THE $\mathcal{G}$-INVARIANT AND CATENARY DATA OF A MATROID

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ABSTRACT. The catenary data of a matroid M of rank r on n elements is the vector $(\nu(M; a_0, a_1, \ldots, a_r))$, indexed by compositions $(a_0, a_1, \ldots, a_r)$, where $a_0 \geq 0$, $a_i > 0$ for $i \geq 1$, and $a_0 + a_1 + \cdots + a_r = n$, with the coordinate $\nu(M; a_0, a_1, \ldots, a_r)$ equal to the number of maximal chains or flags $(X_0, X_1, \ldots, X_r)$ of flats or closed sets such that X_i has rank i, $|X_0| = a_0$, and $|X_i - X_{i-1}| = a_i$. We show that the catenary data of M contains the same information about M as its $\mathcal{G}$-invariant, which was defined by H. Derksen [*J. Algebr. Combin.* 30 (2009) 43–86]. The Tutte polynomial is a specialization of the $\mathcal{G}$-invariant. We show that many known results for the Tutte polynomial have analogs for the $\mathcal{G}$-invariant. In particular, we show that for many matroid constructions, the $\mathcal{G}$-invariant of the construction can be calculated from the $\mathcal{G}$-invariants of the constituents and that the $\mathcal{G}$-invariant of a matroid can be calculated from its size, the isomorphism class of the lattice of cyclic flats with lattice elements labeled by the rank and size of the underlying set. We also show that the number of flats and cyclic flats of a given rank and size can be derived from the $\mathcal{G}$-invariant, that the $\mathcal{G}$-invariant of M is reconstructible from the deck of $\mathcal{G}$-invariants of restrictions of M to its copoints, and that, apart from free extensions and coextensions, one can detect whether a matroid is a free product from its $\mathcal{G}$-invariant.

1. THE $\mathcal{G}$-INVARIANT

Motivated by work on the F-invariant by Billera, Jia, and Reiner [1], Derksen introduced the $\mathcal{G}$-invariant for matroids and polymatroids in [9]. Together with Fink, Derksen showed in [10] that it is a universal valuative invariant for subdivisions of matroid base polytopes.

Let M be a rank-r matroid with rank function r and closure operator cl on the set $\{1, 2, \ldots, n\}$. The *rank sequence* $\underline{r}(\pi) = r_1 r_2 \ldots r_n$ of a permutation π on $\{1, 2, \ldots, n\}$ is the sequence defined by $r_1 = r(\{\pi(1)\})$ and for $j \geq 2$,

$$r_j = r(\{\pi(1), \pi(2), \ldots, \pi(j)\}) - r(\{\pi(1), \pi(2), \ldots, \pi(j-1)\}).$$

In a rank sequence, r_j is 0 or 1, there are exactly r 1's, and the set $\{\pi(j) : r_j = 1\}$ is a basis of M. The rank sequence $\underline{r}(\pi)$ is an (n, r)-*sequence*, that is, a sequence of n terms, r of which are 1 and the other $n - r$ of which are 0.

Let $[\underline{r}]$ be a variable or formal symbol, one for each (n, r)-sequence, and let $\mathcal{G}(n, r)$ be the vector space (over a field of characteristic zero) of dimension $\binom{n}{r}$ consisting of all formal linear combinations of such symbols. The $\mathcal{G}$-*invariant* and its coefficients $g_{\underline{r}}(M)$ are defined by

$$\mathcal{G}(M) = \sum_{\underline{r}} g_{\underline{r}}(M)[\underline{r}] = \sum_{\pi} [\underline{r}(\pi)]$$

where the sum is over all $n!$ permutations of $\{1, 2, \ldots, n\}$. (This definition is essentially Derksen's. We just replace a quasisymmetric function constructed from $\underline{r}$ by the symbol $[\underline{r}]$.) A *specialization* of the $\mathcal{G}$-invariant with values in an abelian group $\mathbb{A}$ is a function

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assigning a value in $\mathbb{A}$ to each symbol $[\underline{r}]$. Specializations may be given directly (as in Theorem 1.1) or indirectly. In particular, if L is a linear transformation from $\mathcal{G}(n,r)$ to a vector space V, then the assignment $[\underline{r}] \mapsto L([\underline{r}])$ gives a specialization.

Recall that the Tutte polynomial $T(M;x,y)$ of a rank-r matroid M on the set E is defined by

$$T(M;x,y) = \sum_{A \subseteq E} (x-1)^{r-r(A)}(y-1)^{|A|-r(A)}.$$

Derksen [9] showed that the $\mathcal{G}$-invariant specializes to the Tutte polynomial. In the next theorem, we state his specialization without using quasisymmetric functions.

Theorem 1.1 (Derksen)**.** *Let M be a rank-r matroid on a set of n elements. Then the Tutte polynomial $T(M;x,y)$ can be obtained from $\mathcal{G}(M)$ by the specialization*

$$[\underline{r}] \mapsto \sum_{m=0}^{n} \frac{(x-1)^{r-\mathrm{wt}(r_1r_2\ldots r_m)}(y-1)^{m-\mathrm{wt}(r_1r_2\ldots r_m)}}{m!(n-m)!},$$

where $\mathrm{wt}(r_1r_2\ldots r_m)$ is the number of 1's in the initial segment $r_1r_2\ldots r_m$ of $\underline{r}$.

In this paper, we study the $\mathcal{G}$-invariant from a combinatorial point of view. After some order-theoretic preliminaries in Section 2, we begin in Section 3 by determining the exact combinatorial information contained in the $\mathcal{G}$-invariant. This information is encoded in the catenary data, a vector or array of integers that records the number of flags or maximal chains of flats with given sizes. The $\mathcal{G}$-invariant and catenary data contain the same information; indeed, the catenary data are the coefficients of the $\mathcal{G}$-invariant when expanded in a new basis of $\mathcal{G}(n,r)$ called the γ-basis. In Section 4, we study how the $\mathcal{G}$-invariant and catenary data behave under matroid constructions. Many of the known results saying that the Tutte polynomial of a construction can be calculated from the Tutte polynomials of its constituents have counterparts for the $\mathcal{G}$-invariant. We show in Section 5 that many parameters of a matroid not derivable from the Tutte polynomial can be derived from the $\mathcal{G}$-invariant. In addition, we show, in Section 6, that the $\mathcal{G}$-invariant can be reconstructed from several decks, giving analogs of reconstructibility results for the Tutte polynomial. In Section 7, extending a result of Eberhardt [12] for the Tutte polynomial, we show that the $\mathcal{G}$-invariant is determined by the isomorphism type of the lattice of cyclic flats, along with the size and rank of the cyclic flat that corresponds to each element in the lattice. In the final section, we show that, except for free extensions and coextensions, whether a matroid is a free product can be detected from its $\mathcal{G}$-invariant.

Although we focus on matroids, we remark that the $\mathcal{G}$-invariant of matroids constructed from graphs, such as cycle or bicircular matroids, should have many applications in graph theory. For example, whether a graph has a Hamiltonian cycle is not deducible from the Tutte polynomial of the graph but is deducible from the $\mathcal{G}$-invariant of its cycle matroid (Corollary 5.3).

2. Partial orders on sequences and compositions

We use the notation 0^a for a sequence of a 0's and 1^b for a sequence of b 1's. An (n,r)-*composition* is a length-$(r+1)$ integer sequence $(a_0,a_1,a_2,\ldots,a_r)$ satisfying the inequalities $a_0 \geq 0$ and $a_j > 0$ for $1 \leq j \leq r$, together with the equality

$$a_0 + a_1 + a_2 + \cdots + a_r = n.$$

The correspondence

$$0^{a_0}10^{a_1-1}10^{a_2-1}\ldots10^{a_r-1} \longleftrightarrow (a_0,a_1,a_2,\ldots,a_r)$$

gives a bijection between the set of (n, r)-sequences and the set of (n, r)-compositions. In this paper, we identify an (n, r)-sequence and its corresponding (n, r)-composition.

We also need a partial order on (n, r)-sequences. If $\underline{s}$ and $\underline{t}$ are (n, r)-sequences, then $\underline{t} \trianglerighteq \underline{s}$ if

$$t_1 + t_2 + \cdots + t_j \geq s_1 + s_2 + \cdots + s_j$$

for every index j; in other words, reading from the left, there are always at least as many 1's in $\underline{t}$ as there are in $\underline{s}$. This order has maximum 1^r0^{n-r} and minimum $0^{n-r}1^r$. Under the bijection, $\trianglerighteq$ defines a partial order on (n, r)-compositions given by $(b_0, b_1, \ldots, b_r) \trianglerighteq (a_0, a_1, \ldots, a_r)$ if and only if for every index j,

$$b_0 + b_1 + \cdots + b_j \leq a_0 + a_1 + \cdots + a_j.$$

This order is a suborder of the reversed dominance order on compositions. While this link plays no role in what we do, we note that the partial order $\trianglerighteq$ on (n, r)-sequences is a distributive lattice that is isomorphic to a sublattice of Young's partition lattice (see [18, p. 288]).

3. Catenary data

Let M be a rank-r matroid on a set E of size n. A *flag* $(X_i) = (X_0, X_1, \ldots, X_r)$ is a maximal chain

$$\mathrm{cl}(\emptyset) = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_{r-1} \subset X_r = E,$$

where X_j is a rank-j flat of M. The *composition* $\mathrm{comp}((X_i))$ of the flag (X_i) is the (n, r)-composition $(a_0, a_1, a_2, \ldots, a_r)$, where $a_0 = |X_0|$, and for positive j,

$$a_j = |X_j - X_{j-1}|.$$

Thus, a_0 is the number of loops in M. Also, the rank sequence $0^{a_0}10^{a_1-1}\ldots10^{a_r-1}$ corresponding to $\mathrm{comp}((X_i))$ is the rank sequence of each of the $a_0!a_1!\cdots a_r!$ permutations that, for all positive j, puts the elements of X_{j-1} before those of the set-difference $X_j - X_{j-1}$.

We use the notation $(t)_k$ for the falling factorial $t(t-1)(t-2)\cdots(t-k+1)$, the number of sequences of k distinct objects chosen from a set of t objects. For an (n, r)-sequence $\underline{s}$ with composition $(a_0, a_1, \ldots, a_r)$, let $\gamma(a_0, a_1, \ldots, a_r)$, and its coefficients $c_{(a_0,a_1,\ldots,a_r)}(b_0, b_1, \ldots, b_r)$ be defined by

$$\begin{aligned}\gamma(a_0, a_1, \ldots, a_r) &= \sum_{(b_0,b_1,\ldots,b_r)} c_{(a_0,a_1,\ldots,a_r)}(b_0, b_1, \ldots, b_r)[0^{b_0}10^{b_1-1}\ldots10^{b_r-1}] \\ &= \sum_{(b_j)\trianglerighteq(a_j)} (a_0)_{b_0}\left(\prod_{j=1}^{r} a_j\left(a_j - 1 + \left(\sum_{i=0}^{j-1} a_i - b_i\right)\right)_{b_j-1}\right)[0^{b_0}10^{b_1-1}\ldots10^{b_r-1}],\end{aligned}$$

where the second sum ranges over all compositions with $(b_0, b_1, \ldots, b_r) \trianglerighteq (a_0, a_1, \ldots, a_r)$. For example,

$$\begin{aligned}\gamma(0, 1, 1, 4) &= 24\,[111000], \\ \gamma(0, 1, 2, 3) &= 36\,[111000] + 12\,[110100], \\ \gamma(0, 1, 3, 2) &= 36\,[111000] + 24\,[110100] + 12\,[110010], \\ \gamma(0, 1, 4, 1) &= 24\,[111000] + 24\,[110100] + 24\,[110010] + 24\,[110001].\end{aligned}$$

Note that, by definition, the coefficient $c_{(a_0,a_1,\ldots,a_r)}(b_0, b_1, \ldots, b_r)$ is non-zero if and only if $(b_0, b_1, \ldots, b_r) \trianglerighteq (a_0, a_1, \ldots, a_r)$. Hence the matrix $(c_{(a_0,a_1,\ldots,a_r)}(b_0, b_1, \ldots, b_r))$ is

a lower triangular matrix. In particular, the linear combinations $\gamma(a_0, a_1, \ldots, a_r)$, where $(a_0, a_1, \ldots, a_r)$ ranges over all (n, r)-compositions, form a basis, called the γ-*basis*, for the vector space $\mathcal{G}(n, r)$.

With each permutation π, we get an ordered basis $(\pi(i_1), \pi(i_2), \ldots, \pi(i_r))$ from the terms in its rank sequence $r_1 r_2 \ldots r_n$ with $r_{i_j} = 1$, where $i_1 < i_2 < \cdots < i_r$. This basis determines a flag, denoted $\mathrm{flag}(\pi)$, obtained by setting $X_0 = \mathrm{cl}(\emptyset)$ and

$$X_j = \mathrm{cl}(\{\pi(i_1), \pi(i_2), \ldots, \pi(i_j)\}).$$

Lemma 3.1. *Let (X_i) be a flag of M with composition $(a_0, a_1, \ldots, a_r)$. Then*

$$\sum_{\pi\,:\,\mathrm{flag}(\pi)=(X_i)} [\underline{r}(\pi)] = \gamma(a_0, a_1, \ldots, a_r).$$

In particular, the sum on the left hand side depends only on $\mathrm{comp}((X_i))$.

