

Not Sini: Washin' ton

2021

Amusing Representations

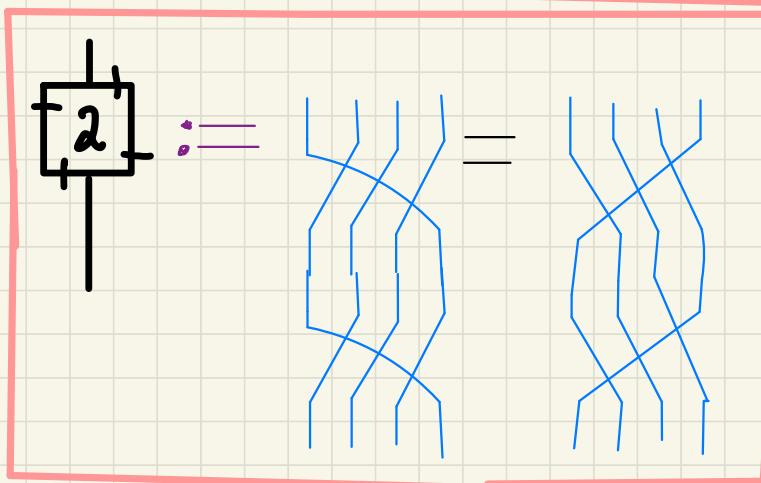
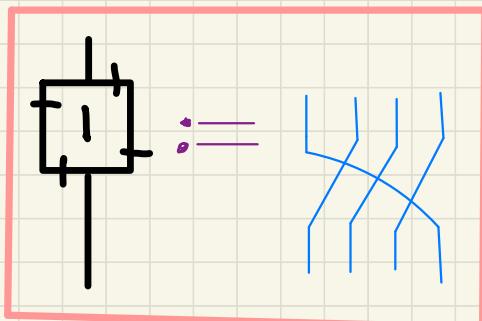
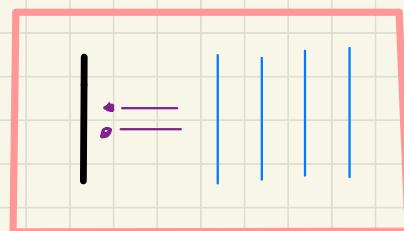
2021
Scott
Carter



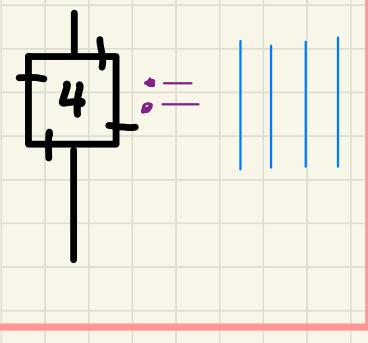
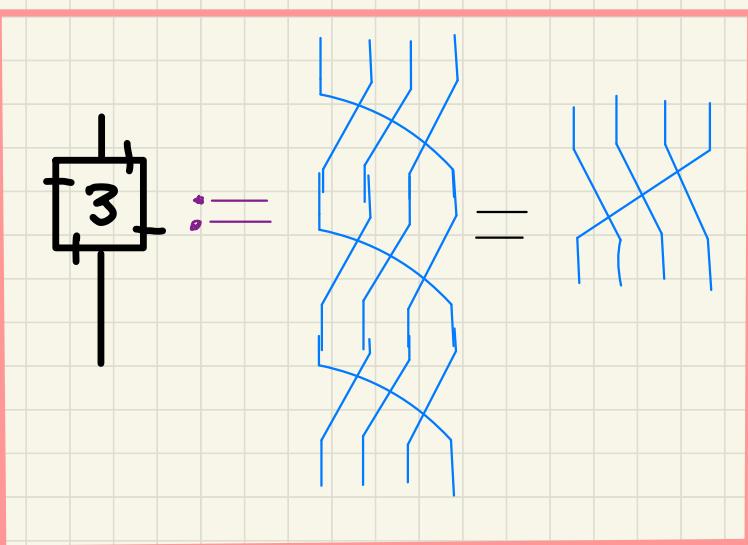
Based on joint work
with Yongju Bae
+ Byoerhi Kim

Start by playing

Dictionary:

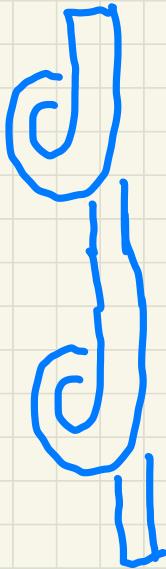


So just
 $\mathbb{Z}/4$
w/ 4 half twists
being none.



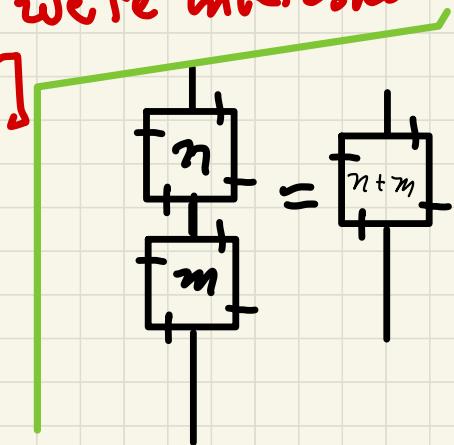
So just $\mathbb{Z}/4$ w/ 4 half twists being none.

Cf.

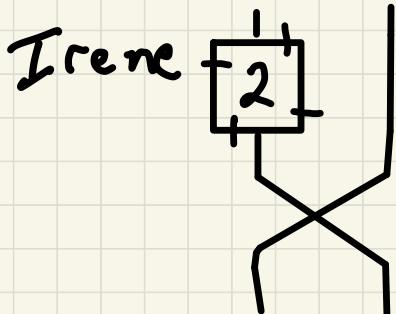


$$\text{L } \pi_1(SO(3)) = \mathbb{Z}/2.$$

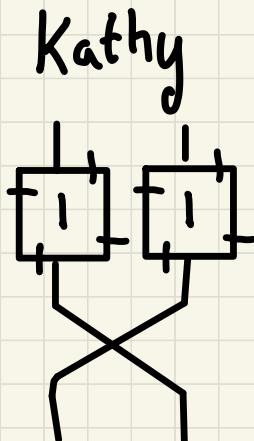
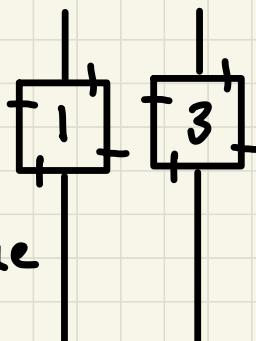
[After all we're interested
in $SU(2)$]



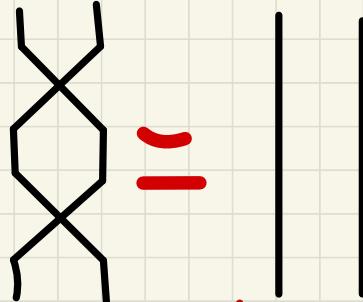
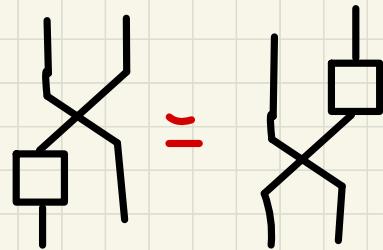
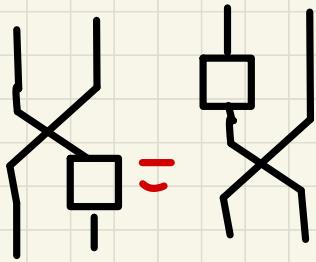
Introduce:



