An Introduction to Matroid Theory Through Lattice Paths

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These slides and related papers are available at
http://home.gwu.edu/∼jbonin/

The material on lattice path matroids
in the first part of this talk
is based on joint work with

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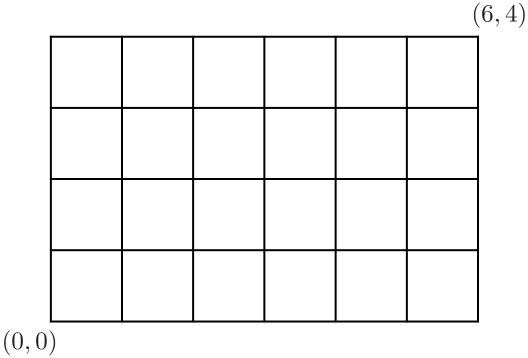
Departament de Matemátiques
Universitat Politècnica de Catalunya, Barcelona

Lattice paths are sequences of east and north steps of unit length, starting at $(0,0)$.



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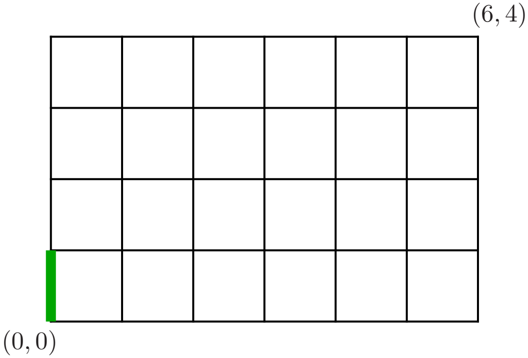

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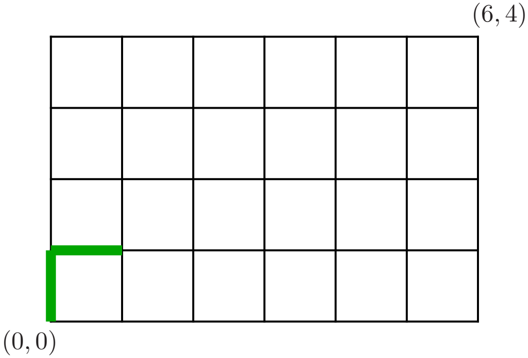

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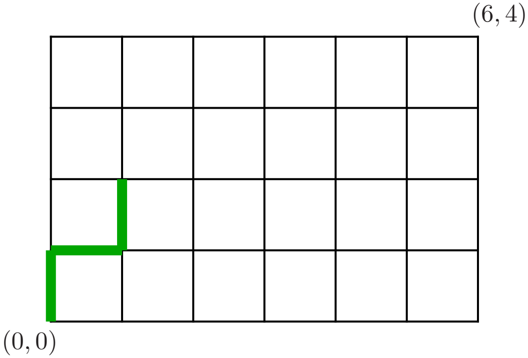

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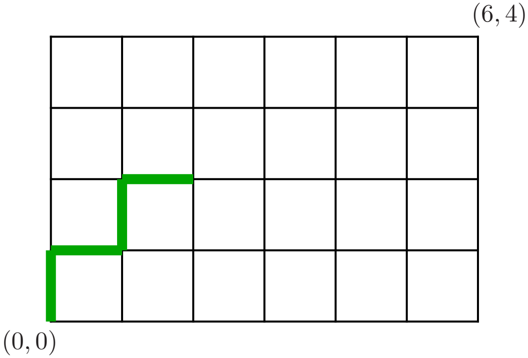

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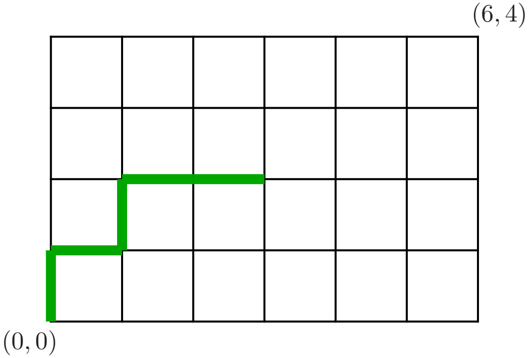

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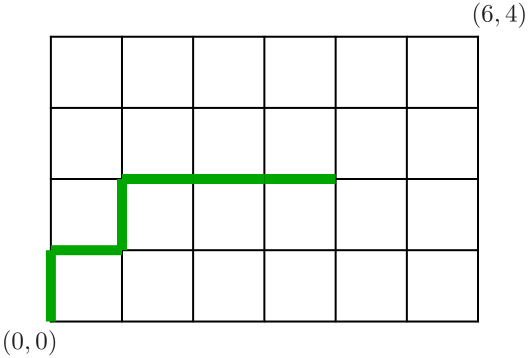

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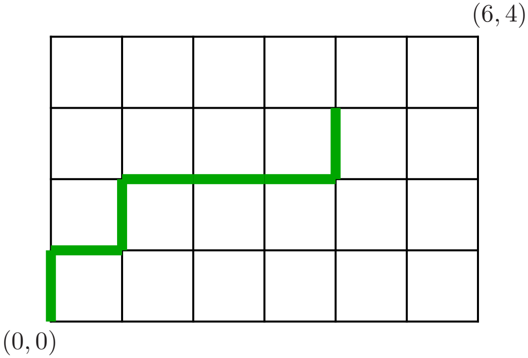

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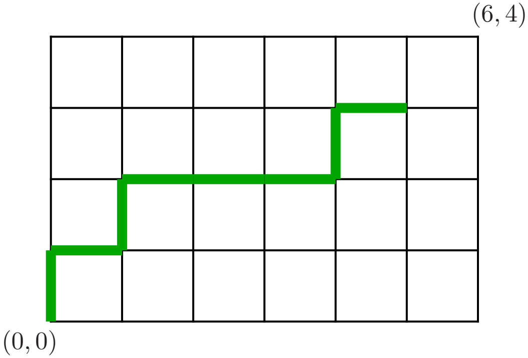