Proof. Let π be a permutation whose rank sequence $\underline{r}(\pi)$ equals $0^{b_0}10^{b_1-1}10^{b_2-1}\ldots 10^{b_r-1}$. Then, by the Mac Lane-Steinitz exchange property for closure, $\mathrm{flag}(\pi) = (X_i)$ if and only if

$$\begin{aligned}
&\pi(1), \pi(2), \ldots, \pi(b_0) \in X_0,\\
&\pi(b_0+1) \in X_1 - X_0,\\
&\pi(b_0+2), \ldots, \pi(b_0+b_1) \in X_1,\\
&\quad\vdots\\
&\pi(b_0+b_1+\cdots+b_{j-1}+1) \in X_j - X_{j-1},\\
&\pi(b_0+b_1+\cdots+b_{j-1}+2), \ldots, \pi(b_0+b_1+\cdots+b_{j-1}+b_j) \in X_j,\\
&\quad\vdots\\
&\pi(b_0+b_1+\cdots+b_{r-1}+1) \in X_r - X_{r-1},\\
&\pi(b_0+b_1+\cdots+b_{r-1}+2), \ldots, \pi(n) \in X_r.
\end{aligned}$$

Such a permutation π exists if and only if $(b_0, b_1, \ldots, b_r) \trianglerighteq (a_0, a_1, \ldots, a_r)$ and all such permutations can be found by choosing a length-b_0 sequence of distinct elements from X_0 (a set of size a_0), an element from $X_1 - X_0$ (a set of size a_1), a length-(b_1-1) sequence of distinct elements from the subset of elements in $X_1 \cup X_0$ not chosen earlier (a set of size $(a_1-1)+(a_0-b_0)$), an element from $X_2 - X_1$ (a set of size a_2), a length-(b_2-1) sequence of distinct elements from the subset of elements in $X_2 \cup X_1 \cup X_0$ not chosen earlier (a set of size $(a_2-1)+(a_1+a_0)-(b_1+b_0)$), and so on. Thus there are $c_{(a_0,a_1,\ldots,a_r)}(b_0, b_1, \ldots, b_r)$ such permutations π. □

The ideas in the proof of Lemma 3.1 give the next result.

Lemma 3.2. *For a flag (X_i) of M with composition $(a_0, a_1, \ldots, a_r)$, there are $a_1 a_2 \cdots a_r$ ordered bases $(e_1, e_2, \ldots, e_r)$ of M with* $\mathrm{cl}(\{e_1, e_2, \ldots, e_i\}) = X_i$ *for* $1 \le i \le r$.

For an (n, r)-composition $(a_0, a_1, \ldots, a_r)$, let $\nu(M; a_0, a_1, \ldots, a_r)$ be the number of flags (X_i) in M with composition $(a_0, a_1, \ldots, a_r)$. The *catenary data* of M is the $\binom{n}{r}$-dimensional vector $(\nu(M; a_0, a_1, \ldots, a_r))$ indexed by (n, r)-compositions. When giving the catenary data of a specific matroid, we usually give only the coordinates that might be non-zero; when a coordinate is not given, it is zero.

Theorem 3.3. *The $\mathcal{G}$-invariant of M is determined by its catenary data and conversely. In particular,*

$$\mathcal{G}(M) = \sum_{(a_0,a_1,\ldots,a_r)} \nu(M; a_0, a_1, \ldots, a_r)\, \gamma(a_0, a_1, \ldots, a_r). \tag{3.1}$$

Proof. Partitioning the permutations π on E according to their flags, using Lemma 3.1, and then partitioning the flags according to their compositions, we have

$$\begin{aligned}
\mathcal{G}(M) &= \sum_{(X_i)} \left(\sum_{\mathrm{flag}(\pi)=(X_i)} [\underline{r}(\pi)] \right) \\
&= \sum_{(X_i)} \gamma(|X_0|, |X_1 - X_0|, \ldots, |X_r - X_{r-1}|) \\
&= \sum_{(a_0,a_1,\ldots,a_r)} \nu(M; a_0, a_1, \ldots, a_r)\, \gamma(a_0, a_1, \ldots, a_r).
\end{aligned}$$

□

Since equation (3.1) is obtained by partitioning terms in a sum (with no cancellation), writing the $\mathcal{G}$-invariant in the symbol basis requires at least as many terms as writing it in the γ-basis. The next example shows that the γ-basis may require far fewer terms than the symbol basis.

Example 3.4. A perfect matroid design (Young and Edmonds [21]; see also Welsh [19, Section 12.5]) is a matroid in which flats of the same rank have the same size. Thus, for such a matroid M, there is a sequence $(\alpha_0, \alpha_1, \ldots, \alpha_r)$, where $r = r(M)$, such that $|F| = \alpha_i$ for all flats F of rank i in M. For example, the rank-r projective geometry $\mathrm{PG}(r-1, q)$ over the finite field $\mathrm{GF}(q)$ is a perfect matroid design with the sequence of flat sizes being

$$(0, 1, q+1, q^2+q+1, \ldots, q^{r-1}+q^{r-2}+\cdots+q+1).$$

Another example is the rank-r affine geometry $\mathrm{AG}(r-1, q)$ over $\mathrm{GF}(q)$, with its sequence being $(0, 1, q, q^2, \ldots, q^{r-1})$. If a rank-$i$ flat X is contained in exactly t flats of rank $i+1$, say X_j, for $1 \le j \le t$, then the t differences $X_j - X$ partition the set of elements not in X, and $|X_j - X| = \alpha_{i+1} - \alpha_i$, so t is $(\alpha_r - \alpha_i)/(\alpha_{i+1} - \alpha_i)$. Therefore

$$\nu(M; \alpha_0, \alpha_1 - \alpha_0, \alpha_2 - \alpha_1, \ldots, \alpha_r - \alpha_{r-1}) = \prod_{i=0}^{r-1} \frac{\alpha_r - \alpha_i}{\alpha_{i+1} - \alpha_i},$$

and this accounts for all flags in M. (Indeed, a matroid is a perfect matroid design if and only if exactly one coefficient $\nu(M; a_0, a_1, \ldots, a_r)$ is nonzero.) Thus,

$$\mathcal{G}(M) = \left(\prod_{i=0}^{r-1} \frac{\alpha_r - \alpha_i}{\alpha_{i+1} - \alpha_i} \right) \gamma(\alpha_0, \alpha_1 - \alpha_0, \alpha_2 - \alpha_1, \ldots, \alpha_r - \alpha_{r-1}).$$

In particular,

$$\mathcal{G}\big(\mathrm{PG}(r-1, q)\big) = \left(\prod_{i=0}^{r-1} \frac{q^{r-i} - 1}{q - 1} \right) \gamma(0, 1, q, q^2, \ldots, q^{r-1}),$$

and

$$\mathcal{G}\big(\mathrm{AG}(r-1, q)\big) = q^{r-1} \left(\prod_{i=1}^{r-1} \frac{q^{r-i} - 1}{q - 1} \right) \gamma(0, 1, q-1, q^2-q, \ldots, q^{r-1}-q^{r-2}).$$

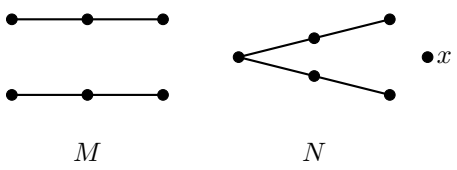


FIGURE 1. The smallest pair of matroids with the same $\mathcal{G}$-invariant and catenary data.

Example 3.5. The matroids M and N in Figure 1, the two smallest matroids that have the same Tutte polynomial [4], also have the same catenary data, namely,

$$\nu(\,\cdot\,;0,1,2,3) = 6, \qquad \nu(\,\cdot\,;0,1,1,4) = 18.$$

This agrees with Derksen's calculation in [9] that their $\mathcal{G}$-invariant equals

$$72[110100] + 648[111000].$$

The following proposition (whose proof is immediate) gives a recursion on restrictions to copoints (that is, rank-$(r-1)$ flats) for catenary data.

Proposition 3.6. *Let $(a_0, a_1, \ldots, a_r)$ be an (n, r)-composition. Then*

$$\nu(M; a_0, a_1, \ldots, a_{r-1}, a_r) = \sum_{X \text{ a copoint}, |E-X|=a_r} \nu(M|X; a_0, a_1, \ldots, a_{r-1}).$$

Proposition 3.6 allows us to compute the catenary data of paving matroids (see also [13]). Recall that a rank-r matroid M is *paving* if all circuits have r or $r+1$ elements. Note that a matroid M is paving if and only if every symbol $[\underline{r}]$ that occurs with non-zero coefficient in its $\mathcal{G}$-invariant starts with $r-1$ 1's.

Theorem 3.7. *Let M be a rank-r paving matroid on $\{1, 2, \ldots, n\}$ with $f_{r-1}(m)$ copoints with m elements. Then M has catenary data:*

$$\nu(M; 0, \underbrace{1, 1, , \ldots, 1}_{r-2}, m-r+2, n-m) = f_{r-1}(m) \cdot (m)_{r-2}.$$

All simple rank-3 matroids are paving, so we have explicit formulas for their $\mathcal{G}$-invariants and catenary data. This includes the matroids in Example 3.5.

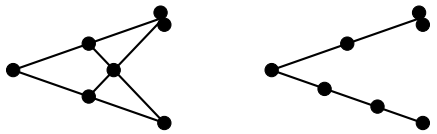
Example 3.8. The cycle matroid $M(K_4)$ of the complete graph K_4, has three 2-point lines and four 3-point lines; hence

$$\nu(M(K_4); 0, 1, 1, 4) = 6, \qquad \nu(M(K_4); 0, 1, 2, 3) = 12.$$

and

$$\mathcal{G}(M(K_4)) = 6\,\gamma(0,1,1,4) + 12\,\gamma(0,1,2,3) = 576\,[111000] + 144\,[110100].$$

We end this section with two more examples, the matroids M_1 and M_2 in Figure 2. Brylawski [4, p. 268] showed that they are the smallest pair of non-paving matroids with the same Tutte polynomial, but Derksen [9] showed that their $\mathcal{G}$-invariants are distinct. The catenary data of these two matroids are given in the table



M_1 M_2

FIGURE 2. The matroids M_1 and M_2.

	M_1	M_2
$\nu(M_i;0,1,1,5)$	4	6
$\nu(M_i;0,1,2,4)$	7	3
$\nu(M_i;0,1,3,3)$	4	6
$\nu(M_i;0,2,1,4)$	1	3
$\nu(M_i;0,2,2,3)$	2	1

and they are indeed different.

4. CONSTRUCTIONS ON MATROIDS

In many cases, the $\mathcal{G}$-invariant of a construction is determined by the $\mathcal{G}$-invariants of its constituents. In this section, we give an account of currently known results of this type, all of which originated from results on Tutte polynomials. Specifically, we discuss deletion, contraction, dual, truncation, Higgs lift, direct sum, free extension, free coextension, free product, q-cone, and circuit-hyperplane relaxation.

4.1. **Deletion and contraction.** The $\mathcal{G}$-invariant does not satisfy a deletion-contraction rule since it is strictly stronger than the Tutte polynomial, which is universal for deletion-contraction invariants. Still, we get a formula involving all single-element deletions, and likewise for contractions. First, for two $0,1$-sequences $\underline{r}$ and $\underline{r}'$, define the *concatenation* $\diamond$ of the corresponding symbols $[\underline{r}]$ and $[\underline{r}']$ by $[\underline{r}]\diamond[\underline{r}'] = [\underline{r}\underline{r}']$ and extend $\diamond$ by bilinearity. Putting the permutations into two groups according to whether the last element is a coloop of M gives

$$\mathcal{G}(M) = \sum_{\substack{a\in E,\\ \text{not a coloop}}} \mathcal{G}(M\backslash a)\diamond[0] + \sum_{\substack{a\in E,\\ \text{a coloop}}} \mathcal{G}(M\backslash a)\diamond[1].$$

Similarly,

$$\mathcal{G}(M) = \sum_{\substack{a\in E,\\ \text{not a loop}}} [1]\diamond\mathcal{G}(M/a) + \sum_{\substack{a\in E,\\ \text{a loop}}} [0]\diamond\mathcal{G}(M/a).$$

4.2. **Dual.** The next proposition, due to Derksen [9], is included for completeness.