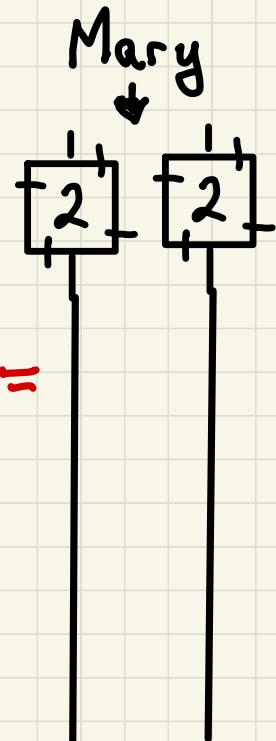
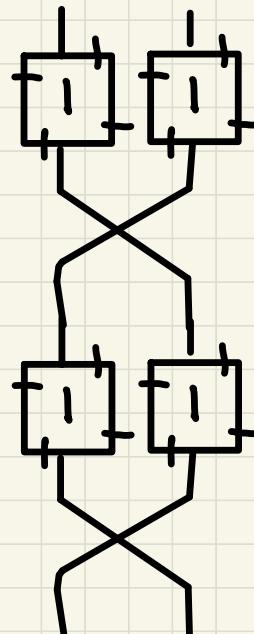
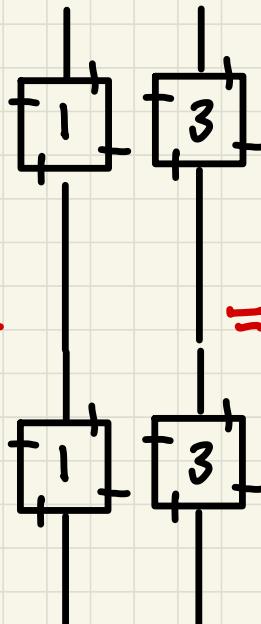
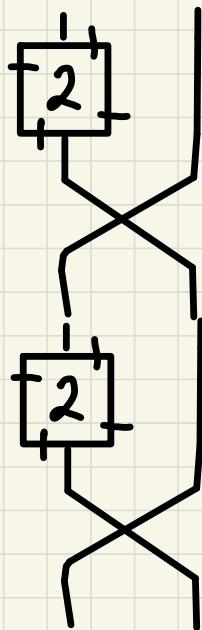
Jane



Rules:



If Vertical Stacking.



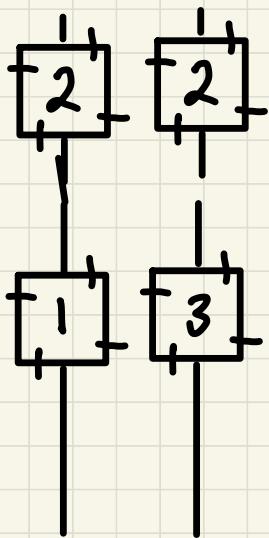
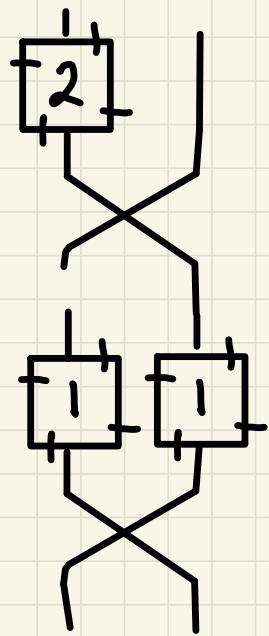
$(\text{Irene})^2 = (\text{June})^2 = (\text{Kathy})^2 = \text{Mary}$