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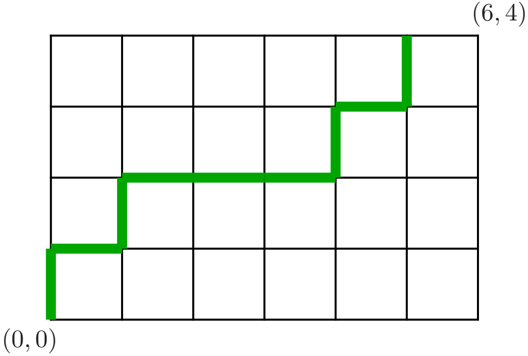

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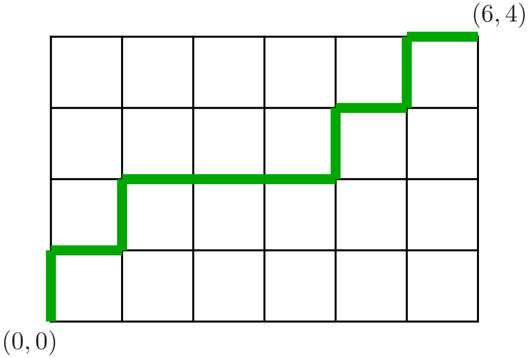
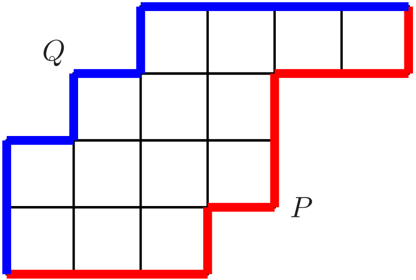


Fix lattice paths P and Q from $(0, 0)$ to (m, r) with P never going above Q.

Let $\mathcal{P}$ be the set of lattice paths from $(0, 0)$ to (m, r) that stay in the region that P and Q bound.



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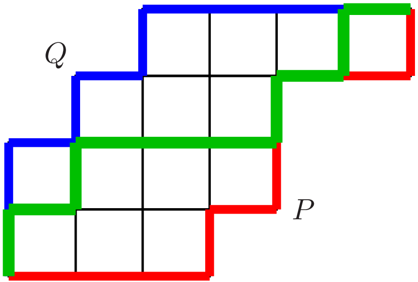
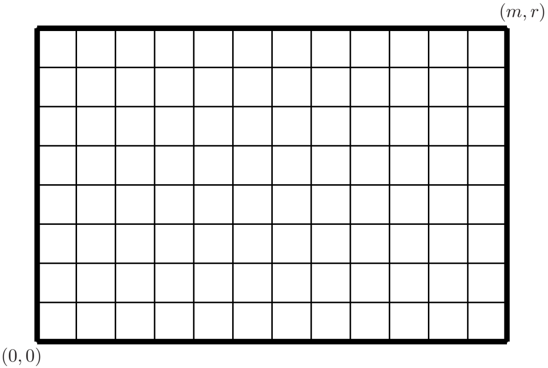


What is $|\mathcal{P}|$?

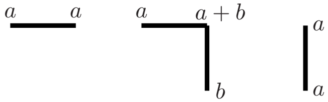
A Simple Example

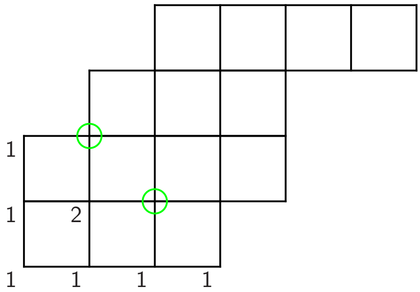
For $P = E^m N^r$
and $Q = N^r E^m$,
we get

$$|\mathcal{P}| = \binom{m+r}{r}.$$



$|\mathcal{P}|$ can be computed recursively.





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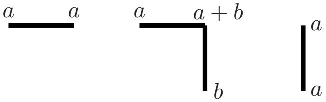

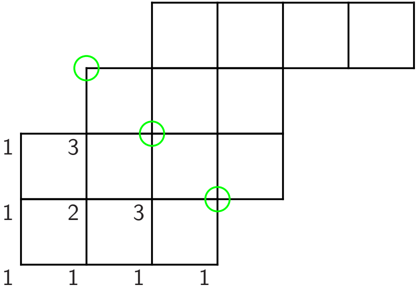

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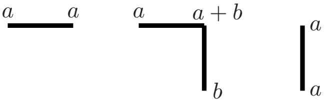
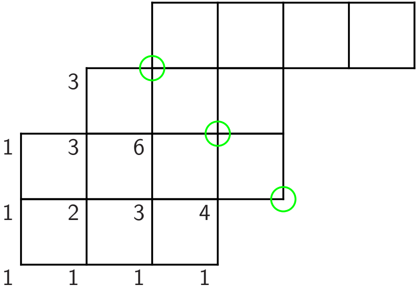
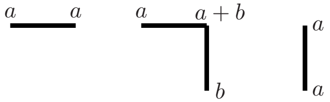
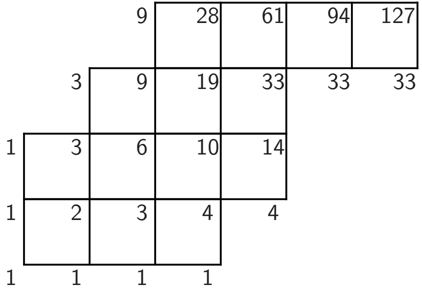




Finding the Number of Paths Recursively

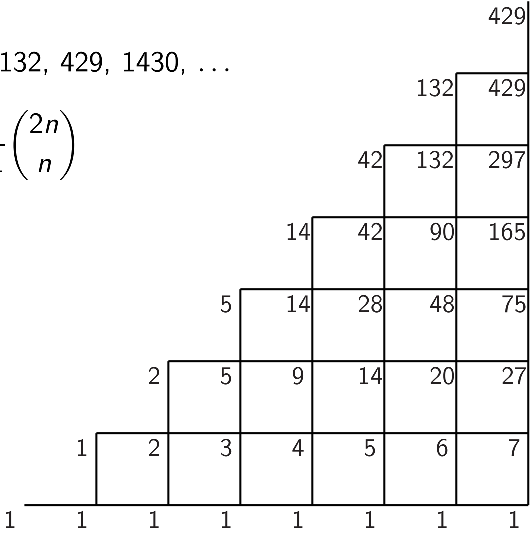
$|\mathcal{P}|$ can be computed recursively.