Proposition 4.1. *The $\mathcal{G}$-invariant $\mathcal{G}(M^*)$ of the dual of M can be obtained from $\mathcal{G}(M)$ by the specialization*

$$[r_1r_2\dots r_{n-1}r_n] \mapsto [\bar{r}_n\bar{r}_{n-1}\dots\bar{r}_2\bar{r}_1],$$

where $\bar{}$ switches 0's and 1's, that is, $\bar{0}=1$ and $\bar{1}=0$.

There is no simple version of Proposition 4.1 for the γ-basis.

4.3. **Truncation and Higgs lift.** The *truncation* $\mathrm{Trun}(M)$ of a matroid M of positive rank is the matroid on the same set whose bases are the independent sets of M that have size $r(M)-1$. The construction dual to truncation is the *free or Higgs lift* [14] defined by $\mathrm{Lift}(M) = (\mathrm{Trun}(M^*))^*$ for a matroid M with at least one circuit.

Proposition 4.2. *Let M be a matroid having positive rank. Then $\mathcal{G}(\mathrm{Trun}(M))$ can be obtained from $\mathcal{G}(M)$ by the specialization $[\underline{r}] \mapsto [\underline{r}^{\downarrow}]$, where $\underline{r}^{\downarrow}$ is the sequence obtained from $\underline{r}$ by replacing the right-most 1 by 0. Expressed in the γ-basis, $\mathcal{G}(\mathrm{Trun}(M))$ can be obtained by the specialization given by the linear transformation $\mathcal{G}(n,r) \to \mathcal{G}(n,r-1)$ defined on the γ-basis by $\gamma(a_0,a_1,\ldots,a_r) \mapsto \gamma(a_0,a_1,\ldots,a_{r-1}+a_r)$.*

Let M have at least one circuit. Then $\mathcal{G}(\mathrm{Lift}(M))$ can be obtained from $\mathcal{G}(M)$ by the specialization $[\underline{r}] \mapsto [\underline{r}^{\uparrow}]$, where $\underline{r}^{\uparrow}$ is obtained from $\underline{r}$ by replacing the left-most 0 by 1.

4.4. **Direct sum.** We first define a shuffle. Let $(\alpha_1,\alpha_2,\ldots,\alpha_m)$ and $(\beta_1,\beta_2,\ldots,\beta_n)$ be sequences and P be a subset of $\{1,2,\ldots,m+n\}$ of size m. The *shuffle* $\mathrm{sh}((\alpha_i),(\beta_j);P)$ is the length-$(m+n)$ sequence obtained by inserting the sequence (α_i) in order into the positions in P and the sequence (β_j) in order into the remaining positions. For example,

$$\mathrm{sh}((\alpha_1,\alpha_2,\alpha_3,\alpha_4),(\beta_1,\beta_2,\beta_3);\{1,3,4,7\}) = (\alpha_1,\beta_1,\alpha_2,\alpha_3,\beta_2,\beta_3,\alpha_4).$$

If $[\underline{r}]$ and $[\underline{s}]$ are symbols, the first in $\mathcal{G}(n_1,r_1)$ and the second in $\mathcal{G}(n_2,r_2)$, then their *shuffle product* $[\underline{r}] \wr [\underline{s}]$ is the following linear combination in $\mathcal{G}(n_1+n_2,r_1+r_2)$:

$$[\underline{r}] \wr [\underline{s}] = \sum_{P\subseteq\{1,2,\ldots,n_1+n_2\},|P|=n_1} [\mathrm{sh}(\underline{r},\underline{s};P)].$$

The shuffle product $\wr$ is extended to $\mathcal{G}(n_1,r_1)\times\mathcal{G}(n_1,r_1)$ by bilinearity.

The *direct sum* $M_1\oplus M_2$ of the matroids M_i with rank and rank functions r_i on disjoint sets E_i is the matroid on the set $E_1\cup E_2$ with rank function $r_{M_1\oplus M_2}$ where, for $X_i\subseteq E_i$,

$$r_{M_1\oplus M_2}(X_1\cup X_2) = r_1(X_1)+r_2(X_2). \tag{4.1}$$

From the fact that the rank sequence of a shuffle of two permutations is the shuffle of their rank sequences, we obtain the following result.

Proposition 4.3. *The $\mathcal{G}$-invariant of the direct sum $M_1\oplus M_2$ is given by the formula*

$$\mathcal{G}(M_1\oplus M_2) = \mathcal{G}(M_1)\wr\mathcal{G}(M_2).$$

The next proposition follows easily from the fact that the lattice of flats $L(M_1\oplus M_2)$ of the direct sum is the direct product $L(M_1)\times L(M_2)$ of the lattices of flats of the summands, together with formula (4.1).

Proposition 4.4. *The catenary data of the direct sum $M_1\oplus M_2$ can be calculated in the following way: for an $(|E_1|+|E_2|, r_1+r_2)$-composition $(a_0+b_0,c_1,c_2,\ldots,c_{r_1+r_2})$, where a_0 and b_0 are the numbers of loops in M_1 and M_2, respectively, we have*

$$\nu(M_1\oplus M_2;a_0+b_0,c_1,\ldots,c_{r_1+r_2}) = \sum_{((a_i),(b_j),P)} \nu(M_1;a_0,a_1,\ldots,a_{r_1})\,\nu(M_2;b_0,b_1,\ldots,b_{r_2})$$

where the sum is over all triples $((a_1,\ldots,a_{r_1}),(b_1,\ldots,b_{r_2}),P)$ with $|P|=r_1$ and

$$\mathrm{sh}((a_1,\ldots,a_{r_1}),(b_1,\ldots,b_{r_2});P) = (c_1,c_2,\ldots,c_{r_1+r_2}).$$

The next result treats the effect on the $\mathcal{G}$-invariant of adding a coloop or loop.

Corollary 4.5. *The $\mathcal{G}$-invariant $\mathcal{G}(M \oplus U_{1,1})$ can be obtained from $\mathcal{G}(M)$ by the specialization*

$$[r_1 r_2 \dots r_n] \mapsto [1 r_1 r_2 \dots r_n] + \sum_{j=1}^{n} [r_1 \dots r_j 1 \dots r_n]$$

and $\mathcal{G}(M \oplus U_{0,1})$ can be obtained from $\mathcal{G}(M)$ by the specialization

$$[r_1 r_2 \dots r_n] \mapsto [0 r_1 r_2 \dots r_n] + \sum_{j=1}^{n} [r_1 \dots r_j 0 \dots r_n].$$

There is also a simple description of the effect of adding or removing loops from a matroid using the γ-basis.

Proposition 4.6. *Let M be a loopless rank-r matroid. Then when expressed in the γ-basis, the $\mathcal{G}$-invariants $\mathcal{G}(M \oplus U_{0,h})$ and $\mathcal{G}(M)$ can be obtained from each other by the specializations*

$$\gamma(h, a_1, a_2, \dots, a_r) \longleftrightarrow \gamma(0, a_1, a_2, \dots, a_r).$$

While M has exactly a_0 loops if and only if some term of the form $\nu(M; a_0, a_1, \dots, a_r)$ is positive, the number of coloops is the greatest integer k for which some composition ending in k ones has $\nu(M; a_0, a_1, \dots, 1, 1) > 0$.

In the cycle matroid $M(K_{r+1})$ of the complete graph, there are $2^r - 1$ copoints. For $1 \le m < \lfloor (r+1)/2 \rfloor$, there are $\binom{r+1}{m}$ copoints isomorphic to $M(K_{r+1-m}) \oplus M(K_m)$, with an additional $\frac{1}{2}\binom{r+1}{(r+1)/2}$ copoints isomorphic to $M(K_{(r+1)/2}) \oplus M(K_{(r+1)/2})$ if $r+1$ is even. Hence, by Propositions 3.6 and 4.4, the catenary data of $M(K_{r+1})$ can be obtained recursively from the catenary data of lower-rank cycle matroids $M(K_m)$. This recursion is straightforward but complicated. Similar recursions exist for the catenary data of other matroids (such as bicircular matroids) constructed from complete graphs.

Example 4.7. Consider $M(K_5)$. In $M(K_5)$, there are five copoints isomorphic to $M(K_4)$ and ten isomorphic to $M(K_3) \oplus M(K_2)$. The catenary data of $M(K_4)$ is given in Example 3.8. Since $M(K_3) \oplus M(K_2) \cong U_{2,3} \oplus U_{1,1}$, Proposition 4.4 yields its catenary data:

$$\nu(M(K_3) \oplus M(K_2); 0, 1, 1, 2) = 6, \quad \nu(M(K_3) \oplus M(K_2); 0, 1, 2, 1) = 3.$$

We can now use Proposition 3.6 to obtain the catenary data of $M(K_5)$:

$$\begin{aligned}
\nu(M(K_5); 0, 1, 2, 3, 4) &= 5 \cdot 12 = 60, \\
\nu(M(K_5); 0, 1, 1, 4, 4) &= 5 \cdot 6 \ = 30, \\
\nu(M(K_5); 0, 1, 1, 2, 6) &= 10 \cdot 6 = 60, \\
\nu(M(K_5); 0, 1, 2, 1, 6) &= 10 \cdot 3 = 30.
\end{aligned}$$

The Dowling matroids $Q_r(G)$ based on the finite group G generalize the cycle matroids $M(K_{r+1})$ (see [11]). In $Q_r(G)$, there are $[(|G|+1)^r - 1]/|G|$ copoints. When $|G| \ge 2$ and $0 \le m \le r-1$, there are $\binom{r}{m}|G|^{r-m-1}$ copoints isomorphic to $Q_m(G) \oplus M(K_{r-m})$. Thus, by Lemma 3.6, the catenary data of $Q_r(G)$ can be obtained from lower-rank Dowling matroids or cycle matroids of complete graphs by a recursion depending only on the order $|G|$. The next result follows by induction starting with the observation that $Q_1(G)$ is isomorphic to $U_{1,1}$.

Proposition 4.8. *The $\mathcal{G}$-invariant of the Dowling matroid $Q_r(G)$ depends only on r and $|G|$.*

4.5. **Free extension and coextension.** Given a matroid M and an element x not in $E(M)$, the *free extension* $M + x$ is the matroid on $E(M) \cup x$ whose bases are the bases of M or the sets of the form $B \cup x$ where B is a basis of a copoint of M. Equivalently,

$$M + x = \mathrm{Trun}(M \oplus U_{1,1}),$$

where $U_{1,1}$ is the rank-1 matroid on the set $\{x\}$. The construction dual to free extension is the *free coextension* $M \times x$, defined by $M \times x = (M^* + x)^*$. This equals the Higgs lift $\mathrm{Lift}(M \oplus U_{0,1})$, where $U_{0,1}$ is the rank-0 matroid on the set $\{x\}$.

Proposition 4.9. *The $\mathcal{G}$-invariant of the free extension $M + x$ is obtained from $\mathcal{G}(M)$ by the specialization*

$$[r_1 r_2 \dots r_n] \mapsto [1 r_1 r_2 \dots r_n]^{\downarrow} + \sum_{j=1}^{n} [r_1 \dots r_j 1 \dots r_n]^{\downarrow},$$

and the $\mathcal{G}$-invariant of the free coextension $M \times x$ is obtained by

$$[r_1 r_2 \dots r_n] \mapsto [0 r_1 r_2 \dots r_n]^{\uparrow} + \sum_{j=1}^{n} [r_1 \dots r_j 0 \dots r_n]^{\uparrow}.$$

A freedom (or nested) matroid is obtained from a loop or a coloop by a sequence of free extensions or additions of a coloop. Thus, one can recursively compute the $\mathcal{G}$-invariant of a freedom matroid using Corollary 4.5 and Proposition 4.9.

Example 4.10. The matroid N in Figure 1 is the free extension of $N\backslash x$. The $\mathcal{G}$-invariant of $N\backslash x$ is $96\,[11100] + 24\,[11010]$. Each occurrence of $[11100]$ in $\mathcal{G}(N\backslash x)$ gives rise to six terms in the free extension in the following way. First insert a $\mathbf{1}$ to obtain

$$[\mathbf{1}11100] + [1\mathbf{1}1100] + [11\mathbf{1}100] + [111\mathbf{1}00] + [1110\mathbf{1}0] + [11100\mathbf{1}]$$

and then change the right-most 1 to $\mathbf{0}$

$$[111\mathbf{0}00] + [111\mathbf{0}00] + [111\mathbf{0}00] + [111\mathbf{0}00] + [1110\mathbf{0}0] + [11100\mathbf{0}].$$

Likewise, each occurrence of $[11010]$ gives rise to six terms:

$$[111000] + [111000] + [111000] + [110100] + [110100] + [110100].$$

From this, we obtain

$$\mathcal{G}(N) = (96 \cdot 6 + 3 \cdot 24)\,[111000] + 3 \cdot 24\,[110100] = 648\,[111000] + 72\,[110100].$$

4.6. **Free product.** The free product is a non-commutative matroid operation defined by Crapo and Schmitt [7]. Given an ordered pair M_1 and M_2 of matroids of ranks r_1 and r_2 on disjoint sets E_1 and E_2, the *free product* $M_1 \boxdot M_2$ is the matroid on $E_1 \cup E_2$ whose bases are the subsets B of $E_1 \cup E_2$ of size $r_1 + r_2$ such that $B \cap E_1$ is independent in M_1 and $B \cap E_2$ spans M_2. The rank function of $M_1 \boxdot M_2$ is given as follows: for $X_i \subseteq E_i$,

$$r_{M_1 \boxdot M_2}(X_1 \cup X_2) = \min\{r_1(E_1) + r_2(X_2), r_1(X_1) + |X_2|\}. \tag{4.2}$$

The free product $U_{1,1} \boxdot M$ is the free coextension of M, while $M \boxdot U_{0,1}$ is the free extension of M.