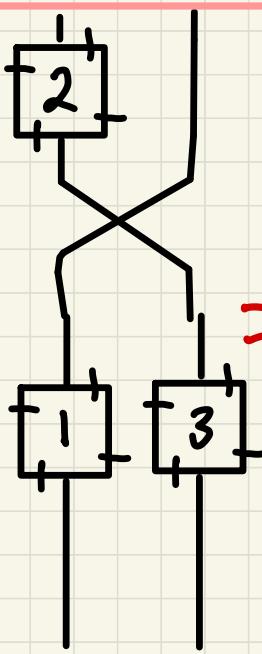
Irene

Jane

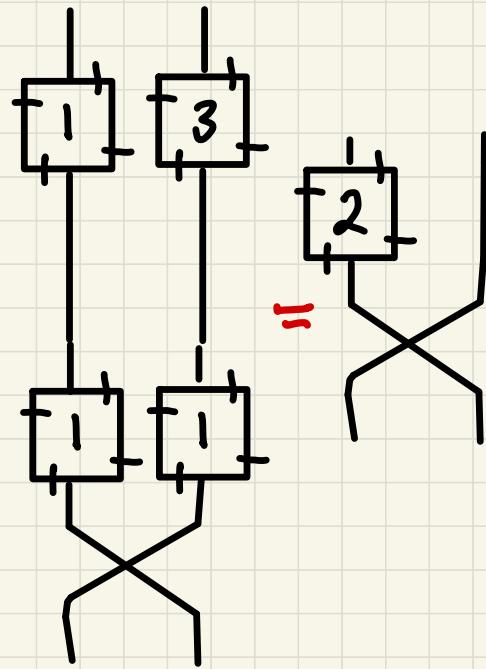
Kathy

Mary

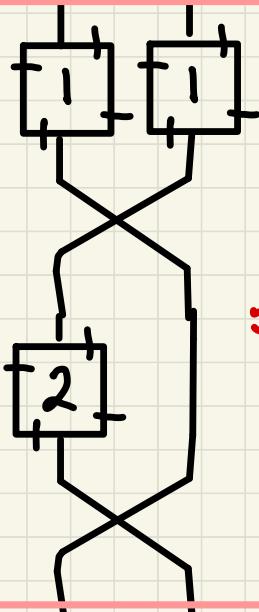




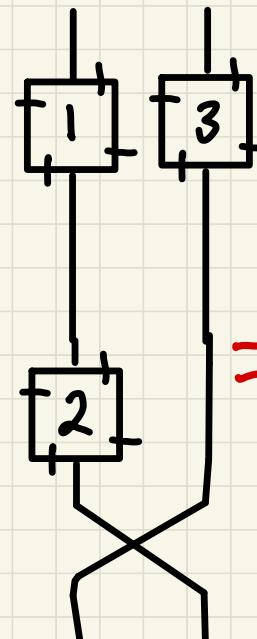
Irene · Jane
= Kathy



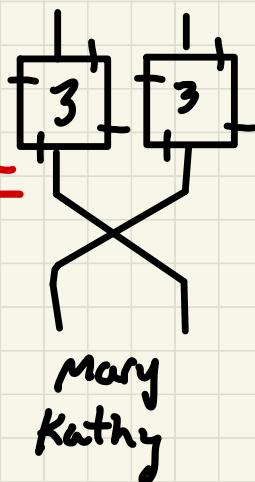
Jane · Kathy = Irene



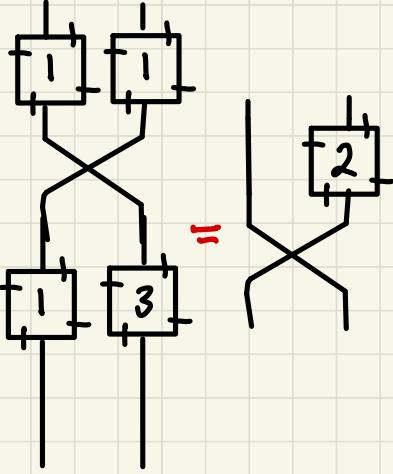
Kathy · Irene
= Jane



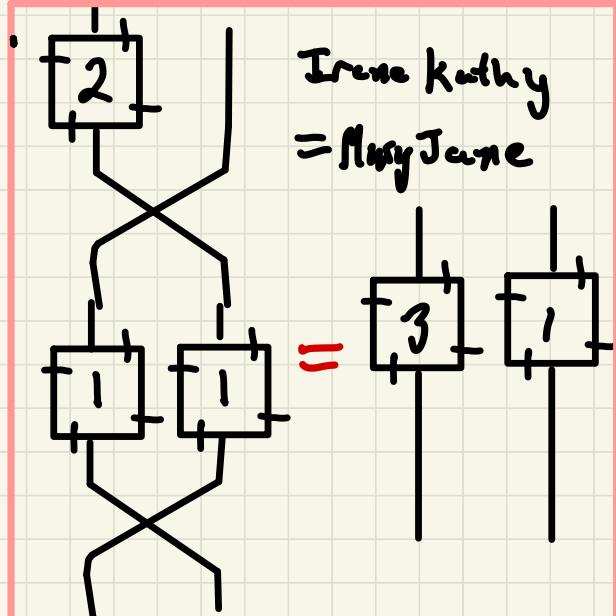
Jane · Irene



Mary ·
Kathy



Irene
Kathy
Jane
= Mary Irene



Irene Kathy
= Mary Jane

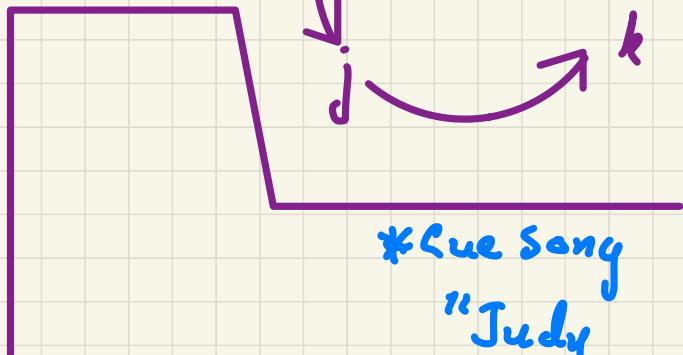
Summary:

$$1, i, j, k, -1, -i, -j, -k$$

$$ij = k = -ji$$

$$jk = i = -kj$$

$$ki = j = -ik$$



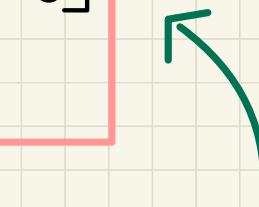
*Lue Song

"Judy
in disguise"

So Irene, Jane, & Kathy were unit quaternions in disguise.

$$\begin{array}{c} \text{Diagram with two boxes labeled } 2 \text{ and } 1 \\ \longleftrightarrow \end{array} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

i



$$\begin{array}{c} \text{Diagram with three boxes labeled } 2, 2, 1 \\ \longleftrightarrow \end{array} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

k

$$\begin{array}{c} \text{Diagram with two boxes labeled } 1, 3 \\ \longleftrightarrow \end{array} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

j

lie alg.

(2×2) trace=0, $\det = 1 \therefore$ basis for $\mathfrak{su}(2)$

Divide by $\frac{1}{2}$ (norm=1). Lie bracket

$[A, B] = AB - BA$ is the standard cross product

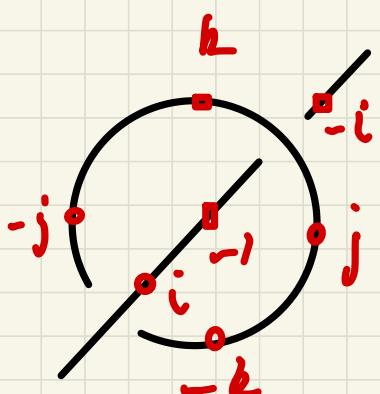
from cal III. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & +i \\ i & 0 \end{bmatrix}$

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

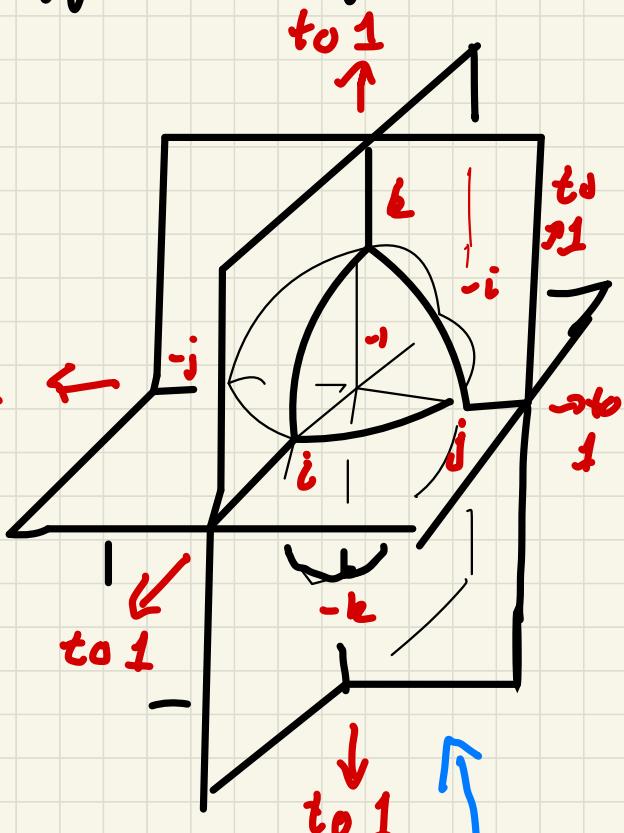
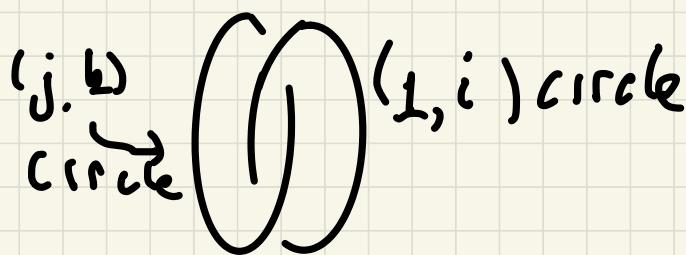
$$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Reminder

So $\mathfrak{su}(2)$ -Lie alg. is the tangent space at identity for $SU(2)$ which topologically is $S^3 = \{w+xi+yj+zk : x^2+y^2+z^2+w^2=1\}$



↓ Hopf link

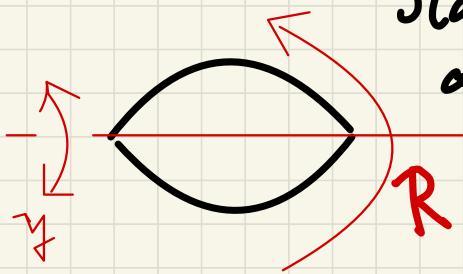


S^3 in
stereographic
projection

Meanwhile, $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is
 Dic_2 , where

$$\text{Dic}_n = \langle \rho, x : \rho^{2n} = 1, x^2 = \rho^{-1}, \rho x = x\rho^{-1} \rangle.$$

is the 2-fold extension of the dihedral group — the group of symmetries of an n -gon.