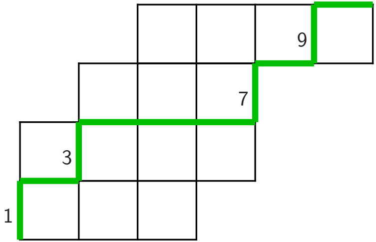
1, 1, 2, 5, 14, 42, 132, 429, 1430, ...

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$



Each path in $\mathcal{P}$ is determined by its north steps.

The lattice path
with north steps
1, 3, 7, 9.



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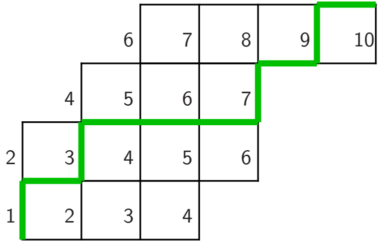

Label each north step by its position in each path it is in.

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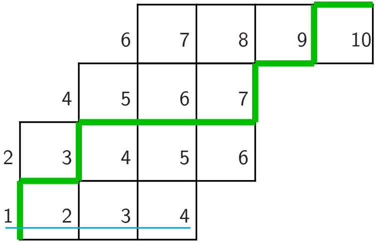

Label each north step by its position in each path it is in.

$N_1 = \{1, 2, 3, 4\}$

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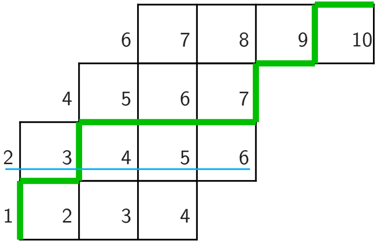
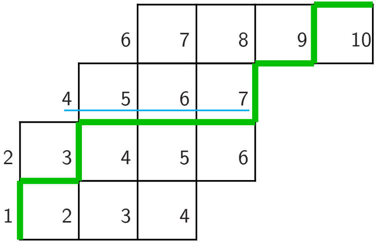


Label each north step by its position in each path it is in.

$N_1 = \{1, 2, 3, 4\}$
$N_2 = \{2, 3, 4, 5, 6\}$

Each path in $\mathcal{P}$ is determined by its north steps.

The lattice path with north steps 1, 3, 7, 9.



Label each north step by its position in each path it is in.

$N_1 = \{1, 2, 3, 4\}$
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$N_3 = \{4, 5, 6, 7\}$

Each path in $\mathcal{P}$ is determined by its north steps.

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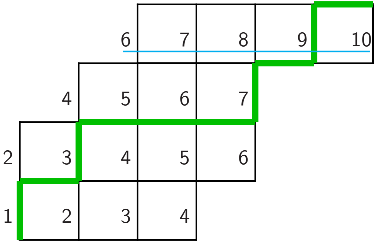

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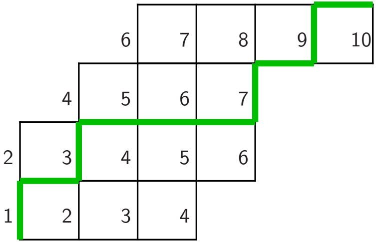

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For $1 \leq i \leq r$, let N_i be the set of all labels on the north steps in row i of the diagram (from the bottom).

A set system is a (finite) multiset $\mathcal{A} = (A_j : j \in J)$ of subsets of a (finite) set S.

Example: $S = \{a, b, c, d, e, f, g, h\}$ and $\mathcal{A} = (A, B, C, D)$ with

$A = \{a, b, e, f, h\}$, $B = \{b, c, g\}$, $C = \{d, e, g, h\}$, $D = \{d, f, h\}$

A **set system** is a multiset $\mathcal{A} = (A_j : j \in J)$ of subsets of a set S.

$A = \{a, b, e, f, h\}, \ B = \{b, c, g\}, \ C = \{d, e, g, h\}, \ D = \{d, f, h\}$

A transversal of $\mathcal{A}$ is a set $\{x_j : j \in J\}$ of $|J|$ distinct elements with $x_j \in A_j$ for all $j \in J$.

Some transversals: $\{a, b, e, h\}$, $\{e, f, g, h\}$.

$A = \{a, b, \mathbf{e}, f, h\}, \ B = \{b, c, \mathbf{g}\}, \ C = \{d, e, g, \mathbf{h}\}, \ D = \{d, \mathbf{f}, h\}$
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Caution: a transversal is just a set whose elements can be matched with the sets in $\mathcal{A}$; the matchings are not part of the transversal.

$$A = \{a, b, e, f, h\}, \;\; B = \{b, c, g\}, \;\; C = \{d, e, g, h\}, \;\; D = \{d, f, h\}$$

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A partial transversal of $\mathcal{A}$ is a transversal of some subsystem $(A_k : k \in K)$ with $K \subseteq J$.

Some partial transversals: $\{a, b, e\}$, $\{c\}$, $\emptyset$.

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Some sets that are not partial transversals: $\{a, b, c\}$, $\{a, b, e, g\}$.

$$A = \{a, b, e, f, h\}, \;\; B = \{b, c, g\}, \;\; C = \{d, e, g, h\}, \;\; D = \{d, f, h\}$$

$$\begin{array}{c c} & \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} * & * & 0 & 0 & * & * & 0 & * \\ 0 & * & * & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & * & 0 & * & 0 & * \end{pmatrix} \end{array}$$

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$*$s are algebraically independent.

$$A = \{a, b, e, f, h\},\ \ B = \{b, c, g\},\ \ C = \{d, e, g, h\},\ \ D = \{d, f, h\}$$

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$*$s are algebraically independent.