Proposition 4.11. *The $\mathcal{G}$-invariant of the free product $M_1 \boxdot M_2$ can be calculated from the $\mathcal{G}$-invariants of M_1 and M_2.*

Proof. Let $E_1 = \{1, 2, \ldots, m\}$ and $E_2 = \{m+1, m+2, \ldots, m+n\}$. Let $[\underline{r}_i]$ be symbols that occur in $\mathcal{G}(M_i)$, so $\underline{r}_i$ is the rank sequence of a permutation π_i on E_i. While $\mathcal{G}(M_i)$ does not give π_i, we can calculate the rank sequence of a shuffle $\pi = \mathrm{sh}(\pi_1, \pi_2; P)$ from the triple $(\underline{r}_1, \underline{r}_2, P)$. Given j with $1 \leq j \leq m+n$, let $X_1 = \{\pi(1), \pi(2), \ldots, \pi(j)\} \cap E_1$ and $X_2 = \{\pi(1), \pi(2), \ldots, \pi(j)\} \cap E_2$, so the set $\{\pi(1), \pi(2), \ldots, \pi(j)\}$ is the disjoint union of X_1 and X_2. Note that $|X_1| = |P \cap \{1, 2, \ldots, j\}|$ and $|X_2| = j - |X_1|$, and that $r_i(X_i)$ is the number of 1's in the first $|X_i|$ positions in $\underline{r}_i$. By equation (4.2), the rank of $\{\pi(1), \pi(2), \ldots, \pi(j)\}$ in $M_1 \,\square\, M_2$ can be found from $r_1(E_1)$, $r_2(X_2)$, $r_1(X_1)$, and $|X_2|$, and hence, from $(\underline{r}_1, \underline{r}_2, P)$. Applying this procedure starting from $j = 1$ yields the rank sequence of $\mathrm{sh}(\pi_1, \pi_2; P)$ from the triple $(\underline{r}_1, \underline{r}_2, P)$.

Each permutation of $\{1, 2, \ldots, m+n\}$ has exactly one representation as a shuffle of permutations π_i of E_i, so the multiset of rank sequences $\underline{r}(\pi)$, over all permutations π $\{1, 2, \ldots, m+n\}$, can be obtained by calculating the multiset of rank sequences obtained from the triples $(\underline{r}_1, \underline{r}_2, P)$, where $\underline{r}_i$ ranges over the multiset of rank sequences that occur in $\mathcal{G}(M_i)$ and P ranges over all m-subsets of $\{1, 2, \ldots, m+n\}$. In this way we obtain $\mathcal{G}(M_1 \,\square\, M_2)$ from $\mathcal{G}(M_1)$ and $\mathcal{G}(M_2)$. □

Example 4.12. Let $M_1 = U_{1,2} \oplus U_{1,1}$ and $M_2 = U_{1,3} \oplus U_{1,2}$ on disjoint sets. Then, $\mathcal{G}(M_1) = 4\,[110] + 2\,[101]$ and $\mathcal{G}(M_2) = 72\,[11000] + 36\,[10100] + 12\,[10010]$. The free product $M_1 \,\square\, M_2$ is a rank-4 matroid on a set of size 8. By the computation shown in the table below, the triple $(101, 10010, \{3, 4, 5\})$ gives the rank sequence 11100010 in $M_1 \,\square\, M_2$. Repeating this procedure $2 \cdot 3 \cdot 56$ times, we obtain $\mathcal{G}(M_1 \,\square\, M_2)$.

j	$\lvert X_1\rvert$	$\lvert X_2\rvert$	$r_1(X_1)$	$r_2(X_2)$	$\min\{r_1(E_1) + r_2(X_2), r_1(X_1) + \lvert X_2\rvert\}$
1	0	1	0	1	$\min\{2+1, 0+1\} = 1$
2	0	2	0	1	$\min\{2+1, 0+2\} = 2$
3	1	2	1	1	$\min\{2+1, 1+2\} = 3$
4	2	2	1	1	$\min\{2+1, 1+2\} = 3$
5	3	2	2	1	$\min\{2+1, 2+2\} = 3$
6	3	3	2	1	$\min\{2+1, 2+3\} = 3$
7	3	4	2	2	$\min\{2+2, 2+4\} = 4$
8	3	5	2	2	$\min\{2+2, 2+5\} = 4$

4.7. **q-cone.** In his study of tangential blocks, Whittle [20] introduced q-cones (originally called q-lifts) of $\mathrm{GF}(q)$-representable simple matroids.

Definition 4.13. *Let M be a simple $\mathrm{GF}(q)$-representable rank-r matroid and choose a representation of M in the rank-$(r+1)$ projective geometry $\mathrm{PG}(r, q)$, that is, a set E of points in $\mathrm{PG}(r, q)$ so that the restriction $\mathrm{PG}(r, q)|E$ is isomorphic to M. Let $A \mapsto \overline{A}$ be the closure operator of $\mathrm{PG}(r, q)$. Choose a point a in $\mathrm{PG}(r, q)$ not in the linear hyperplane $\overline{E}$ and let E' be the union*

$$\bigcup_{p \in E} \overline{\{a, p\}}.$$

The matroid $\mathrm{PG}(r, q)|E'$ is the q-cone *of M with* base E *and* apex a.

A $\mathrm{GF}(q)$-representable matroid M may have inequivalent representations, so different choices of E may yield non-isomorphic q-cones of M: Oxley and Whittle [17] gave examples of matroids with inequivalent representations that yield non-isomorphic q-cones. However, using a formula for the characteristic polynomial of q-cones due to Kung [15, Section 8.6], Bonin and Qin [3] showed that the Tutte polynomial of a q-cone of M can

be calculated from the Tutte polynomial of M and so depends only on M. We next treat a similar result for the $\mathcal{G}$-invariant.

Proposition 4.14. *Let M be a simple $\mathrm{GF}(q)$-representable matroid. The catenary data of a q-cone M' of M can be calculated from the catenary data of M. Thus, the $\mathcal{G}$-invariant of a q-cone M' depends only on $\mathcal{G}(M)$ (not on M or the representation).*

Proof. Let M' be the q-cone of M with base E and apex a. Identify M with the restriction $\mathrm{PG}(r,q)|E$. For a flag

$$\emptyset = Y_0 \subset Y_1 \subset Y_2 \subset \cdots \subset Y_{r-1} \subset Y_r \subset Y_{r+1} = E'$$

of M', let

$$X_i = \mathrm{cl}_{M'}(Y_i \cup a) \cap E.$$

Since E is closed in M', the set X_i is a flat of M. The *jump* of the flag (Y_i) is the least j with $a \in Y_j$. It follows that $X_{j-1} = X_j$ and that $(X_0, X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_{r+1})$ is a flag in M, which we call the *projection of* (Y_i) *onto* M.

Given a flag (X_i) in M with composition $(a_0, a_1, \ldots, a_r)$, we obtain all flags (Y_i) in M' with projection (X_i) and jump j as follows. For a fixed ordered basis $(b_1, b_2, \ldots, b_r)$ of M with $X_i = \mathrm{cl}_M(\{b_1, b_2, \ldots, b_i\})$, choose a sequence $b'_1, b'_2, \ldots, b'_{j-1}$ with $b'_i \in \overline{\{a, b_i\}} - a$, and set

$$Y_i = \begin{cases} \mathrm{cl}_{M'}(\{b'_1, b'_2, \ldots, b'_i\}) & \text{if } 0 \le i \le j-1, \\ \mathrm{cl}_{M'}(X_{i-1} \cup a) & \text{if } j \le i \le r+1. \end{cases}$$

It is easy to check that all flags in M' with projection (X_i) and jump j arise exactly once this way, and, due to the choice of $b'_1, b'_2, \ldots, b'_{j-1}$, there are q^{j-1} such flags. Also,

$$|Y_i| = \begin{cases} |X_i| & \text{if } 0 \le i \le j-1, \\ q|X_{i-1}| + 1 & \text{if } j \le i \le r+1, \end{cases}$$

so a flag (X_i) in M with composition $(a_0, a_1, \ldots, a_r)$ is the projection of q^{j-1} flags in M' with jump j, and these flags all have composition

$$(a_0, a_1, \ldots, a_{j-1}, (a_0 + a_1 + \cdots + a_{j-1})(q-1) + 1, a_j q, a_{j+1} q, \ldots, a_r q).$$

Thus, we get the catenary data of M' from that of M. □

Example 4.15. The matroids M and N in Figure 1 are representable over the field $\mathrm{GF}(q)$ whenever q is a prime power with $q > 3$. They have the same catenary data, so their q-cones have the same catenary data. The catenary data of M is $\nu(M; 0, 1, 2, 3) = 6$ and $\nu(M; 0, 1, 1, 4) = 18$. Each of the six flags with composition $(0, 1, 2, 3)$ is the projection of

(1) one flag with jump 1 and composition $(0, 1, q, 2q, 3q)$,
(2) q flags with jump 2 and composition $(0, 1, q, 2q, 3q)$,
(3) q^2 flags with jump 3 and composition $(0, 1, 2, 3q-2, 3q)$,
(4) q^3 flags with jump 4 and composition $(0, 1, 2, 3, 6q-5)$.

This and a similar calculation for the other 18 flags of M give the catenary data for a q-cone M':

$$\begin{aligned} \nu(M'; 0, 1, q, 2q, 3q) &= 6(1+q), & \nu(M'; 0, 1, q, q, 4q) &= 18(1+q), \\ \nu(M'; 0, 1, 2, 3q-2, 3q) &= 6q^2, & \nu(M'; 0, 1, 1, 2q-1, 4q) &= 18q^2, \\ \nu(M'; 0, 1, 2, 3, 6q-5) &= 6q^3, & \nu(M'; 0, 1, 1, 4, 6q-5) &= 18q^3. \end{aligned}$$

4.8. **Circuit-hyperplane relaxation.** Recall that if X is a both a circuit and a hyperplane (that is, copoint) of M, then the corresponding *circuit-hyperplane relaxation* is the matroid on the same set whose bases are those of M along with X. The next result is easy and extends a well-known result about Tutte polynomials.

Proposition 4.16. *If M' is obtained from the rank-r matroid M on an n-element set by relaxing a circuit-hyperplane, then*

$$\mathcal{G}(M') - \mathcal{G}(M) = r!\,(n-r)!\,([1^r0^{n-r}] - [1^{r-1}010^{n-r-1}]).$$

Equivalently, the only compositions for which $\nu(M)$ and $\nu(M')$ differ are the following:

$$\nu(M'; 0, 1, \ldots, 1, n-r+1) = \nu(M; 0, 1, \ldots, 1, n-r+1) + r!$$

and

$$\nu(M'; 0, 1, \ldots, 1, 2, n-r) = \nu(M; 0, 1, \ldots, 1, 2, n-r) - \frac{r!}{2}.$$

5. Deriving matroid parameters from the $\mathcal{G}$-invariant

Since the Tutte polynomial is a specialization of the $\mathcal{G}$-invariant, any matroid parameter that is derivable from the Tutte polynomial is derivable from the $\mathcal{G}$-invariant or catenary data. An easy but important example is the number $b(M)$ of bases of a matroid M. Indeed, the coefficient of the maximum symbol $[1^r0^{n-r}]$ in $\mathcal{G}(M)$ equals $r!(n-r)!b(M)$. Alternatively, by Lemma 3.2,

$$b(M) = \frac{1}{r!} \sum_{(a_0, a_1, \ldots, a_r)} \nu(M; a_0, a_1, \ldots, a_r)\ a_1 a_2 \cdots a_r.$$

In this section, we identify some of the parameters of a matroid that are derivable from the $\mathcal{G}$-invariant but not from the Tutte polynomial.

The result motivating this section is that the number $f_k(s)$ of flats of rank k and size s can be derived from the $\mathcal{G}$-invariant. As the matroid M_1 in Figure 2 has two 2-point lines whereas M_2 has three, but $T(M_1; x, y) = T(M_2; x, y)$, the Tutte polynomial does not determine all numbers $f_k(s)$. However, for a rank-r matroid and a given rank k, the maximum size m_k of a rank-k flat and the number $f_k(m_k)$ is derivable from $T(M; x, y)$. Indeed, m_k is the greatest integer m for which the monomial $(x-1)^{r-k}(y-1)^{m-k}$ occurs in $T(M; x, y)$ with non-zero coefficient and $f_k(m_k)$ is that coefficient. As shown in Section 5 of [6], $f_k(s)$ is derivable from $T(M; x, y)$ for each s with $m_{k-1} < s \leq m_k$.