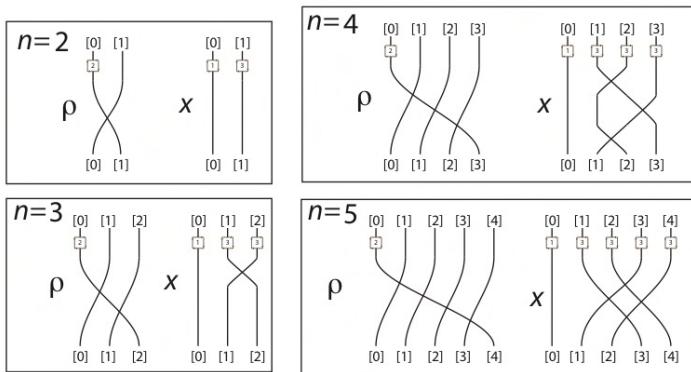
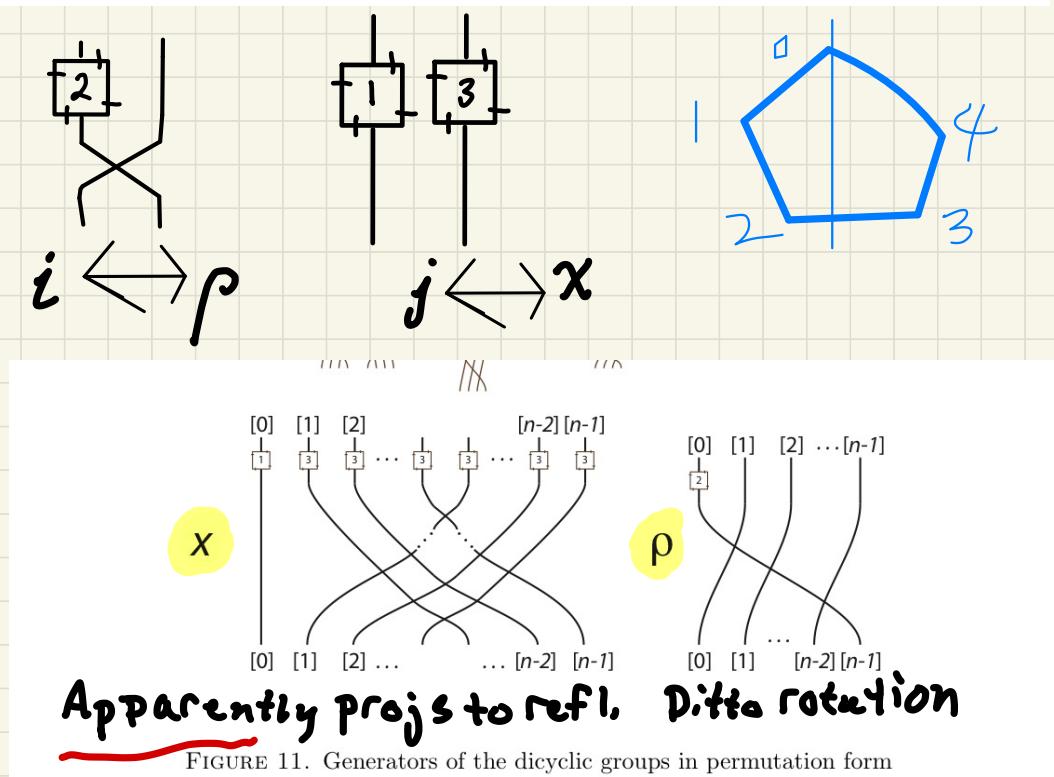
Start from sym.s of a bigon

[leave the gratuitous pun in the chat]

ρ projects to R
 x " to y .

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \text{Dic}_n \longrightarrow \text{Dihedral}_n \rightarrow 0$$

$$\text{Dic}_n = \langle \rho, x : \rho^{2n} = 1, x^2 = \rho^n, \rho x = x\rho^{-1} \rangle.$$



But there's a better presentation.

Rather than doing this in general, I'll look @ $n=3$.

$$\gamma = e^{\frac{\pi i}{3}}$$

$$= \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)$$

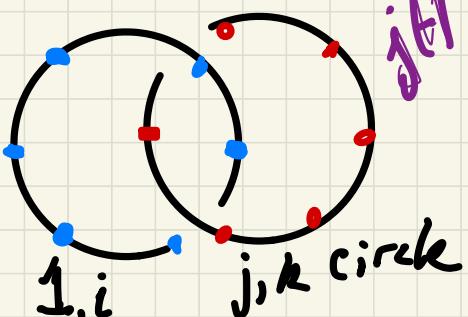
$$H = (\gamma^l : l=0, \dots, 5)$$

an ordered subgroup circle

— only ordered as a set.

$$\begin{aligned} p &\mapsto \gamma \\ x &\mapsto j \end{aligned}$$

jH = a coset
with induced
order.



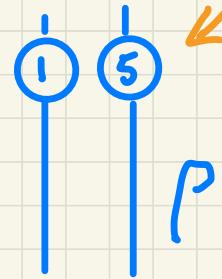
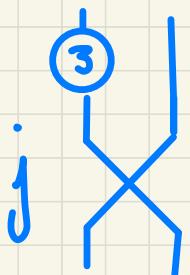
$$\text{Dic}_n = \langle \rho, x : \rho^{2n} = 1, x^2 = \rho^n, \rho x = x \rho^{-1} \rangle.$$

ρ rotates $(1, i)$ circle one way
& the (j, k) " the other way.

x swaps them, but

$$j(j \cos(\theta) + k \sin(\theta)) = i \sin(\theta) - \cos(\theta)$$

So...

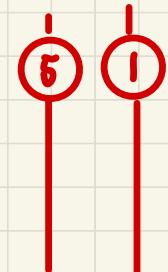
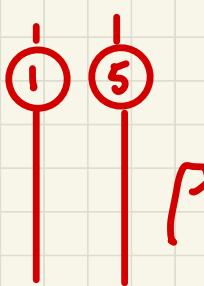
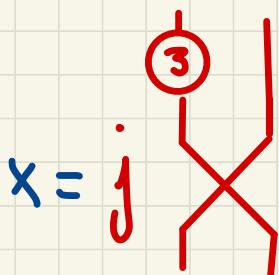
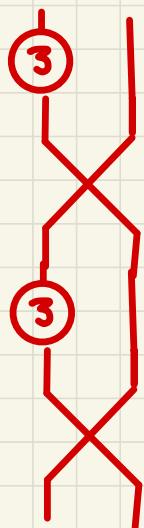


read mod
6

$$j \leftrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\rho \leftrightarrow \begin{bmatrix} 5 & 0 \\ 0 & 5^5 \end{bmatrix}$$