The determinant of this submatrix is

$$-x_{A,b}x_{B,g}x_{C,e}x_{D,h} - x_{A,e}x_{B,b}x_{C,g}x_{D,h}.$$

$$A = \{a, b, e, f, h\},\ \ B = \{b, c, g\},\ \ C = \{d, e, g, h\},\ \ D = \{d, f, h\}$$

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Recall, $\det(a_{i,j}) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$.

The Permutation Expansion of Determinants Reflects Matchings

$$\det \begin{pmatrix} x_{A,b} & x_{A,e} & 0 & x_{A,h} \\ x_{B,b} & 0 & x_{B,g} & 0 \\ 0 & x_{C,e} & x_{C,g} & x_{C,h} \\ 0 & 0 & 0 & x_{D,h} \end{pmatrix} = -x_{A,b}x_{B,g}x_{C,e}x_{D,h} - x_{A,e}x_{B,b}x_{C,g}x_{D,h}$$

The determinant of a square submatrix with rows indexed by R and columns indexed by C is the sum, over all matchings $\phi : C \to R$, of the product arising from the matching, i.e., $\pm \prod_{i \in C} x_{\phi(i),i}$.

(Non-matchings (where $i \notin \phi(i)$ for some $i \in C$) have a zero in the corresponding product and so make no contribution.)

The nonzero entries are algebraically independent, so there is no cancellation, so partial transversals correspond to linearly independent sets of columns.

A matroid M is a finite set S and a set $\mathcal{I}$ of subsets of S, the independent sets, such that
(i) $\emptyset \in \mathcal{I}$,
(ii) if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$, and
(iii) if $A, B \in \mathcal{I}$ and $|B| < |A|$, then $B \cup x \in \mathcal{I}$ for some $x \in A - B$.

(Whitney, 1935)

Ex. Let the elements of S label the columns of a matrix over a field $\mathbb{F}$.
A subset A of S is in $\mathcal{I}$ iff the corresponding columns are distinct and linearly independent.

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Theorem

The partial transversals of a set system $\mathcal{A}$ are the independent sets of a matroid. *(Edmonds and Fulkerson, 1965)*

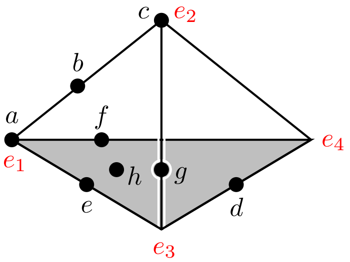
$\mathcal{A}$ is a presentation of this transversal matroid.

Each transversal matroid has a representing matrix in which all entries are nonnegative real numbers.

Scale each nonzero column so the column sum is 1.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{pmatrix} 1 & p & 0 & 0 & r & s & 0 & u \\ 0 & 1-p & 1 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & q & 1-r & 0 & 1-t & v \\ 0 & 0 & 0 & 1-q & 0 & 1-s & 0 & 1-u-v \end{pmatrix} \end{array}$$

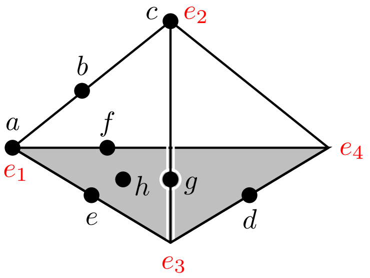
Nonzero columns in such a matrix are points in the convex hull of the standard basis vectors, i.e., a simplex.



Theorem

A matroid is transversal if and only if it has a geometric realization on a simplex with all dependence arising from the structure of the simplex. (Brylawski, 1975)

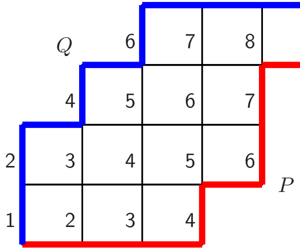
Respecting the structure of the simplex means, for example, if $\{a, b, c\}$ is dependent but all of its 2-subsets are independent, then a, b, c must be on an edge of the simplex.



Let P and Q be lattice paths from $(0, 0)$ to (m, r) with P never going above Q.

Let $\mathcal{P}$ and $N_1, N_2, \ldots, N_r$ be as before.

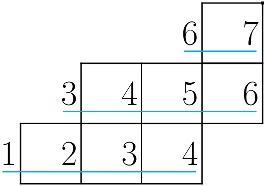
$N_4 = \{6, 7, 8, 9, 10\}$
$N_3 = \{4, 5, 6, 7\}$
$N_2 = \{2, 3, 4, 5, 6\}$
$N_1 = \{1, 2, 3, 4\}$



The matroid $M[P, Q]$ is the transversal matroid on $\{1, 2, \ldots, m + r\}$ with presentation $(N_1, N_2, \ldots, N_r)$.

A lattice path matroid is a matroid that is isomorphic to $M[P, Q]$ for some such P and Q.

An Example in Detail

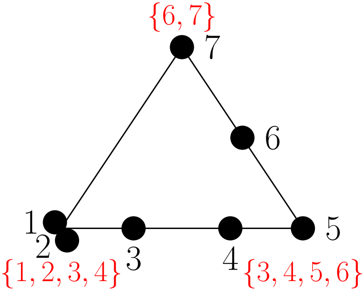


$$N_3 = \{6, 7\}$$

$$N_2 = \{3, 4, 5, 6\}$$

$$N_1 = \{1, 2, 3, 4\}$$

$$\begin{array}{c} \\ N_3 \\ N_2 \\ N_1 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & * & * & * & * & 0 \\ * & * & * & * & 0 & 0 & 0 \end{pmatrix} \end{array}$$



The bases of a matroid are its maximal independent sets.

Theorem

$R \mapsto \{i \mid \text{the } i\text{-th step of } R \text{ is north}\}$ is a bijection between $\mathcal{P}$ and the set of bases of $M[P, Q]$.

Corollary

The number of bases of $M[P, Q]$ is $|\mathcal{P}|$.