Let M be a rank-r matroid on n elements. Fix integers h and k with $0 \leq h \leq k \leq r$, and let $(s_h, s_{h+1}, \ldots, s_k)$ be a sequence of positive integers. For $h < j \leq k$, let $s'_j = s_j - s_{j-1}$. An $(h, k; s_h, s_{h+1}, \ldots, s_k)$-*chain* is a (saturated) chain $(X_h, X_{h+1}, \ldots, X_k)$ of flats such that $r(X_j) = j$ and $|X_j| = s_j$ for $h \leq j \leq k$. For a rank-h flat X with $|X| = s_h$ and a rank-k flat Y with $|Y| = s_k$, let $F_{X,Y}(s_h, s_{h+1}, \ldots, s_k)$ be the number of $(h, k; s_h, s_{h+1}, \ldots, s_k)$-chains $(X, X_{h+1}, \ldots, X_{k-1}, Y)$ starting with the flat X and ending with the flat Y. Finally, let

$$F_{h,k}(s_h, s_{h+1}, \ldots, s_k) = \sum_{|X|=s_h, |Y|=s_k} F_{X,Y}(s_h, s_{h+1}, \ldots, s_k),$$

so $F_{h,k}(s_h, s_{h+1}, \ldots, s_k)$ is the number of $(h, k; s_h, s_{h+1}, \ldots, s_k)$-chains. When $h = k$, each chain collapses to a rank-k flat, and $f_k(s) = F_{k,k}(s)$, where $s = s_k$.

In the next lemma, we shall think of elements in the γ-basis as variables and linear combinations in $\mathcal{G}(n, r)$ as polynomials in those variables, so one can multiply them as polynomials.

Lemma 5.1. *Let $0 \le h \le k \le r$. Then*

$$\sum_{\substack{X \text{ a flat}, r(X)=h, |X|=s_h \\ Y \text{ a flat}, r(Y)=k, |Y|=s_k}} \mathcal{G}(M|X) F_{X,Y}(s_h, s_{h+1}, \dots, s_{k-1}, s_k) \mathcal{G}(M/Y)$$
$$= \sum_{(a_j)} \nu(M; a_0, a_1, \dots, a_h, s'_{h+1}, \dots, s'_k, a_{k+1}, \dots, a_r) \gamma(a_0, a_1, \dots, a_h) \gamma(0, a_{k+1}, \dots, a_r),$$

where the sum ranges over all (n, r)-compositions

$$(a_0, \dots, a_h, s'_{h+1}, \dots, s'_{k-1}, s'_k, a_{k+1}, \dots, a_r)$$

such that $a_0 + a_1 + \cdots + a_h = s_h$ and $a_{k+1} + a_{k+2} + \cdots + a_r = n - s_k$.

Proof. For a rank-h flat X with $|X| = s_h$ and a rank-k flat Y with $|Y| = s_k$, let

$$\nu(M; a_0, a_1, \dots, a_{h-1}, X, s'_{h+1}, \dots, s'_{k-1}, Y, a_{k+1}, \dots, a_r)$$

be the number of chains (X_i) in M satisfying $r(X_j) = j$, $|X_0| = a_0$, $|X_j - X_{j-1}| = a_j$ for $1 \le j < h$ and $k < j \le r$, $X_h = X$, $X_k = Y$, and $|X_j - X_{j-1}| = s'_j$ for $h < j \le k$. Then

$$\nu(M; a_0, a_1, \dots, a_{h-1}, X, s'_{h+1}, \dots, s'_{k-1}, Y, a_{k+1}, \dots, a_r)$$
$$= \nu(M|X; a_0, a_1, \dots, a_{h-1}, a_h) F_{X,Y}(s_h, s_{h+1}, \dots, s_{k-1}, s_k) \nu(M/Y; 0, a_{k+1}, \dots, a_r).$$

Hence,

$$\sum_{(a_j)} \nu(M; a_0, a_1, \dots, a_{h-1}, X, s'_{h+1}, \dots, s'_{k-1}, Y, a_{k+1}, \dots, a_r) \gamma(a_0, a_1, \dots, a_h) \gamma(0, a_{k+1}, \dots, a_r)$$
$$= \mathcal{G}(M|X) F_{X,Y}(s_h, s_{h+1}, \dots, s_{k-1}, s_k) \mathcal{G}(M/Y).$$

Summing over all flats X and Y having the stated rank and size, we obtain

$$\sum_{X,Y} \mathcal{G}(M|X) F_{X,Y}(s_h, s_{h+1}, \dots, s_k) \mathcal{G}(M/Y)$$
$$= \sum_{(a_j)} \left(\sum_{X,Y} \nu(M; a_0, a_1, \dots, a_{h-1}, X, s'_{h+1}, \dots, s'_{k-1}, Y, a_{k+1}, \dots, a_r) \right)$$
$$\times \gamma(a_0, a_1, \dots, a_h) \gamma(0, a_{k+1}, \dots, a_r)$$
$$= \sum_{(a_j)} \nu(M; a_0, a_1, \dots, a_{h-1}, a_h, s'_{h+1}, \dots, s'_{k-1}, s'_k, a_{k+1}, \dots, a_r)$$
$$\times \gamma(a_0, a_1, \dots, a_h) \gamma(0, a_{k+1}, \dots, a_r). \quad \square$$

Proposition 5.2. *The numbers $F_{h,k}(s_h, s_{h+1}, \dots, s_k)$ and in particular, $f_k(s)$, are derivable from the catenary data. Also, if $f_k(s) = 1$ and F is the unique flat of rank k and size s, then both $\mathcal{G}(M|F)$ and $\mathcal{G}(M/F)$ can be derived from $\mathcal{G}(M)$.*

Proof. For the first assertion, specialize all the symbols in the equation in Lemma 5.1 to 1. Then we have $\mathcal{G}(M|X) = s_h!$ and $\mathcal{G}(M/Y) = (n - s_k)!$, and the sum on the left equals $s_h!(n - s_k)! F_{h,k}(s_h, s_{h+1}, \dots, s_k)$. Since the sum on the right can be calculated from the catenary data of M, we can derive $F_{h,k}(s_h, s_{h+1}, \dots, s_k)$. For the second assertion, to get $\mathcal{G}(M|F)$, take $X = F$ and $Y = E(M)$; to get $\mathcal{G}(M/F)$, take $X = \mathrm{cl}(\emptyset)$ and $Y = F$. $\square$

Corollary 5.3. *The number of cocircuits of size s, the number of circuits of size s, and the number of cyclic sets (that is, unions of circuits) of size s and rank j are derivable from the catenary data. In particular, one can deduce whether the matroid has a spanning circuit, so one can determine whether a graph is Hamiltonian from the $\mathcal{G}$-invariant of its cycle matroid.*

Proof. A cocircuit is the complement of a copoint; hence, the number of cocircuits of size t equals $f_{r-1}(n-t)$. Circuits in M are cocircuits in the dual M^*. Finally, a set is cyclic if and only if it is a union of cocircuits in M^*, that is, its complement is a flat of M^*. □

The matroids in Figure 2 show that none of the parameters in Corollary 5.3 is determined by the Tutte polynomial.

Proposition 5.2 can be used to derive $f_k(s,c)$, the number of rank-k flats X of size s such that the restriction $M|X$ has exactly c coloops. The number $f_k(s,0)$ is the number of *cyclic* flats (that is, flats without coloops) of rank k and size s. We use the following easy lemma.

Lemma 5.4. *Let $(X_0, X_1, \ldots, X_r)$ be a flag with composition $(a_0, a_1, \ldots, a_r)$. Then the restriction $M|X_{i+1}$ is the direct sum of $M|X_i$ and a coloop if and only if $a_{i+1} = 1$.*

Proposition 5.5. *The number $f_k(s,c)$ is derivable from the $\mathcal{G}$-invariant.*

Proof. For j with $0 \le j \le k$, the numbers $f_k(s,j)$ satisfy the linear equations, one for each c with $0 \le c \le k$,

$$\sum_{j=c}^{k} f_k(s,j)\frac{j!}{(j-c)!} = F_{k-c,k}(s-c, s-c+1, s-c+2, \ldots, s),$$

where the sequence $s-c, s-c+1, s+c+2, \ldots, s$ increases by 1 at each step. To see this, note that the chains that $F_{k-c,k}(s-c, s-c+1, \ldots, s)$ counts are obtained by picking a rank-k size-s flat that has exactly j coloops with $c \le j \le k$, and going down from rank k to rank $k-c$ by deleting c of the j coloops one by one in some order. As the system of linear equations is triangular with diagonal entries equal to $c!$, we can solve it to get $f_k(s,c)$. □

Much of the interest in the $\mathcal{G}$-invariant has centered on the fact that it is a universal valuative invariant, so we end this section by relating our work to that part of the theory. For brevity, we address these remarks to readers who are already familiar with valuative invariants as discussed in [9]. We show that the parameters studied in this section are valuative invariants.

To make explicit the dependence on the matroid M, we shall write, for example, $f_k(M;s)$ instead of $f_k(s)$. We shall use two results from the theory of valuative invariants. The first is the basic theorem of Derksen [9] that specializations of the $\mathcal{G}$-invariant, in particular, the $\mathcal{G}$-invariant coefficients $g_{\underline{r}}(M)$, are valuative invariants. The second is the easy lemma that if u and v are valuative invariants on size-n rank-r matroids, then so is any linear combination $\alpha_{nr}u + \beta_{nr}v$, where α_{nr} and β_{nr} depend only on n and r.

Theorem 5.6. *The following parameters are valuative invariants:*

$$\nu(M;a_i),\ F_{h,k}(M;s_i),\ f_k(M;s), \text{ and } f_k(M;s,c).$$

Proof. The catenary data is obtained from the $\mathcal{G}$-invariant by a change of basis in $\mathcal{G}(n,r)$. Hence, $\nu(M;a_i)$ is a linear combination of $g_{\underline{r}}(M)$; explicitly, $\nu(M;a_i)$ is obtained from the vector of $\mathcal{G}$-invariant coefficients by applying the inverse of $[c_{(a_0,a_1,\ldots,a_r)}(b_0,b_1,\ldots,b_r)]$,

a matrix depending only on n and r. Hence, $\nu(M; a_i)$ is a valuative invariant. In the notation of Lemma 5.1, the number $F_{h,k}(M; s_h, \ldots, s_k)$ is given in terms of the catenary data by

$$\frac{1}{s_h!(n-s_k)!}\sum_{(a_j)}\gamma_1(a_0,\ldots,a_h)\gamma_1(0,a_{k+1},\ldots,a_r)\nu(M;a_0,\ldots,a_h,s'_{h+1},\ldots,s'_k,a_{k+1},\ldots,a_r),$$

where the numbers $\gamma_1(a_0,\ldots,a_h)$ and $\gamma_1(0,a_{k+1},\ldots,a_r)$ are obtained from $\gamma(a_0,\ldots,a_h)$ and $\gamma(0,a_{k+1},\ldots,a_r)$ by specializing all symbols to 1, and depend only on h, k, s_h, s_k, n, and r. Hence $F_{h,k}(M; s_i)$ and $f_k(M; s)$ are valuative invariants. Finally, the numbers $f_k(M; s, c)$ are obtained from the numbers $F_{h,k}(M; s_i)$ by solving a system of equations with coefficients depending only on k, s, and c; hence, they are valuative invariants as well. □

6. Reconstructing the $\mathcal{G}$-invariant from decks

Let $\bar{\mathcal{G}}(n,r)$ be the subspace of $\mathcal{G}(n,r)$ that is spanned by the γ-basis elements that are indexed by compositions that start with 0. The *circle product* $\odot$ is the binary operation from $\mathcal{G}(n_1,r_1)\times\bar{\mathcal{G}}(n_2,r_2)$ to $\mathcal{G}(n_1+n_2,r_1+r_2)$ defined by

$$\gamma(a_0,a_1,\ldots,a_{r_1})\odot\gamma(0,b_1,\ldots,b_{r_2})=\gamma(a_0,a_1,\ldots,a_{r_1},b_1,b_2,\ldots,b_{r_2})$$

on γ-basis elements and extended to $\mathcal{G}(n_1,r_1)\times\bar{\mathcal{G}}(n_2,r_2)$ by bilinearity. Let $\mathcal{F}_k$ denote the set of all rank-k flats in a matroid M. The next lemma shows that only the sets $\mathcal{F}_k$ have a property that is a key to the work in this section.