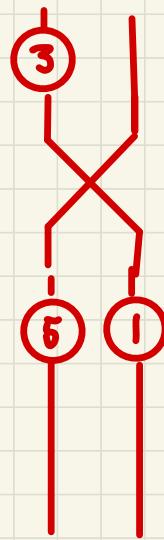
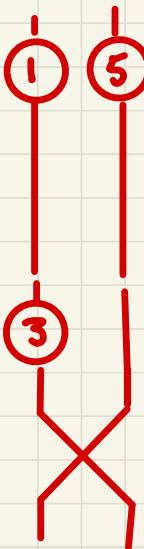
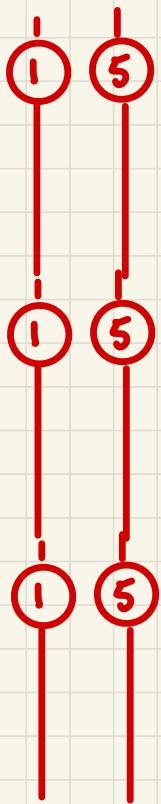
Play area:



$$\rho^{-1}$$

--- o --- j --- r ---

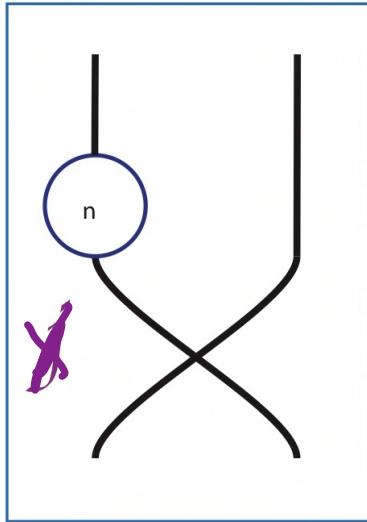
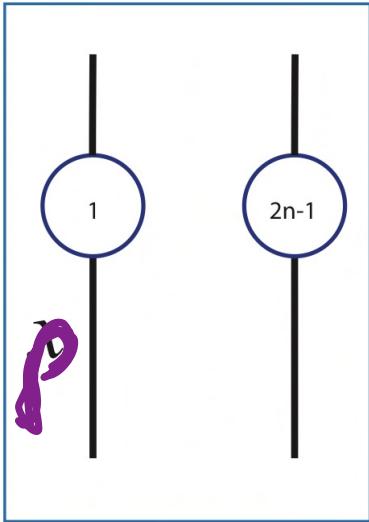
$$\text{Dic}_n = \langle \rho, x : \rho^{2n} = 1, x^2 = \rho^n, \rho x = x \rho^{-1} \rangle.$$



In general Dic_n, let
 $\zeta = e^{\frac{\pi i}{n}}$

Since $\zeta = e^{(\pi i)/n}$, there is a corresponding matrix representation

$$\rho \Rightarrow \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}, \quad j \Rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$



& get these diagrams & matrices

What's going on:

Krasner-Kaloujnine Theorem:

appears in section 1

THEOREM 2. Let G denote a finite group of order nk . Let H denote a subgroup of order k . Then there is an inclusion $G \subset H^n \rtimes \Sigma_n$ where the second factor permutes the coordinates of H^n .

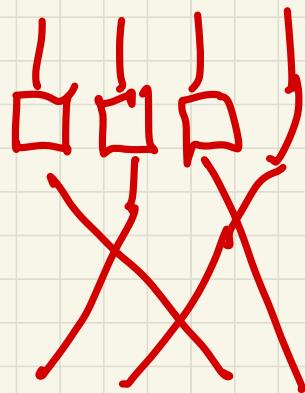
Sketch: Order $H = (h_1, \dots, h_k)$.

$(g_i H, \dots, g_i H)$

$[] [] []$

g

$H \quad g_1 H \quad \dots \quad g_{n-1} H$



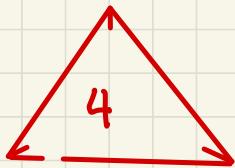
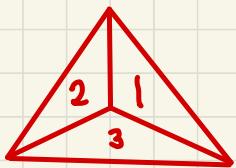
$[]$

Goal to make use of this theorem to give cool descriptions of favorite groups.

Digress:

$$\mathfrak{S} A_4 = \text{Inn}(Q(4,1))$$

$$\left[\begin{matrix} (123), (134), (243), (142) \\ \parallel \quad \parallel \quad \parallel \quad \parallel \\ 4 \quad 2 \quad 1 \quad 3 \end{matrix} \right] = Q(4,1)$$



Row & Col

4	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

There is a 2-fold extension of $Q(4,1)$: known as $Q(8,1)$. Vendramin's $\text{Inn}(Q(8,1)) = \tilde{A}_4$ the binary tetrated group.

$$\text{Inn}(Q(4,1)) = A_4$$

HMM...

$$\begin{array}{ccccccc}
 0 & \rightarrow & \mathbb{Z}/2 & \xrightarrow{\sim} & \tilde{A}_4 & \rightarrow & A_4 \rightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 & & \mathbb{Z}/2 & \hookrightarrow & Q(8,1) & \rightarrow & Q(4,1) \rightarrow 0
 \end{array}$$

But by that we had to determine
 $\text{Inn}(Q(8,1))$. It is generated by
these permutations

$$\begin{aligned}a &\equiv (04)(123567); \\b &\equiv (37)(052416); \\c &\equiv (15)(036472); \\d &\equiv (26)(075431).\end{aligned}$$

& $\text{Inn}(Q(4,1))$ is

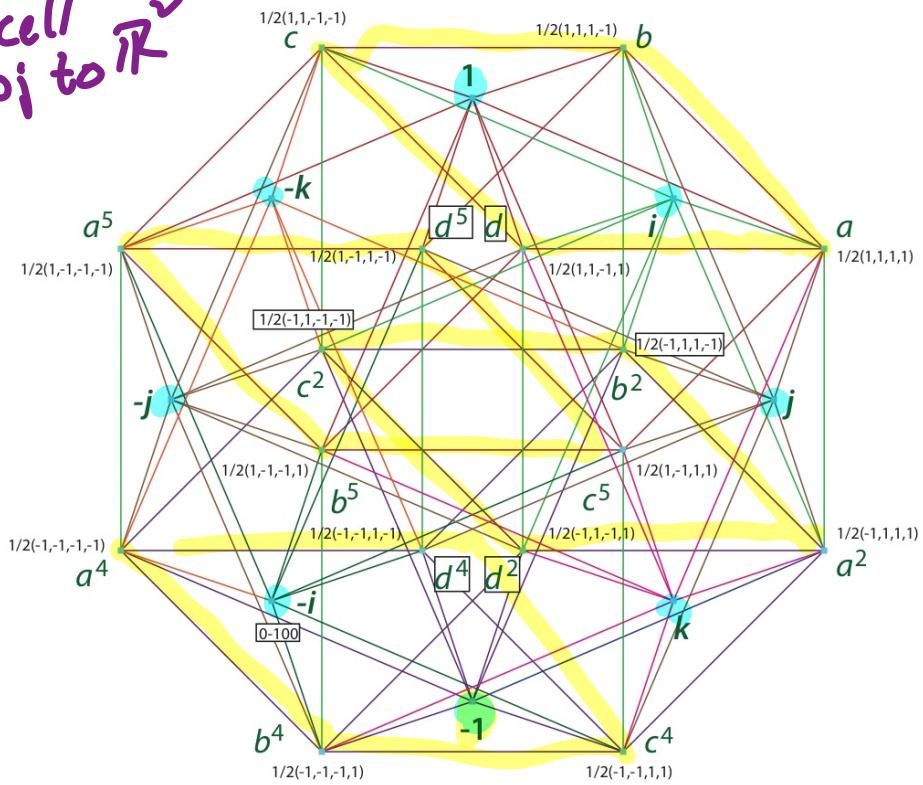
$$\langle (123), (032), (013), (021) \rangle$$

We looked @ subgroups gen by
 a, b, c, d . Saw they were $\mathbb{Z}/6$ & then
looked up the group ~

$$SL_2(\mathbb{Z}/3) \cong A_4$$

— the binary tetrahedral group.

