( \text{a} \).

20. The angle between the angle bisector of \( A \) and \( AB \) is \( \frac{A}{2} \). The angle between the altitude of \( A \) and \( AB \) is \( 90 - B \). Thus, the angle between the angle bisector of \( A \) and the altitude at \( A \) is

\[
\left| \frac{A}{2} - 90 + B \right| = \left| \frac{A + 2B - 180}{2} \right| = \left| \frac{B - C}{2} \right| = \frac{C - B}{2}
\]

By assumption we have \( C - B = 12 \) and \( C - A = 24 \). Subtracting we obtain \( B - A = 12 \). Therefore, the angle between the angle bisector and the altitude at \( C \) is \( 6^\circ \). The answer is \( \text{c} \).

21. Dividing both sides by \( x^2 \) we obtain

\[
a x^2 + b x + 1 + \frac{b}{x} + \frac{a}{x^2} = 0.
\]

Setting \( S = x + 1/x \) we obtain \( S^2 = x^2 + 1/x^2 + 2 \). This yields,

\[
a(S^2 - 2) + bS + 1 = 0 \Rightarrow aS^2 + bS - 2a + 1 = 0.
\]

Since \( x \) is positive \( S \geq 2 \). Therefore, the equation \( aS^2 + bS - 2a + 1 = 0 \) must have a root not less than 2. Note that for \( S = 0 \), the quadratic \( aS^2 + bS - 2a + 1 \) is \( -2a + 1 \) which is negative. Therefore, this quadratic must be nonpositive at \( S = 2 \). This implies \( 4a + 2b - 2a + 1 \leq 0 \), which shows \( a + b \leq -1/2 \). When \( x = 1 \), we obtain \( a + b = -1/2 \). The answer is \( \text{d} \).

22. Call the hexagon \( ABCDEF \). Let \( M, N, P, \) and \( Q \) be vertices of the square on sides \( AB, BC, DE, \) and \( EF \), respectively. Assume \( s \) is the side length of the hexagon and \( x \) be the distance \(|AM|\). Dropping a perpendicular from \( A \) to \( MQ \) we obtain a \( 30 - 60 - 90 \) triangle. Therefore, \(|MQ| = s + x \). Similarly, droppign a perpendicular from \( B \) to \( MN \) we obtain \(|MN| = \sqrt{3}(s - x) \). Setting these equal we obtain \( x = s(2 - \sqrt{3}) \). Therefore, the area of the square is \( (s + x)^2 = s^2(12 - 6\sqrt{3}) \). On the other hand the hexagon can be divided into six equilateral triangles with side length \( s \). Therefore, the area of the hexagon is \( 1 = 6 \cdot \frac{s^2\sqrt{3}}{4} = \frac{3s^2\sqrt{3}}{2} \). Since the area of the hexagon is 1, we conclude \( s^2 = \frac{2}{3\sqrt{3}} \). This gives us \( s^2(12 - 6\sqrt{3}) = \frac{24 - 12\sqrt{3}}{3\sqrt{3}} = \frac{8\sqrt{3} - 12}{3} \). The answer is \( \text{b} \).

23. This integer can be written as:

\[
64^7 + 128^3 + 1 = 2^{42} + 2^{21} + 1 = \frac{2^{63} - 1}{2^{21} - 1}
\]
The numerator is divisible by $2^9 - 1 = 7 \cdot 73$, while taking the denominator mod $2^9 - 1$ we obtain

$$2^{21} - 1 = (2^9)^2 \cdot 2^3 - 1 \equiv 8 - 1 = 7 \mod 2^9 - 1 \Rightarrow 2^{21} - 1 \equiv 7 \mod 73.$$ 

Therefore, the numerator is divisible by 73, while the denominator is not. Thus, 73 divides this number. Its sum of digits is 10. The answer is d.

24. Multiplying the first equation by 2 and subtracting from the second we obtain $(a-d)^2 + (b+c)^2 = 0$. Since $a, b, c, d$ are all real, we obtain $a = d$ and $b = -c$. Substituting into the first and last equation, we obtain $d^2 + c^2 = 1$ and $5d + c = 5$. Solving this we obtain $c = 5 - 5d$. This implies $25 + 25d^2 - 50d + d^2 = 1$. Solving for $d$ we obtain $a = d = 1, 12/13$. The sum of numerator and denominator of 12/13 is 25. The answer is c.

25. The equality $\varphi(n) = \omega(n) + 1$ holds if and only if no composite number less than $n$ is relatively prime to $n$. The smallest composite number relatively prime to $n$ is $p^2$, where $p$ is the smallest prime not dividing $n$. Therefore, $n < p^2$.

If $p = 2$, then $1 < n < 4$. Therefore, $n = 2, 3$.

If $p = 3$, then $n < 9$ and $n$ is divisible by 2. This gives us $n = 4, 8$.

If $p = 5$, then $n < 25$ and $n$ must be divisible by 6. This gives us $n = 6, 12, 18, 24$.

If $p = 7$, then $n < 49$ and $n$ must be divisible by $2 \cdot 3 \cdot 5 = 30$. This yields $n = 30$.

If $p = 11$, then $n < 121$ and $n$ must be divisible by $2 \cdot 3 \cdot 5 \cdot 7 = 210$. Thus, no such solutions exist.

If $p = 13$, then $n < 169$ and $n$ must be divisible by $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$. No such integer exists.

To summarize there are nine such integers: 2, 3, 4, 6, 8, 12, 18, 24, 30. The answer is c.