Lemma 6.1. *For a set $\mathcal{A}$ of flats of a matroid M, the following statements are equivalent:*

(1) *$\mathcal{A}=\mathcal{F}_k$ for some k with $0\le k\le r(M)$, and*
(2) *each flag of M contains exactly one flat in $\mathcal{A}$.*

Proof. It is immediate that statement (1) implies statement (2). Now assume that statement (2) holds. We claim that statement (1) holds where k is $\max\{r(X) : X\in\mathcal{A}\}$. A routine exchange argument shows that if S and T are distinct flats in $\mathcal{F}_k$, then there is a sequence $S,U,\ldots,T$ of flats in $\mathcal{F}_k$ such that the intersection of each pair of consecutive flats has rank $k-1$. If $\mathcal{A}\neq\mathcal{F}_k$, then from such a sequence with $S\in\mathcal{F}_k\cap\mathcal{A}$ and $T\in\mathcal{F}_k-\mathcal{A}$, we get flats $X\in\mathcal{F}_k\cap\mathcal{A}$ and $Y\in\mathcal{F}_k-\mathcal{A}$ with $r(X\cap Y)=k-1$. Extend a flag (Z_i) of $M|(X\cap Y)$ to a flag (X_i) of M that contains X, and, separately, to a flag (Y_i) of M that contains Y. Observe that at least one of (X_i) or (Y_i) contradicts statement (2); thus, statement (1) holds. □

Theorem 6.2 (The slicing formula)**.** *Let M be a rank-r matroid M. For k with $0\le k\le r$,*

$$\mathcal{G}(M)=\sum_{X\in\mathcal{F}_k}\mathcal{G}(M|X)\odot\mathcal{G}(M/X).$$

Thus,

$$\mathcal{G}(M)=\frac{1}{r+1}\sum_{X\in L(M)}\mathcal{G}(M|X)\odot\mathcal{G}(M/X).$$

Proof. A flat X in $\mathcal{F}_k$ determines the set

$$\mathcal{F}_X=\{(X_i): X_k=X\}$$

of flags intersecting $\mathcal{F}_k$ at X. These subsets partition the set of all flags into $|\mathcal{F}_k|$ blocks. Using this partition and Theorem 3.3, we obtain

$$\begin{aligned}\mathcal{G}(M) &= \sum_{X\in\mathcal{F}_k}\left(\sum_{(X_i)\in\mathcal{F}_X}\gamma(|X_0|,|X_1-X_0|,\ldots,|X_r-X_{r-1}|)\right)\\ &= \sum_{X\in\mathcal{F}_k}\left(\sum_{(X_i)\in\mathcal{F}_X}\gamma(|X_0|,|X_1-X_0|,\ldots,|X-X_{k-1}|)\odot\gamma(0,|X_{k+1}-X|,\ldots,|X_r-X_{r-1}|)\right)\\ &= \sum_{X\in\mathcal{F}_k}\mathcal{G}(M|X)\odot\mathcal{G}(M/X).\end{aligned}$$

□

Corollary 6.3. *For any k with $0\le k\le r(M)$, the $\mathcal{G}$-invariant of M can be reconstructed from the deck, or unlabeled multiset,*

$$\{(\mathcal{G}(M|X),\mathcal{G}(M/X)):\ X\in\mathcal{F}_k\}$$

of ordered pairs of $\mathcal{G}$-invariants.

To get the dual result, recall that deletion and contraction are dual operations, that is, $(M\backslash X)^*=M^*/X$ and $(M/X)^*=M^*\backslash X$. As noted in the proof of Corollary 5.3, a set X is a flat of M if and only if its complement, $E-X$, is a cyclic set of M^*. Also, $r_M(X)=k$ if and only if $r(M)-k=|E-X|-r_{M^*}(E-X)$, where the right side is the *nullity* of $E-X$ in M^*. If we replace M with its dual M^*, set $Y=E-X$, and apply Proposition 4.1, we get the following reconstruction result.

Corollary 6.4. *For any j with $0\le j\le r(M^*)$, the $\mathcal{G}$-invariant of M can be reconstructed from the deck*

$$\{(\mathcal{G}(M|Y),\mathcal{G}(M/Y)):\ Y \text{ a cyclic set of } M \text{ of nullity } j\}.$$

Two special cases of Corollary 6.3 occur at the bottom and top of the lattice $L(M)$ of flats. Recall that the *girth* of a matroid is the minimum size of a circuit in it. If M has girth at least $g+2$ and X is a rank-g flat, then $M|X$ is isomorphic to $U_{g,g}$ and

$$\mathcal{G}(M|X)=g!\gamma(0,\underbrace{1,1,\ldots,1}_{g}).$$

By Theorem 6.2,

$$\mathcal{G}(M)=g!\gamma(0,1,1,\ldots,1)\odot\left(\sum_{X\in\mathcal{F}_g}\mathcal{G}(M/X)\right).$$

Corollary 6.5. *If M has girth at least $g+2$, then $\mathcal{G}(M)$ can be reconstructed from the deck*

$$\{(\mathcal{G}(M/X):\ X\in\mathcal{F}_g\}.$$

In particular, if M is a simple matroid, then $\mathcal{G}(M)$ can be reconstructed from the deck $\{\mathcal{G}(M/x)\ :\ x \text{ an element of } M\}$. Dually, if M has n elements and each cocircuit of M has more than t elements, so each set of size $n-t$ spans M, then $\mathcal{G}(M)$ can be reconstructed from the deck

$$\{(\mathcal{G}(M|Y):\ Y \text{ a cyclic set of } M \text{ with } |Y|=n-t\}.$$

The other special case gives another perspective on Proposition 3.6, which is equivalent to the corollary that we derive next. If M has n elements and X is a copoint of M, then $M/X=U_{1,n-|X|}$ and $\mathcal{G}(M/X)=\gamma(0,n-|X|)$. Hence we get the following result.

Corollary 6.6. *The $\mathcal{G}$-invariant of M can be reconstructed from the number n of elements in M and the deck of $\mathcal{G}$-invariants of copoints.*

Corollary 6.6 is motivated by the theorem of Brylawski [5] that the Tutte polynomial is reconstructible from the deck of unlabeled restrictions to copoints. With the next lemma, we can strengthen Corollary 6.6.

Lemma 6.7. *The number n of elements of M can be reconstructed from the deck of $\mathcal{G}$-invariants of copoints.*

Proof. The proof is an adaptation of Brylawski's argument. We begin with an identity. Using the notation in Section 5,

$$F_{h,k-1}(s_h, s_{h+1}, \ldots, s_{k-1})(n - s_{k-1}) = \sum_{s_k} F_{h,k}(s_h, s_{h+1}, \ldots, s_{k-1}, s_k)(s_k - s_{k-1}).$$

This identity holds because both sides equal the number of triples

$$((X_h, X_{h+1}, \ldots, X_{k-1}), x, X),$$

where $|X_i| = s_i$, the element x is not in X_{k-1}, and $X = \mathrm{cl}(X_{k-1} \cup x)$. In particular,

$$f_h(s_h)(n - s_h) = \sum_{s_{h+1}} F_{h,h+1}(s_h, s_{h+1})(s_{h+1} - s_h).$$

Iterating the identity, we obtain

$$\begin{aligned} f_h(s_h) &= \sum_{s_{h+1}} F_{h,h+1}(s_h, s_{h+1}) \frac{(s_{h+1} - s_h)}{(n - s_h)} \\ &= \sum_{s_{h+1}, s_{h+2}} F_{h,h+2}(s_h, s_{h+1}, s_{h+2}) \frac{(s_{h+1} - s_h)(s_{h+2} - s_{h+1})}{(n - s_h)(n - s_{h+1})} \\ &\vdots \\ &= \sum_{s_{h+1}, s_{h+2}, \ldots, s_{r-1}} F_{h,r-1}(s_h, s_{h+1}, \ldots, s_{r-1}) \left(\prod_{j=h}^{r-2} \frac{s_{j+1} - s_j}{n - s_j} \right). \end{aligned}$$

Let s_0 be the number of loops in M. Then, since $f_0(s_0) = 1$, we have

$$1 = \sum_{s_1, s_2, \ldots, s_{r-1}} F_{0,r-1}(s_0, s_1, s_2, \ldots, s_{r-1}) \left(\prod_{j=0}^{r-2} \frac{s_{j+1} - s_j}{n - s_j} \right). \tag{6.1}$$

In addition, we also have

$$F_{0,r-1}(s_0, s_1, s_2, \ldots, s_{r-1}) = \sum_{\text{copoint } X,\, |X| = s_{r-1}} \nu(M|X; s_0, s_1 - s_0, s_2 - s_1, \ldots, s_{r-1} - s_{r-2}),$$

and hence, we can calculate the right-hand side of equation (6.1) given the deck $\{\mathcal{G}(M|X) : X \in \mathcal{F}_{r-1}\}$, and solve for n. The number n of elements is a solution greater than $\max\{s_{r-1}\}$, the maximum number of elements in a copoint, and this solution is unique since the right-hand side is a strictly decreasing function in n. In other words, equation (6.1), and hence the deck, determines the number of elements in M. □

Corollary 6.6 and Lemma 6.7 yield the following result.

Theorem 6.8. *The $\mathcal{G}$-invariant of M can be reconstructed from the deck $\{\mathcal{G}(M|X) : X$ a copoint$\}$. Dually, we can reconstruct $\mathcal{G}(M)$ from the deck $\{\mathcal{G}(M/Y) : Y$ a circuit$\}$.*

The proof of the first assertion in Theorem 6.8 does not use the $\mathcal{G}$-invariants $\mathcal{G}(M|X)$ individually but the sums

$$\mathcal{H}(M;s) = \sum_{X \text{ a copoint}, |X|=s} \mathcal{G}(M|X).$$

Hence, we have the following theorem giving the exact information needed to reconstruct $\mathcal{G}(M)$ from a deck derived from copoints.

Theorem 6.9. *The $\mathcal{G}$-invariant $\mathcal{G}(M)$ and the deck $\{\mathcal{H}(M;s)\}$ (consisting of the sums $\mathcal{H}(M;s)$ that are non-zero) can be constructed from each other.*

7. Configurations

The main result in this section, Theorem 7.3, extends a result by Eberhardt [12] for the Tutte polynomial (Theorem 7.1 below) to the $\mathcal{G}$-invariant. We also show that the converse of Theorem 7.3 is false.

These results use the configuration of a matroid, which Eberhardt defined. Recall that a set X in a matroid M is *cyclic* if X is a union of circuits, or, equivalently, $M|X$ has no coloops. The set $\mathcal{Z}(M)$ of cyclic flats of M, ordered by inclusion, is a lattice. The join $X \vee Y$ in $\mathcal{Z}(M)$ is $\mathrm{cl}(X \cup Y)$, as in the lattice of flats; the meet $X \wedge Y$ is the union of the circuits in $X \cap Y$. It is well-known that a matroid M is determined by $E(M)$ and the pairs $(X, r(X))$ for $X \in \mathcal{Z}(M)$. If deleting all coloops of a matroid M yields M', then we easily get $\mathcal{G}(M)$ from $\mathcal{G}(M')$, so we focus on matroids without coloops. (Similarly, focusing on matroids that also have no loops, as in the proof of Theorem 7.3, is justified.) The *configuration* of a matroid M with no coloops is the triple (L, s, ρ) where L is a lattice that is isomorphic to $\mathcal{Z}(M)$, and s and ρ are functions on L where if $x \in L$ corresponds to $X \in \mathcal{Z}(M)$, then $s(x) = |X|$ and $\rho(x) = r(X)$. The configuration does not record the cyclic flats, and we do not distinguish between L and lattices isomorphic to it, so non-isomorphic matroids (such as the pair in Figure 1) may have the same configuration. Two paving matroids of the same rank r with the same number $f_{r-1}(s)$ of size-s copoints, for each s, have the same configuration. Giménez constructed a set of $n!$ non-paving matroids of rank $2n+2$ on $4n+5$ elements, all with the same configuration (see [2, Theorem 5.7]). We now state Eberhardt's result.

Theorem 7.1. *For a matroid with no coloops, its Tutte polynomial can be derived from its configuration.*

We use Theorem 7.1 to prove Theorem 7.3. (With these techniques, one can give another proof of Theorem 7.1 since knowing $T(M;x,y)$ is equivalent to knowing, for each pair i, j, the number of subsets of $E(M)$ of size i and rank j, and an inclusion/exclusion argument like that in the proof of Lemma 7.3.2 shows that the configuration gives the number of such subsets. We use only a special case of Theorem 7.1, namely, the configuration gives the number of bases.) We also use the following elementary lemma about cyclic flats.

Lemma 7.2. *Assume that M has neither loops nor coloops. If X is any cyclic flat of M, then $\mathcal{Z}(M|X)$ is the interval $[\emptyset, X]$ in $\mathcal{Z}(M)$, and*

$$\mathcal{Z}(M/X) = \{F - X \,:\, F \in \mathcal{Z}(M) \text{ and } X \subseteq F\},$$

so the lattice $\mathcal{Z}(M/X)$ is isomorphic to the interval $[X, E(M)]$ in $\mathcal{Z}(M)$. Thus, from the configuration of M, we get the configuration of any restriction to, or contraction by, a cyclic flat of M. Likewise, the configuration of the truncation $\mathrm{Trun}(M)$ can be derived from that of M.