24 cell
proj to \mathbb{R}^5



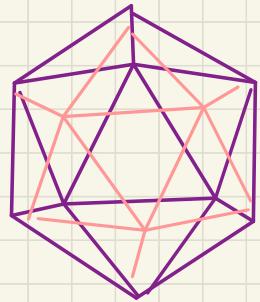
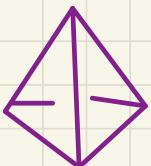
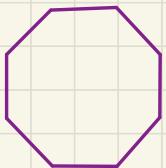
$$\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k).$$

$$a = \frac{1}{2}(1 + i + j + k), \quad b = \frac{1}{2}(1 + i + j - k),$$

$$c = \frac{1}{2}(1 + i - j - k), \quad \text{and} \quad d = \frac{1}{2}(1 + i - j + k)$$

$SU(2) \rightarrow\!\!\! \rightarrow SO(3)$

$SU(3)$ a group of rot's of 3-D space.
Finite subgroups therein



Dihedral:
groups of
sym of
polygons

tetrahedral
 A_4

Octahedral
 S_4

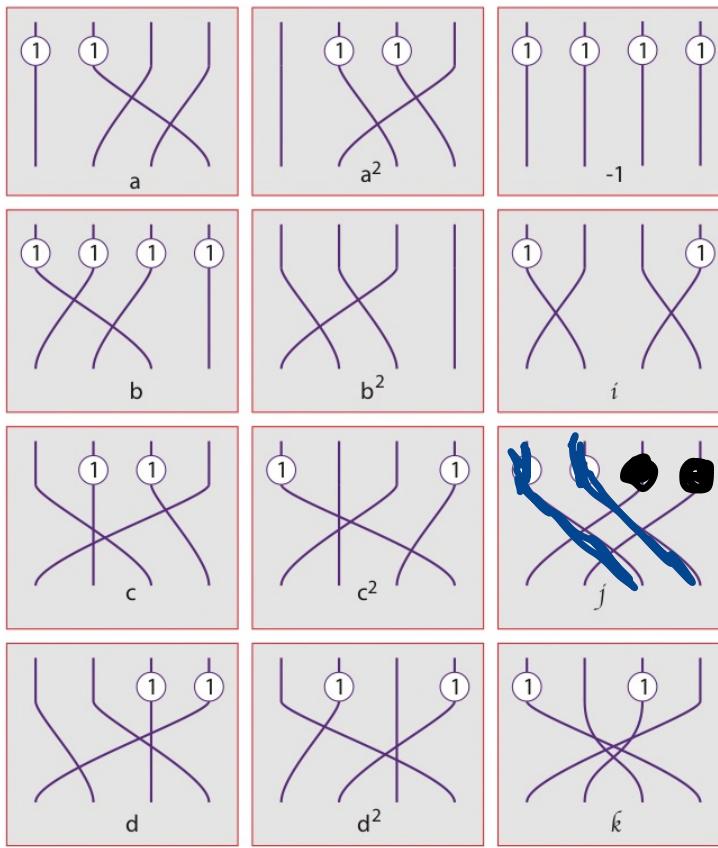
Icosahedral
 A_5



120_{cell}

* 2-fold extensions inside $S^3 = SU(2)$
Binary Dicyclic Binary tet. Binary Oct. Binary icos.

* $0 \rightarrow \mathbb{Z}/2 \rightarrow \tilde{A}_4 \rightarrow A_4 \rightarrow 0$
eg



$\sim \tilde{A}_4$

$| = ||$

$\times = ||$

subject to the relations:

$$\begin{array}{c} \text{Diagram showing two strands with circles labeled 1 crossing, equal to the strands crossed in the opposite direction.} \\ | \quad | \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \text{Diagram showing two strands with circles labeled 1 crossing, equal to the strands crossed in the opposite direction.} \\ | \quad | \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array}$$

$$\begin{array}{c} \text{Diagram showing two strands with circles labeled 1 crossing, equal to a single strand with a circle labeled 1.} \\ | \quad | \\ \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} = \begin{array}{c} \text{Diagram showing a single strand with a circle labeled 1.} \\ | \end{array}$$

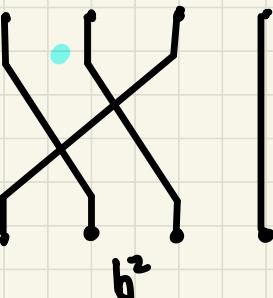
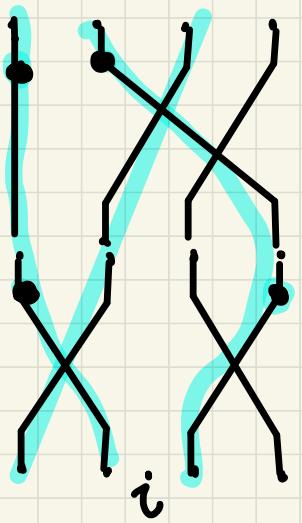
$$\begin{array}{c} \text{Diagram showing a single strand with a circle labeled 1, equal to an empty vertical line.} \\ | \quad | \\ | \quad | \\ | \quad | \end{array} = \begin{array}{c} \text{Empty vertical line} \\ | \end{array}$$

In terms of the
Krasner-Kaloujnine Theorem:

N is a subgroup. Cosets correspond.
as indicated

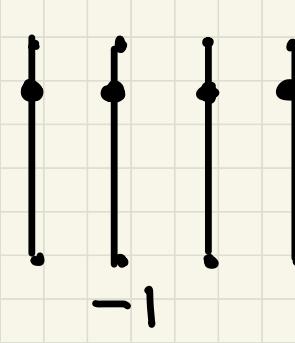
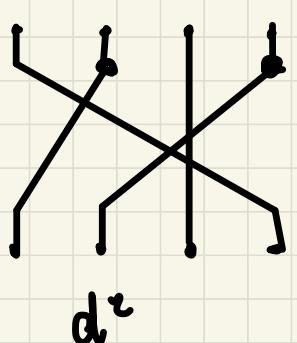
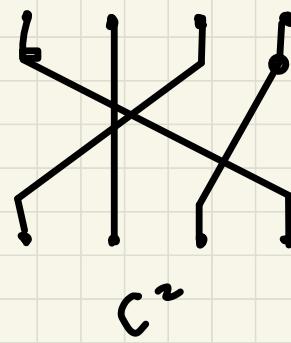
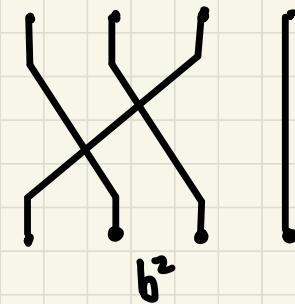
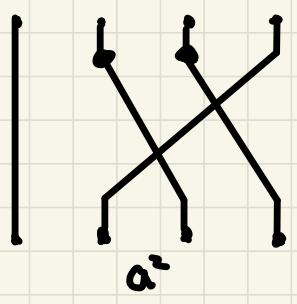
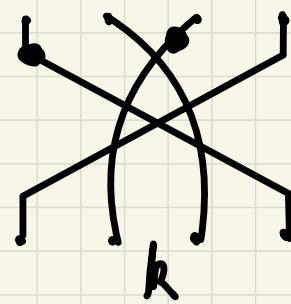
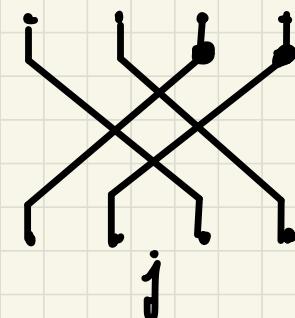
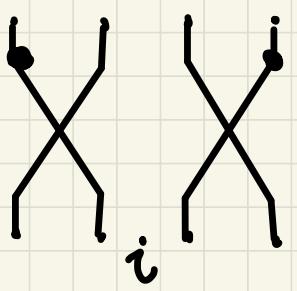
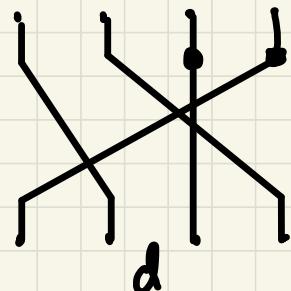
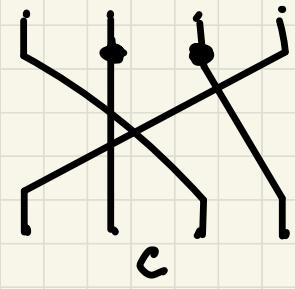
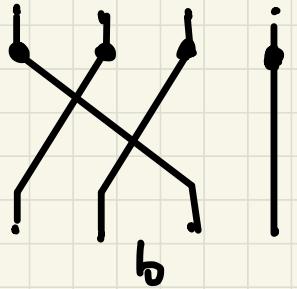
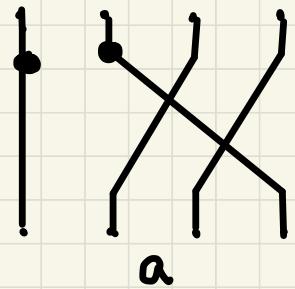
$0 \leftrightarrow N = (1, a^2, a^4)$	$4 \leftrightarrow aN = (a, -1, a^5)$
$1 \leftrightarrow iN = (i, b^4, d^5)$	$5 \leftrightarrow iaN = (d^2, -i, b)$
$2 \leftrightarrow jaN = (b^2, -j, c^5)$	$6 \leftrightarrow jN = (j, c^2, b^5)$
$3 \leftrightarrow kN = (k, d^4, c)$	$7 \leftrightarrow kaN = (c^4, -k, d)$