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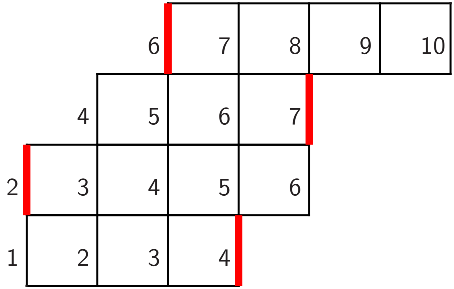

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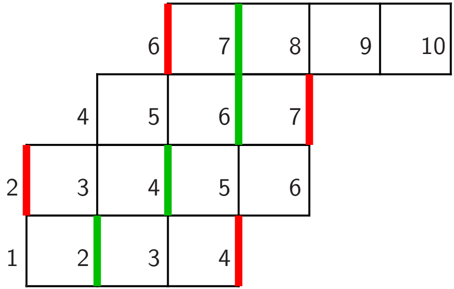

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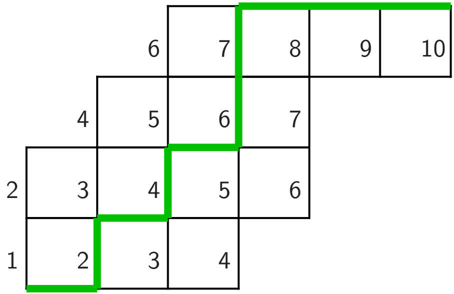
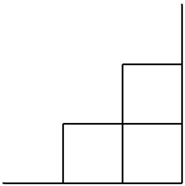


The *n*-th Catalan matroid M_n is $M[P, Q]$ with

$$P = \underbrace{EEE \cdots E}_{n} \underbrace{NNN \cdots N}_{n} = E^n N^n$$

and

$$Q = \underbrace{ENENEN \cdots EN}_{n \text{ pairs}} = (EN)^n.$$

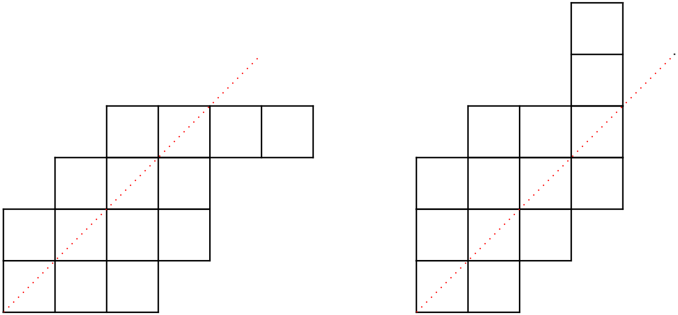


Theorem

The number of bases of M_n is the n-th Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

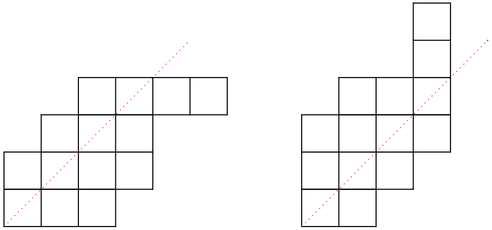
Duality



The dual M^* of M has $\{S - B \mid B \text{ is a basis of } M\}$ as its set of bases.

Theorem

The class of lattice path matroids is closed under duality.



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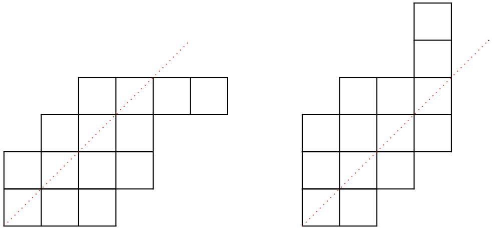
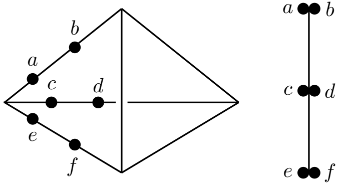
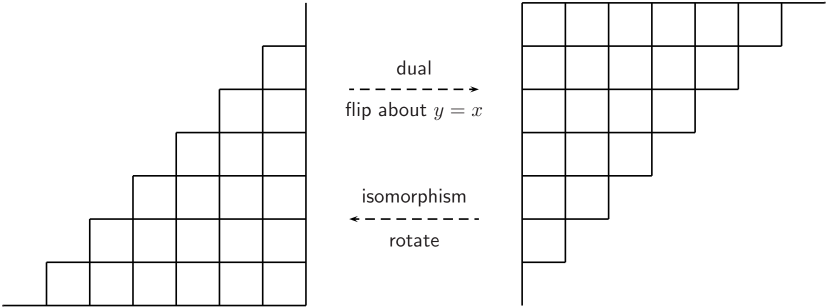


The class of transversal matroids is **not** closed under duality.



Catalan Matroids are Isomorphic to their Duals

Catalan matroids are self-dual, but not identically self-dual.



Rotating the diagram 180° gives an isomorphic matroid; the natural order of the elements is reversed.

Given a matroid M on S and $x \in S$, the deletion $M \backslash x$ is given by

(i) ground set: $S - x$

(ii) independent sets: independent sets I of M with $x \notin I$.

If x is not in all bases of M, then the bases of $M \backslash x$ are the bases B of M with $x \notin B$.