Theorem 7.3. *For a matroid M with no coloops, its catenary data, and so $\mathcal{G}(M)$, can be derived from its configuration.*

Proof. As noted above, we can make the further assumption that M has no loops. Let $\iota(N)$ denote the number of independent copoints of a matroid N, and $b(N)$ denote its number of bases. The proof is built on two lemmas.

Lemma 7.3.1. If, for each chain $\emptyset = D_0 \subset D_1 \subset \cdots \subset D_t = E(M)$ in $\mathcal{Z}(M)$, we have each $r(D_i)$ and $\iota(M|D_i/D_{i-1})$, then we can compute the catenary data of M.

Proof. Let $\emptyset = X_0 \subset X_1 \subset \cdots \subset X_r = E(M)$ be a flag of flats of M, so $r = r(M)$. Each flat X_i is the disjoint union of the set U_i of coloops of $M|X_i$ and a cyclic flat F_i of M. Thus, $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_r$. Since M has neither loops nor coloops, $U_0 = F_0 = \emptyset = U_r$ and $F_r = E(M)$. Let $\emptyset = D_0 \subset D_1 \subset \cdots \subset D_t = E(M)$ be the distinct cyclic flats among $F_0, F_1, \ldots, F_r$, and let $U = U_1 \cup U_2 \cup \cdots \cup U_{r-1}$. Thus, $|U| = r - t$. We call $D_0, D_1, \ldots, D_t$ and U the cyclic flats and set of coloops of the flag. For i with $1 \leq i \leq t$, let $W_i = U \cap (D_i - D_{i-1})$, which may be empty. Fix i with $1 \leq i < t$. Let j be the least integer with $D_i \subseteq X_j$, so $U_j = X_j - D_i$. Delete all elements of U_j from the flats in the chain $X_0 \subset X_1 \subset \cdots \subset X_j$; the distinct flats that remain form a chain, and the greatest is D_i and the one of rank $r(D_i) - 1$ is $D_{i-1} \cup W_i$. Thus, W_i is an independent copoint of $M|D_i/D_{i-1}$.

Conversely, let $\emptyset = D_0 \subset D_1 \subset \cdots \subset D_t = E(M)$ be a chain of cyclic flats. For each i with $1 \leq i \leq t$, let W_i be an independent copoint of $M|D_i/D_{i-1}$ and set $U = W_1 \cup W_2 \cup \cdots \cup W_t$ (so $|U| = r - t$). There are flags of flats of M whose cyclic flats and set of coloops are $D_0, D_1, \ldots, D_t$ and U, and we obtain all such flags in the following way. Take any permutation π of $U \cup \{D_1, \ldots, D_t\}$ in which, for each i, all elements of W_i, and all D_k with $k < i$, are to the left of D_i. To get a flag from π, successively adjoin to $\emptyset$ the elements in initial segments of π.

We next find the contributions of these flags to the catenary data of M. With the type of permutation π described above, there are t associated integer compositions, each with t parts, namely, for each i, we have the integer composition $|W_i| = a_{i1} + a_{i2} + \cdots + a_{it}$ where a_{ij} is the number of elements of W_i that are between D_{j-1} and D_j in π. Thus, if $j > i$, then $a_{ij} = 0$. We call the lower-triangular $t \times t$ matrices $A = (a_{ij})$ the matrix of compositions of π. A lower-triangular $t \times t$ matrix with non-negative integer entries is a matrix of compositions for some such permutation π if and only if its row sums are $|W_1|, |W_2|, \ldots, |W_t|$. Let a'_j be the sum of the entries in column j of A, which is the number of elements between D_{j-1} and D_j in π. The composition of the flag that we get from any permutation π whose matrix of compositions is A is

$$\big(0, \underbrace{1, \ldots, 1}_{a'_1}, |D_1| - |W_1|, \underbrace{1, \ldots, 1}_{a'_2}, |D_2| - |W_1 \cup W_2|, \ldots, \underbrace{1, \ldots, 1}_{a'_t}, |D_t| - (r-t)\big),$$

and the number of such permutations π is

$$\prod_{i=1}^{t} \binom{|W_i|}{a_{i1}, a_{i2}, \ldots, a_{ii}} a'_i!,$$

where the multinomial coefficient accounts for choosing the a_{ij} elements of W_i that will be between D_{j-1} and D_j, and the factorial accounts for permutations of the elements that are between D_{i-1} and D_i.

Thus, what the chain $D_0, D_1, \ldots, D_t$ in $\mathcal{Z}(M)$ and the sets $W_1, W_2, \ldots, W_t$ contribute to the catenary data can be found from just the t numbers $|W_1|, |W_2|, \ldots, |W_t|$. Since

$|W_i| = r(M|D_i/D_{i-1}) - 1$ and W_i can be any independent copoint of $M|D_i/D_{i-1}$, the catenary data of M can be computed from the data $r(D_i)$ and $\iota(M|D_i/D_{i-1})$, over all chains in $\mathcal{Z}(M)$ that include $\emptyset$ and $E(M)$, as Lemma 7.3.1 asserts. □

Lemma 7.3.2. We can compute $\iota(N)$ if we have *(i)* $b(\mathrm{Trun}(N))$ as well as *(ii)* for each chain $\emptyset = F_0 \subset F_1 \subset \cdots \subset F_p \subset E(N)$ in $\mathcal{Z}(N)$, the numbers $b(N|F_k/F_{k-1})$, for $1 \leq k \leq p$, and $b(\mathrm{Trun}(N/F_p))$.

Proof. Note that $b(\mathrm{Trun}(N))$ is the number of independent sets of size $r(N) - 1$ in N. Set $\mathcal{Z}' = \{F \in \mathcal{Z}(N) : 0 < r(F) < r(N)\}$. For $F \in \mathcal{Z}'$, let

$$A_F = \{I \subset E : r(I) = |I| = r(N) - 1 \text{ and } \mathrm{cl}(I \cap F) = F\}.$$

Thus,

$$\iota(N) = b(\mathrm{Trun}(N)) - \Big|\bigcup_{F\in\mathcal{Z}'} A_F\Big|,$$

so by inclusion/exclusion,

$$\iota(N) = b(\mathrm{Trun}(N)) + \sum_{S\subseteq\mathcal{Z}', S\neq\emptyset} (-1)^{|S|}\Big|\bigcap_{F\in S} A_F\Big|. \tag{7.1}$$

Note that if $I \in A_{F_1} \cap A_{F_2}$ for $F_1, F_2 \in \mathcal{Z}'$, then $I \cap (F_1 \vee F_2)$ spans $F_1 \vee F_2$ (recall, $F_1 \vee F_2 = \mathrm{cl}(F_1 \cup F_2)$); thus $r(F_1 \vee F_2) \leq r(I) < r(N)$, so $F_1 \vee F_2 \in \mathcal{Z}'$, and $I \in A_{F_1\vee F_2}$; hence, $A_{F_1} \cap A_{F_2} = A_{F_1} \cap A_{F_2} \cap A_{F_1\vee F_2}$. Now assume that F_1 and F_2 are incomparable, so $F_1 \vee F_2$ properly contains both of them. If F_1 and F_2 are in a subset S of $\mathcal{Z}'$, and if S' is the symmetric difference $S\triangle\{F_1 \vee F_2\}$, then $(-1)^{|S|} = -(-1)^{|S'|}$ and

$$\bigcap_{F\in S} A_F = \bigcap_{F\in S'} A_F,$$

so such terms could cancel in the sum in equation (7.1). Pair off such terms as follows: take a linear extension $\leq$ of the order $\subseteq$ on $\mathcal{Z}'$ and, if a subset S of $\mathcal{Z}'$ contains incomparable cyclic flats, let F_1 and F_2 be such a pair for which (F_1, F_2) is least in the lexicographic order that $\leq$ induces on $\mathcal{Z}' \times \mathcal{Z}'$; cancel the term in the sum in equation (7.1) that arises from S with the one that arises from $S' = S\triangle\{F_1 \vee F_2\}$. These cancellations leave

$$\iota(N) = b(\mathrm{Trun}(N)) + \sum_{\substack{\text{nonempty chains}\\ S\subseteq\mathcal{Z}'}} (-1)^{|S|}\Big|\bigcap_{F\in S} A_F\Big|. \tag{7.2}$$

Now let S be a chain in this sum, say consisting of $F_1 \subset F_2 \subset \cdots \subset F_p$. The independent sets in $\bigcap_{i=1}^p A_{F_i}$ are the union of a basis of $N|F_1$, a basis of $N|F_k/F_{k-1}$, for $1 < k \leq p$, and a basis of $\mathrm{Trun}(N/F_p)$. Thus, $\big|\bigcap_{i=1}^p A_{F_i}\big|$, and so $\iota(N)$, can be found from the data given in Lemma 7.3.2. □

With these lemmas, we now prove Theorem 7.3. Let (L, s, ρ) be the configuration of M. Chains $\emptyset = D_0 \subset D_1 \subset \cdots \subset D_t = E(M)$ in $\mathcal{Z}(M)$ correspond to chains $\hat{0} = x_0 < x_1 < \cdots < x_t = \hat{1}$ in L, where $\hat{0}$ and $\hat{1}$ are the least and greatest elements of L. The configuration of $M|D_i/D_{i-1}$ consists of the interval $[x_{i-1}, x_i]$ in L along with the maps $y \mapsto s(y) - s(x_{i-1})$ and $y \mapsto \rho(y) - \rho(x_{i-1})$. (For what we do with such minors (e.g., construct more such minors and find their configurations), working with the matroid is equivalent to working with its configuration; to make it easier to follow, in the rest of the proof we use the matroid.) The only other data we need in order to apply Lemma 7.3.1 is $\iota(M|D_i/D_{i-1})$. Lemma 7.3.2 reduces this to data that, by Theorem 7.1, can be derived from the configuration of $M|D_i/D_{i-1}$ and hence from that of M. Specifically, letting N

be $M|D_i/D_{i-1}$, we have its configuration, say (L', s', ρ'), and the chains in L' correspond to those in $\mathcal{Z}(N)$, and we get the configurations of $\mathrm{Trun}(N)$ and each $N|F_k/F_{k-1}$ and $\mathrm{Trun}(N/F_p)$ in Lemma 7.3.2. Thus, we get the number of bases of $\mathrm{Trun}(N)$ and each $N|F_k/F_{k-1}$ and $\mathrm{Trun}(N/F_p)$ by Theorem 7.1, so, as needed, by Lemma 7.3.2 we get $\iota(N)$. □

By Proposition 4.8, Dowling matroids of the same rank based on groups of the same finite order have the same $\mathcal{G}$-invariant. Here we prove that, for $r \geq 4$, Dowling matroids based on non-isomorphic finite groups have different configurations; thus, the converse of Theorem 7.3 is false. We first recall the few facts about these matroids that we use. Let G be a finite group and r a positive integer. The set on which the Dowling matroid $Q_r(G)$ is defined consists of the $r + \binom{r}{2}|G|$ elements

(P1) $p_1, p_2, \ldots, p_r$, and
(P2) a_{ij} for each $a \in G$ and $i, j \in \{1, 2, \ldots, r\}$ with $i \neq j$, where $a_{ji} = (a^{-1})_{ij}$.

The lines (rank-2 flats) of $Q_r(G)$ are of three types:

(L1) $\ell_{ij} := \{p_i, p_j\} \cup \{a_{ij} : a \in G\}$, where $|\{i, j\}| = 2$,
(L2) $\{a_{ij}, b_{jk}, (ab)_{ik}\}$, where $a, b \in G$ and $|\{i, j, k\}| = 3$, and
(L3) $\{p_i, a_{jk}\}$ with $|\{i, j, k\}| = 3$, and $\{a_{hi}, b_{jk}\}$ with $|\{h, i, j, k\}| = 4$.

The set $\{p_1, p_2, \ldots, p_r\}$ is a basis of $Q_r(G)$; each element of $Q_r(G)$ is in some line that is spanned by two elements in this basis. For each t with $2 \leq t \leq r$, the restriction of $Q_r(G)$ to the closure of any t-element subset of $\{p_1, p_2, \ldots, p_r\}$ is isomorphic to $Q_t(G)$ and so has $t + \binom{t}{2}|G|$ elements; if $|G| \geq 4$ (the least m for which there are non-isomorphic groups of order m), then all other flats of rank t are strictly smaller.