Play [toys below]

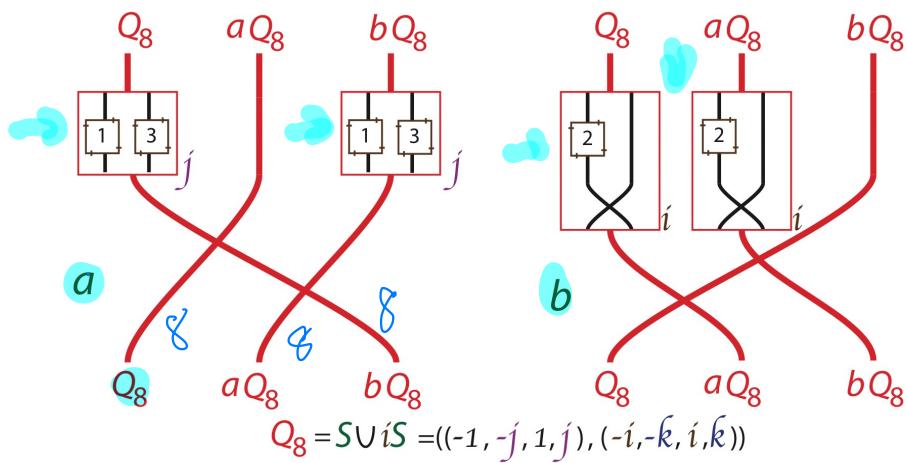


$$ai = b^2, aj = -c, ak = d^2$$

$$ab = j, ba = i, aj = -c, a^{-1}b = c, a^{-1}i = d$$



Other Pictures:



$$Q_8 \quad aQ_8 \quad bQ_8$$

A braid relation diagram involving three strands labeled a , b , and aQ_8 . The strands a and b cross each other, while aQ_8 remains vertical. The strands are represented by boxes containing permutations: a has boxes [1 3] and [2], b has boxes [2] and [2], and aQ_8 has boxes [1 3] and [1]. The result is shown as three strands j with boxes [1 3], [2], and [3 3], labeled (i) and $(-k)$.

$$i \quad j \quad = \quad k \quad (j) \quad (-i)$$

A braid relation diagram involving strands i , j , and k . The strands i and j cross each other, while k remains vertical. The strands are represented by boxes containing permutations: i has boxes [2], [1 3], and [3 3]; j has boxes [1 1], [2], and [3 3]; k has boxes [1 1], [1 3], and [2]. The result is shown as three strands k , (j) , and $(-i)$.

$$Q_8 \quad aQ_8 \quad bQ_8$$

A braid relation diagram involving strands b , a , and bQ_8 . The strands b and a cross each other, while bQ_8 remains vertical. The strands are represented by boxes containing permutations: b has boxes [2] and [2]; a has boxes [1 3] and [1 3]; bQ_8 has boxes [1 3] and [1]. The result is shown as three strands i with boxes [2], [1 1], and [1 3], labeled (k) and (j) .

$$a^{-1} \quad b \quad = \quad c$$

A braid relation diagram involving strands a^{-1} , b , and c . The strands a^{-1} and b cross each other, while c remains vertical. The strands are represented by boxes containing permutations: a^{-1} has boxes [2], [2], and [3 1]; b has boxes [2] and [2]; c has boxes [3 1], [3 1], and [1 1]. The result is shown as three strands c , (j) , and (i) .

$$a^{-1} \quad i \quad = \quad d$$

A braid relation diagram involving strands a^{-1} , i , and d . The strands a^{-1} and i cross each other, while d remains vertical. The strands are represented by boxes containing permutations: a^{-1} has boxes [2], [2], and [3 1]; i has boxes [2], [1 1], and [1 3]; d has boxes [1 1] and [1 1]. The result is shown as two strands d with boxes [1 1] and [1 1].

$$i \quad j \quad = \quad k \quad = \quad -1$$

A braid relation diagram involving strands i , j , k , and -1 . The strands i and j cross each other, while k and -1 remain vertical. The strands are represented by boxes containing permutations: i has boxes [2], [1 1], and [1 3]; j has boxes [1 3], [2], and [3 3]; k has boxes [1 1], [1 3], and [2]; -1 has boxes [2], [2], and [2]. The result is shown as four strands -1 with boxes [2], [2], [2], and [2].

One more repr. of \tilde{A}_4 that uses the ordered subgroup: $A = \langle a \rangle = (1, a, a^2, -1, -a, -a^2)$

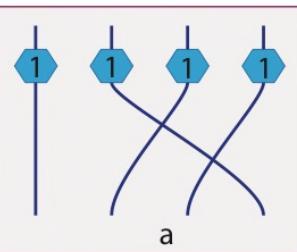
The action of a upon these ordered sets is as follows:

$$\begin{aligned} aA &= (a, a^2, -1, -a, -a^2, 1) = [1/6, A] \\ a(iA) &= (b^2, c^2, -j, -b^2, -c^2, j) = [1/6, jA] \\ a(jA) &= (-c, -d, -k, c, d, k) = [1/6, kA] \\ a(kA) &= (d^2, -b, -i, -d^2, b, i) = [1/6, iA]. \end{aligned}$$

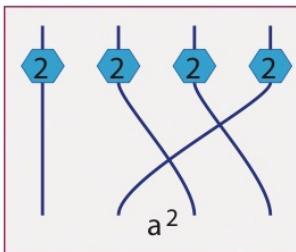
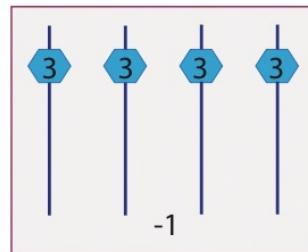
And here is the action of b :

$$\begin{aligned} bA &= (b, i, d^2, -b, -i, -d^2) = [5/6, iA] \\ b(iA) &= (c^2, -j, -b^2, -c^2, j, b^2) = [1/3, jA] \\ b(jA) &= (a^2, -1, -a, -a^2, 1, a) = [1/3, A] \\ b(kA) &= (d, k, -c, -d, -k, c) = [5/6, kA]. \end{aligned}$$