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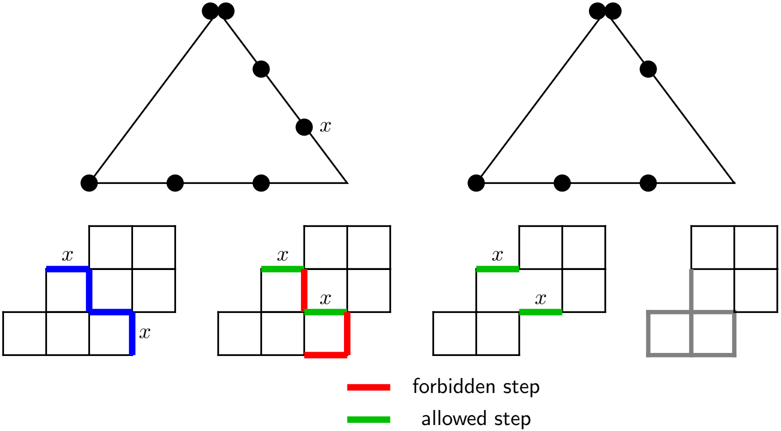
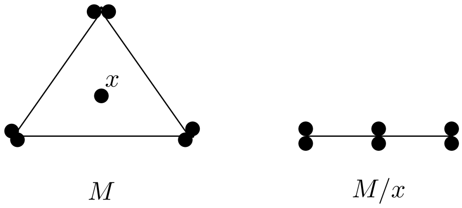


Contraction is the dual operation: $M/x = (M^* \backslash x)^*$.

If $\{x\}$ is independent, then the independent sets of M/x are the sets I for which $I \cup x$ is independent.

Geometrically, M/x projects $M \backslash x$ onto a hyperplane from the perspective of the point x.



Minors result from any combination of deletions and contractions.

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The class of lattice path matroids is closed under minors.

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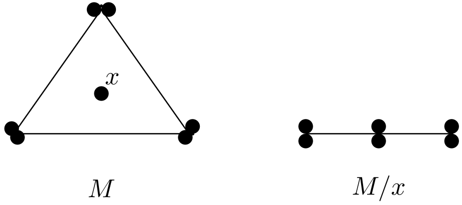
The class of transversal matroids is not closed under contraction.



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$$T(M; x, y) = \sum_{A \subseteq S} (x-1)^{r(S)-r(A)} (y-1)^{|A|-r(A)}.$$

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From the Tutte polynomial, we get

- chromatic and flow polynomials of graphs,
- weight enumerators of linear codes,
- information about arrangements of hyperplanes,
- Jones polynomials of alternating knots,
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- and ... information about lattice paths.

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is an evaluation of the Tutte polynomial.

For (transversal) matroids, computing $T(M;x,y)$ is $\#P$ hard.

Tutte polynomials of lattice path matroids are atypically accessible.

$T(M_5; x, y)$ is

$$xy \left(\begin{array}{lllll} & \mathbf{5}x & +\mathbf{5}x^2 & +3x^3 & +x^4 \\ +\mathbf{5}y & +\mathbf{5}xy & +3x^2y & +x^3y & \\ +\mathbf{5}y^2 & +3xy^2 & +x^2y^2 & & \\ +3y^3 & +xy^3 & & & \\ +y^4 & & & & \end{array} \right)$$

Sum of the coefficients: 42
Sum of the coefficients in the first “row”: 14

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Catalan numbers: 1, 1, 2, 5, 14, 42, 132, . . .

$T(M_6; x, y)$ is

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Theorem

$$T(M; x, y) = \sum_{\text{bases } B} x^{i(B)} y^{e(B)}$$

(Crapo, 1967)

$$T(M; x, y) = \sum_{\text{bases } B} x^{i(B)} y^{e(B)}$$

Corollary

$T(M; 1, 1)$ is the number of bases of M.

Internally active: $b = \min\{x : (B - b) \cup x \text{ is a basis}\}$.

Externally active: $a = \min\{x : (B \cup a) - x \text{ is a basis}\}$.

Lemma

The element b is internally active for B in M
if and only if b is externally active for $S - B$ in M^*.

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Corollary

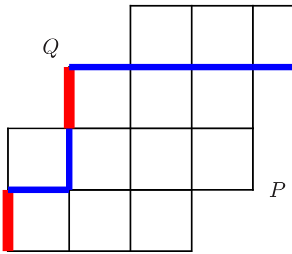
If M is self-dual, then $T(M; x, y)$ is symmetric in x and y.

Order: $1 < 2 < \cdots < m + r$.

Internally active: $b = \min\{x : (B - b) \cup x \text{ is a basis}\}$.

Theorem

The internally active elements in B correspond to the north steps of the associated path P_B that are on the upper bounding path Q.



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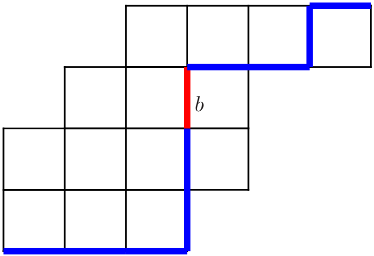

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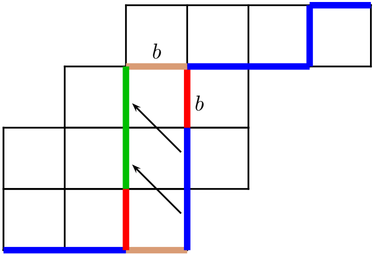

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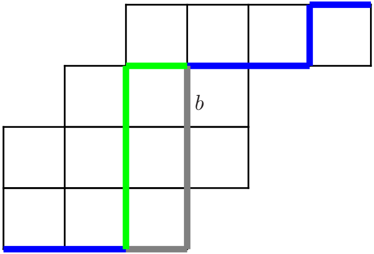

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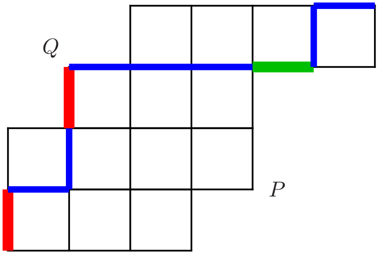


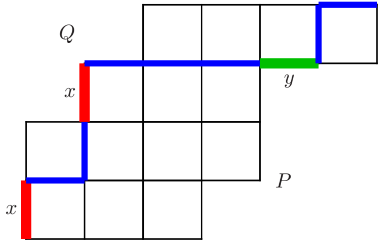
Corollary

$e(B)$ is the number of east steps that P_B shares with the lower bounding path P.