Proposition 7.4. *For $r \geq 4$ and finite groups G and $\hat{G}$, if the Dowling matroids $Q_r(G)$ and $Q_r(\hat{G})$ have the same configuration, then G and $\hat{G}$ are isomorphic.*

Proof. Assume that $|G| = |\hat{G}| \geq 4$. By the hypothesis, there is a lattice isomorphism $\phi : \mathcal{Z}(Q_r(G)) \to \mathcal{Z}(Q_r(\hat{G}))$ that preserves the rank and size of each cyclic flat. Let cl be the closure operator of $Q_r(G)$, and cl' that of $Q_r(\hat{G})$. Applying the last remark before the proposition with $t = 4$ shows that, with relabeling if needed, we may assume that $\phi(\mathrm{cl}(\{p_1, p_2, p_3, p_4\})) = \mathrm{cl}'(\{p_1, p_2, p_3, p_4\})$. By restricting to these flats, we may assume that $r = 4$. Likewise, we may assume that $\phi(\mathrm{cl}(\{p_i, p_j, p_k\})) = \mathrm{cl}'(\{p_i, p_j, p_k\})$ whenever $\{i, j, k\} \subset \{1, 2, 3, 4\}$. We will show that $Q_3(G)$ and $Q_3(\hat{G})$ are isomorphic. The following result by Dowling [11, Theorem 8] then completes the proof: for $r \geq 3$, the matroids $Q_r(G)$ and $Q_r(\hat{G})$ are isomorphic if and only if the groups G and $\hat{G}$ are isomorphic.

The singleton flats $\{a_{ij}\}$ are not in $\mathcal{Z}(Q_4(G))$, but we show that they induce certain partitions of sets of 3-point lines in $\mathcal{Z}(Q_4(G))$. Let i, j, s, t be $1, 2, 3, 4$ in some order. Let P_{ijs} be the set of 3-point lines in the plane $\mathrm{cl}(\{p_i, p_j, p_s\})$. For $a \in G$, set

$$P_s^{a_{ij}} = \{\ell \in P_{ijs} : a_{ij} \in \ell\}.$$

Thus, $\{P_s^{a_{ij}} : a \in G\}$ is a partition of P_{ijs}. With P_{ijt} and $P_t^{b_{ij}}$ defined similarly, take $\ell \in P_s^{a_{ij}}$ and $\ell' \in P_t^{b_{ij}}$. By the remarks above, if $a \neq b$, then $\mathrm{cl}(\ell \cup \ell')$ is $Q_4(G)$, while if $a = b$, then $\mathrm{cl}(\ell \cup \ell')$ is a cyclic flat of rank 3. Now $\mathrm{cl}(\ell \cup \ell') \in \mathcal{Z}(Q_4(G))$ and the same observations hold for $Q_4(\hat{G})$, so the isomorphism ϕ maps the blocks $P_s^{a_{ij}}$ and $P_t^{a_{ij}}$ in the partitions of P_{ijs} and P_{ijt} to their counterparts (written, for instance, as $\hat{P}_s^{c_{ij}}$) in $Q_4(\hat{G})$. Thus, for $\{i, j\} \subset \{1, 2, 3, 4\}$, there is a bijection $\psi_{ij} : G \to \hat{G}$

with $\phi(P_s^{a_{ij}}) = \hat{P}_s^{(\psi_{ij}(a))_{ij}}$ and $\phi(P_t^{a_{ij}}) = \hat{P}_t^{(\psi_{ij}(a))_{ij}}$. Putting these maps together, we have a bijection $\psi : Q_4(G) \to Q_4(\hat{G})$ with $\psi(p_i) = p_i$ and $\psi(a_{ij}) = (\psi_{ij}(a))_{ij}$ for $\{i,j\} \subset \{1,2,3,4\}$ and $a \in G$. A 3-point line $\ell = \{a_{ij}, b_{jk}, (ab)_{ik}\}$ of $Q_4(G)$ is in the equivalence classes $P_k^{a_{ij}}$, $P_i^{b_{jk}}$, and $P_j^{(ab)_{ik}}$, so $\phi(\ell)$ is in the equivalence classes $\hat{P}_k^{\psi(a_{ij})}$, $\hat{P}_i^{\psi(b_{jk})}$, and $\hat{P}_j^{\psi((ab)_{ik})}$, and so $\phi(\ell) = \psi(\ell)$. Thus, since ψ also clearly preserves the lines of type (L1), the restriction of ψ to $\mathrm{cl}(\{p_1, p_2, p_3\})$ shows that $Q_3(G)$ and $Q_3(\hat{G})$ are isomorphic, so, by Dowling's result, G and $\hat{G}$ are isomorphic. □

8. Detecting free products

The Tutte polynomial reflects direct sums in a remarkably faithful way. In [16], Merino, de Mier, and Noy showed that the Tutte polynomial $T(M; x, y)$ factors as $A(x, y)B(x, y)$, for polynomials $A(x, y)$ and $B(x, y)$ over $\mathbb{Z}$, if and only if $A(x, y)$ and $B(x, y)$ are the Tutte polynomials of the constituents in some direct sum factorization of M. By that result and Theorem 1.1, from $\mathcal{G}(M)$ one can deduce whether M is a direct sum; however, we do not know whether the $\mathcal{G}$-invariants of the constituents are determined by $\mathcal{G}(M)$. In this section we prove a result of this type for free products.

As noted earlier, the free product $M \square U_{0,1}$ is the free extension of M, while $U_{1,1} \square M$ is the free coextension. The matroids in Figure 1 show that the $\mathcal{G}$-invariant cannot detect free extensions (hence, coextensions). Thus, we consider only *proper* free products $M_1 \square M_2$, by which we mean that each of M_1 and M_2 has at least two cyclic flats. Below we reduce the problem to *sharp* free products, which are proper free products $M_1 \square M_2$ in which M_1 has no coloops and M_2 has no loops. A *pinchpoint* of the lattice $\mathcal{Z}(M)$ of cyclic flats of M is a cyclic flat, neither the maximum nor the minimum, that each cyclic flat either contains or is contained in. We now state our main result.

Theorem 8.1. *From $\mathcal{G}(M)$, one can deduce whether M is a proper free product. Also, for each pinchpoint X of $\mathcal{Z}(M)$, the $\mathcal{G}$-invariants of the constituents of the corresponding sharp free product factorization of M can be derived from $\mathcal{G}(M)$ and $r(X)$.*

We use [8, Proposition 6.1], which we cite next.

Lemma 8.2. *Let M_1 and M_2 be matroids on disjoint ground sets. Set*

$$\mathcal{Z}' = \big(\mathcal{Z}(M_1) - \{E(M_1)\}\big) \cup \{E(M_1) \cup Y \,:\, Y \in \mathcal{Z}(M_2) - \{\emptyset\}\}.$$

If the factorization $M_1 \square M_2$ is sharp, then $\mathcal{Z}(M)$ is $\mathcal{Z}' \cup \{E(M_1)\}$; otherwise it is $\mathcal{Z}'$.

We first explain a reduction that we use: from each proper factorization of M, we get a sharp factorization of M. Let M be the proper free product $M_1 \square M_2$. It follows from Lemma 8.2 that $\mathcal{Z}(M)$ has a pinchpoint. In particular, if the factorization is sharp, then $E(M_1)$ is a pinchpoint. If M_1 has coloops, then the maximum flat in $\mathcal{Z}(M_1)$ (which is not $E(M_1)$) is a pinchpoint of $\mathcal{Z}(M)$. If Y is any set of coloops of M_1, then another proper factorization of M is $(M_1\backslash Y) \square ((M|Y) \square M_2)$, and $(M|Y) \square M_2$ applies free coextension to M_2 a total of $|Y|$ times. If Y is the set of all coloops of M_1, then the factorization $(M_1\backslash Y) \square ((M|Y) \square M_2)$ of M is sharp. Likewise, if M_2 has loops, then $E(M_1) \cup \mathrm{cl}_{M_2}(\emptyset)$ is a pinchpoint of $\mathcal{Z}(M)$. If $Y \subseteq \mathrm{cl}_{M_2}(\emptyset)$, then another proper factorization of M is $(M_1 \square (M_2|Y)) \square (M_2\backslash Y)$, and $M_1 \square (M_2|Y)$ applies free extension to M_1 a total of $|Y|$ times. If $Y = \mathrm{cl}_{M_2}(\emptyset)$, then the factorization $(M_1 \square (M_2|Y)) \square (M_2\backslash Y)$ of M is sharp. Thus, finding the sharp factorizations of M is the crux of the problem. The next lemma (which recasts [8, Theorem 6.3]) relates this more precisely to the pinchpoints of $\mathcal{Z}(M)$.

Lemma 8.3. *A matroid M is a proper free product if and only if the lattice $\mathcal{Z}(M)$ has a pinchpoint. Furthermore, the map $X \mapsto (M|X) \square (M/X)$ is a bijection from the set of pinchpoints of $\mathcal{Z}(M)$ onto the set of sharp free product factorizations of M.*

We now prove the main result of this section.

Proof of Theorem 8.1. Recall from Section 5 that from $\mathcal{G}(M)$ we can deduce the number $f_k(s)$ of flats, and the number $f_k(s,0)$ of cyclic flats, of rank k and size s in M. Set

$$C = \{k \,:\, 0 < k < r(M) \text{ and } \sum_s f_k(s,0) = 1\},$$

so $k \in C$ if and only if $0 < k < r(M)$ and exactly one cyclic flat of M has rank k. The rank of any candidate pinchpoint is in C, so if $C = \emptyset$, then M is not a proper free product. When $C \neq \emptyset$, we test each $k \in C$ as follows.

Let X be the unique cyclic flat of rank k. Set $s_0 = |X|$. If $f_k(s_0) > 1$, then there is a non-cyclic flat F of rank k and size s_0. The flat Y obtained from F by deleting the coloops of $M|F$ is cyclic, and $s_0 - |Y| = |F - Y| = k - r(Y)$. If $Y \subset X$, then comparing rank and size shows that $M|X$ has coloops (the elements of $X - Y$), contrary to X being cyclic. Hence, $Y \not\subset X$, so X is not a pinchpoint. Thus, if X is a pinchpoint, then $f_k(s_0) = 1$.

Now assume that X is the only flat of rank k and size s_0. By Proposition 5.2, from $\mathcal{G}(M)$, we get $\mathcal{G}(M|X)$ and $\mathcal{G}(M/X)$. Thus, by Proposition 5.5 we can derive the number of cyclic flats in $M|X$ and in M/X. Now X is a pinchpoint of M if and only if (i) the number of cyclic flats in $M|X$ equals the number of cyclic flats of rank at most k in M, and (ii) the number of cyclic flats in M/X equals the number of cyclic flats of rank at least k in M. Thus, we can detect pinchpoints, and for any pinchpoint X, we can derive $\mathcal{G}(M|X)$ and $\mathcal{G}(M/X)$. □

9. Erratum

The description of the $\mathcal{G}$-invariant of the truncation of a matroid given in Proposition 4.2 is incorrect. The correct statement is given in the following proposition.

Proposition 9.1. *Let M be a matroid having positive rank r. For any $(n, r-1)$-composition $(a_0, a_1, \ldots, a_{r-2}, a'_{r-1})$, the number of flags of $\mathrm{Trun}(M)$ that have that composition, that is, $\nu(\mathrm{Trun}(M); a_0, a_1, \ldots, a_{r-2}, a'_{r-1})$, is*

$$\sum_{\substack{a_{r-1}, a_r \in \mathbb{N}: \\ a_{r-1}+a_r = a'_{r-1}}} \nu(M; a_0, a_1, \ldots, a_{r-2}, a_{r-1}, a_r) \cdot \frac{a_{r-1}}{a_{r-1} + a_r}.$$

Thus,

$$\mathcal{G}(\mathrm{Trun}(M)) = \sum \nu(M; a_0, a_1, \ldots, a_r) \cdot \frac{a_{r-1}}{a_{r-1} + a_r} \cdot \gamma(a_0, a_1, \ldots, a_{r-2}, a_{r-1} + a_r),$$

where the sum is over all (n, r)-compositions $(a_0, a_1, \ldots, a_r)$.

Proof. Let P be the set of pairs $((X_i), e)$ where $(X_i) = (X_0, X_1, \ldots, X_{r-2}, E(M))$ is a flag of $\mathrm{Trun}(M)$ with composition $(a_0, a_1, \ldots, a_{r-2}, a'_{r-1})$, and $e \in E(M) - X_{r-2}$. Thus, $|P| = \nu(\mathrm{Trun}(M); a_0, a_1, \ldots, a_{r-2}, a'_{r-1}) \cdot a'_{r-1}$. Now

$$\{\mathrm{cl}_M(X_{r-2} \cup e) - X_{r-2} \,:\, e \in E(M) - X_{r-2}\}$$

is a partition of the set $E(M) - X_{r-2}$, so we can also get the pairs in P by picking a flag of M whose composition has the form $(a_0, a_1, \ldots, a_{r-2}, *, *)$ and choosing an element that

is in the largest two flats of the flag and in none of the smaller flats. Thus,

$$|P| = \sum_{\substack{a_{r-1},a_r\in\mathbb{N}:\\ a_{r-1}+a_r=a'_{r-1}}} \nu(M; a_0, a_1, \ldots, a_{r-2}, a_{r-1}, a_r) \cdot a_{r-1}.$$

Setting the two expressions for $|P|$ equal to each other and dividing by a'_{r-1}, which is $a_{r-1} + a_r$, gives the result. □

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