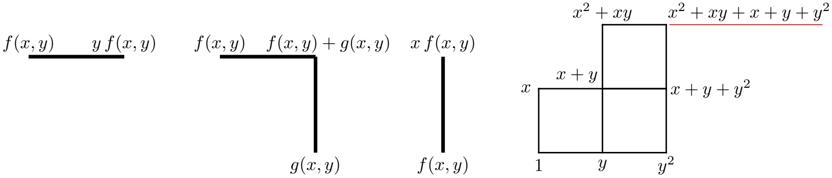
Theorem

The coefficient of $x^i y^j$ in the Tutte polynomial of $M[P, Q]$ is the number of paths in $\mathcal{P}$ sharing i north steps with Q and j east steps with P.

A basis, realized as a path, that contributes x^2y to the Tutte polynomial.



The recurrence to compute Tutte polynomials for lattice path matroids:



Corollary

The Tutte polynomial of a lattice path matroid can be computed in polynomial time.

A Change of Focus:

A Brief Overview of Some of the Many Research Directions in Matroid Theory

A matroid is representable over a field $\mathbb{F}$ if it is isomorphic to the matroid induced on the columns of a matrix over $\mathbb{F}$ by linear independence.

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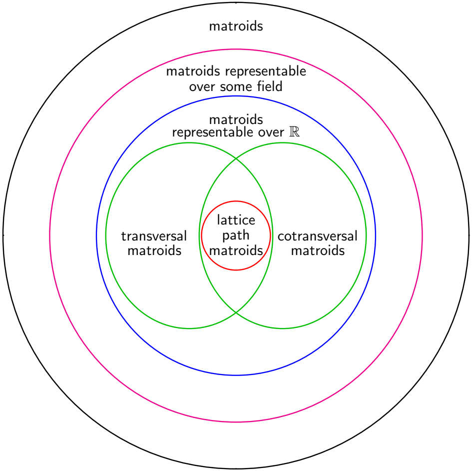
Theorem

A transversal matroid is representable over every sufficiently large field, in particular, over every infinite field.

(Piff and Welsh, 1970)



Conjecture

Asymptotically, almost no matroid is representable over any field. That is, the limit of the ratio

$$\frac{\#\text{ of matroids on } \{1,\ldots,n\} \text{ having matrix representations}}{\#\text{ of matroids on } \{1,\ldots,n\}}$$

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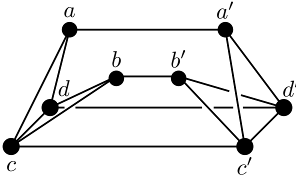
Theorem

For a fixed field $\mathbb{F}$, asymptotically, almost no matroid is representable over $\mathbb{F}$. (Ronyai, Babai, Ganapathy, 2001.)

The matroid below is the Vámos matroid.

Its independent sets are the sets of size at most four except $\{a, a', c, c'\}$, $\{a, a', d, d'\}$, $\{b, b', c, c'\}$, $\{b, b', d, d'\}$, $\{c, c', d, d'\}$.



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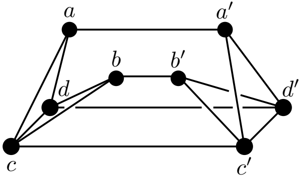

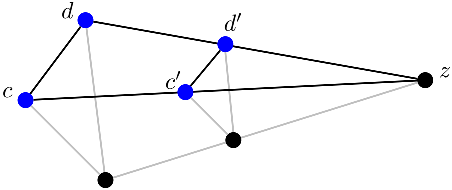

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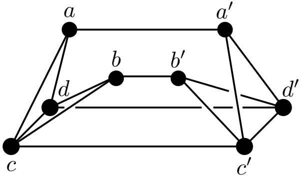

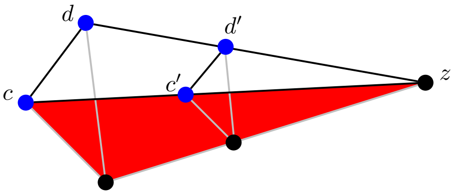

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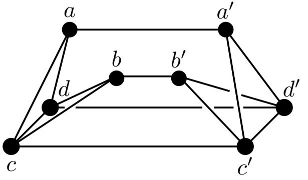

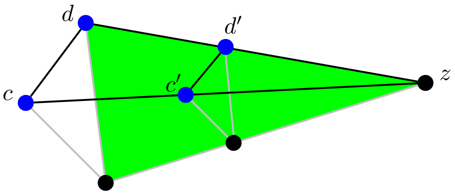

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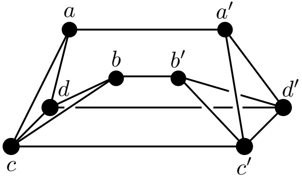

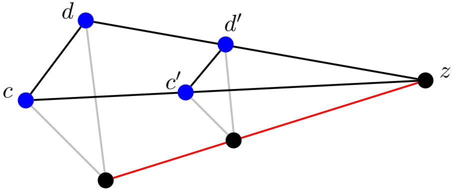

The Problem of Characterizing Representable Matroids

If M is representable over $\mathbb{F}$, then so are its minors.

One way to characterize the matroids are representable over $\mathbb{F}$ is to find the minor-minimal matroids that are not.

These minor-minimal obstructions are the excluded minors.

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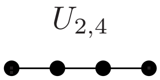
Theorem

A matroid is representable over $\mathbb{F}_2$ (binary) iff it does not have $U_{2,4}$ as a minor.

(Tutte, 1958)



If M is representable over $\mathbb{F}$, then so are its minors.

One way to characterize the matroids are representable over $\mathbb{F}$ is to find the minor-minimal matroids that are not.

These minor-minimal obstructions are the excluded minors.

Theorem

A matroid is representable over $\mathbb{F}_2$ (binary) iff it does not have $U_{2,4}$ as a minor.

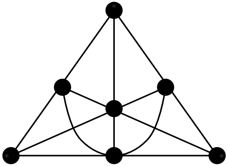
(Tutte, 1958)

$U_{2,4}$

Theorem

A matroid is representable over all fields iff it has none of $U_{2,4}$, F_7, and F_7^* as minors.

(Tutte, 1958)



F_7

Theorem

The excluded minors for representability over $\mathbb{F}_3$ (ternary matroids) are $U_{2,5}$, $U_{3,5}$, F_7, and F_7^*.

(Reid, 1972, unpublished; Seymour, 1979; Bixby, 1979)

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Geelen, Gerards, and Kapoor (2000) proved the excluded minor characterization of representability over $\mathbb{F}_4$.

Currently, over 500 excluded minors for representability over $\mathbb{F}_5$ are known.

Rota's conjecture (1971) has now been proven

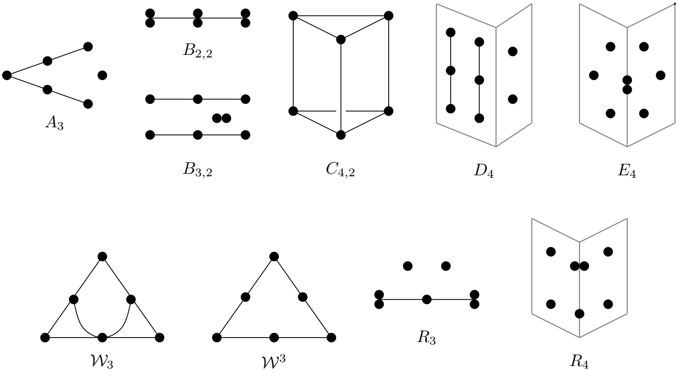
Theorem

If $\mathbb{F}$ is a finite field, then there are only finitely many excluded minors for representability over $\mathbb{F}$.

(Geelen, Gerards, Whittle, announced 2013)

The Excluded Minors for Lattice Path Matroids

The class of lattice path matroids is minor-closed;
the (infinitely many) excluded minors are known. (Bonin, 2010)



The Graph Minors Theorem

In any infinite set of (finite) graphs, some graph is isomorphic to a minor of another. *(N. Robertson and P. Seymour)*

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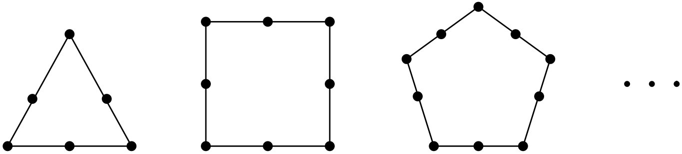

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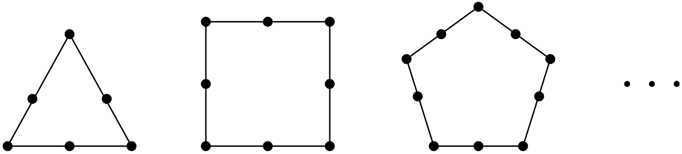

The Matroid Minors Theorem

For any fixed finite field $\mathbb{F}$, in any infinite set of $\mathbb{F}$-representable matroids, some matroid is isomorphic to a minor of another.

(Geelen, Gerards, Whittle, announced, 2013)

Theorem

Let $ex(H; n)$ be the maximum number of edges in a simple graph on n vertices that has no H-subgraph. Then

$$\lim_{n \to \infty} \frac{ex(H; n)}{\binom{n}{2}} = 1 - \frac{1}{\chi(H) - 1}.$$

(Erdös and Stone, 1946)

Theorem

Let $ex(H;n)$ be the maximum number of edges in a simple graph on n vertices that has no H-subgraph. Then

$$\lim_{n\to\infty} \frac{ex(H;n)}{\binom{n}{2}} = 1 - \frac{1}{\chi(H)-1}.$$

(Erdös and Stone, 1946)

Theorem

Let $ex_q(H;n)$ be the maximum number of elements in a simple $\mathbb{F}_q$-representable matroid that has no H-restriction. Then

$$\lim_{n\to\infty} \frac{ex_q(H;n)}{\frac{q^n-1}{q-1}} = 1 - q^{1-c}$$

where c is the minimum number so that H can be partitioned into c restrictions, each of which is affine over $\mathbb{F}_q$. (Geelen and Nelson, 2015)

Theorem

Let $\mathcal{C}$ be any minor-closed class of matroids. Either

1. *$\mathcal{C}$ contains all simple rank-2 matroids, or*
2. *there is a $c \in \mathbb{R}$ with $|E(M)| \leq c \cdot r(M)$ for all simple M in $\mathcal{C}$, or*
3. *$\mathcal{C}$ contains all graphic matroids and there is a $c \in \mathbb{R}$ with $|E(M)| \leq c \cdot (r(M))^2$ for all simple M in $\mathcal{C}$, or*
4. *there is a prime power q and a $c \in \mathbb{R}$ so that $\mathcal{C}$ contains all $\mathbb{F}_q$-representable matroids and $|E(M)| \leq c \cdot q^{r(M)}$ for all simple M in $\mathcal{C}$.*

(Geelen, Kung, and Whittle, 2008)

Theorem

Almost all matroids are 3-connected *(Oxley, Semple, Warshauer, and Welsh)*

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We have no model of a random matroid.

A matroid M is paving if $r(X) = |X|$ whenever $|X| < r(M)$.
It is sparse paving if M and M^* are paving.

Conjecture

Almost all matroids are sparse paving.

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Theorem

Almost all triangle-free graphs are bipartite.

(Erdös, Kleitman, and Rothschild, 1976)

Conjecture

Almost all binary matroids with no three points on a line are affine.

Constructions.

Special classes of matroids.
Binary matroids, graphic matroids, regular matroids, near-regular matroids, transversal matroids, base-orderable matroids, ...

Algebraic matroids.

Applications of the matroid structure theorem (from the matroid minors project).

Tutte polynomials, specializations, and generalizations.

Unimodality.

Generalizations, matroids with extra structure, and variations:
greedoids, jump systems, Coxeter matroids, flag matroids, polymatroids, bimatroids, oriented matroids, antimatroids, ...

... and much, much more.

Thank you for listening.