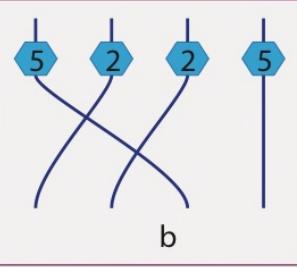
This gives the following set of pictures:



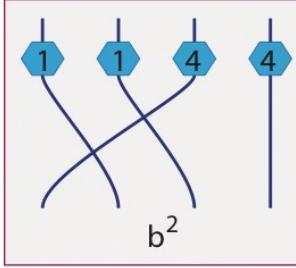
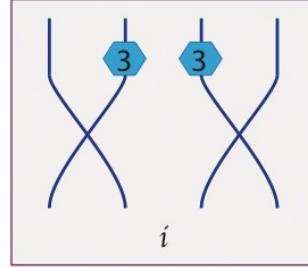
a

 a^2 

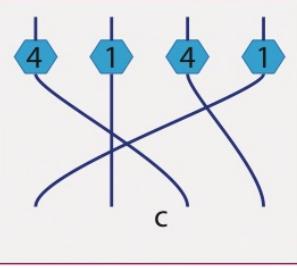
-1



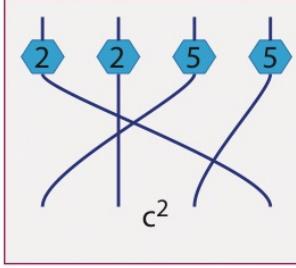
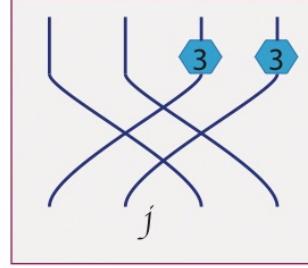
b

 b^2 

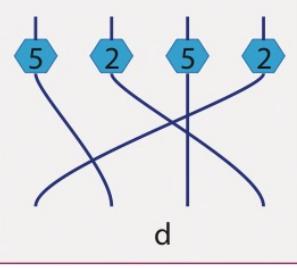
i



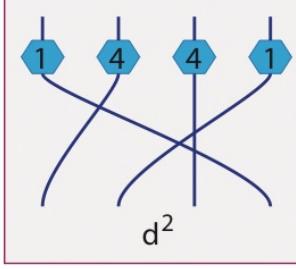
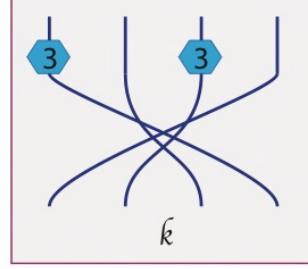
c

 c^2 

j



d

 d^2 

k

subject to the relations:

$$\text{Diagram showing } x = \text{Diagram showing } x$$

$$\text{Diagram showing } x = \text{Diagram showing } x$$

$$\text{Diagram showing } y \\ \text{Diagram showing } x = \text{Diagram showing } x+y$$

modulo 6

In those, we see that a, b, c , & d project to 3 cycles. & Q_8 projects to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ — the Klein 4 group.

$$a \mapsto (234)$$

$$b \mapsto (123)$$

$$c \mapsto (143)$$

$$d \mapsto (142)$$

8. The binary octahedral group

The binary octahedral group, $\widetilde{\Sigma}_4$ is a 2-fold extension of the permutation group Σ_4 . It is given via the presentation

$$\widetilde{\Sigma}_4 = \langle a, f : a^3 = f^4 = (af)^2 \rangle.$$

$$S = (-1, -j, 1, j). \quad a = \frac{1}{2}(1+i+j+k)$$

$$f = (1+i)/\sqrt{2}$$

This is a binary octahedral group, which has 24 (cyclically) ordered cosets. One subgroup, $S = (-1, -j, 1, j)$. Of course the cosets iS , aS , aiS , bS and biS are as before. In $\widetilde{\Sigma}_4$, we have the six additional cosets:

$$fS = ((-1-i), (-j-k), (1+i), (j+k))/\sqrt{2},$$

$$fiS = ((1-i), (j-k), (-1+i), (-j+k))/\sqrt{2},$$

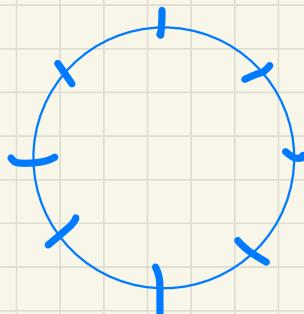
$$faS = ((-i-k), (i-k), (i+k), (-i+k))/\sqrt{2},$$

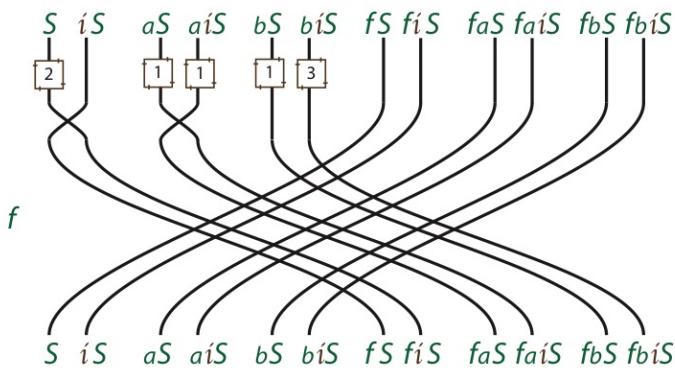
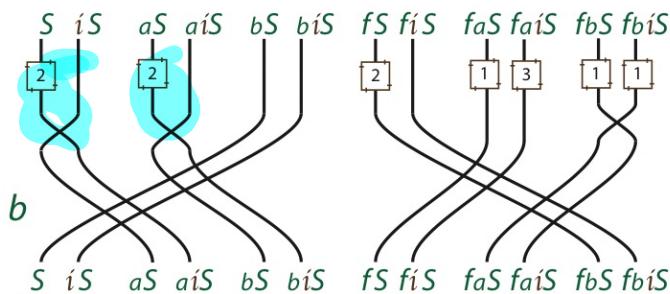
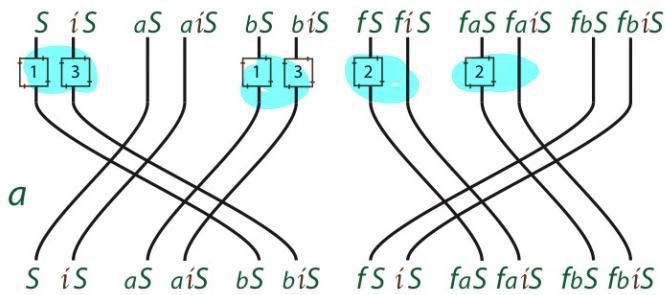
$$faiS = ((1-j), (1+j), (-1+j), (-1-j))/\sqrt{2},$$

$$fbS = ((-i-j), (1-k), (i+j), (-1+k))/\sqrt{2},$$

and

$$fibS = ((1+k), (-i+j), (-1-k), (i-j))/\sqrt{2}.$$





WIP: Rewrite in terms of
quaternions.

$$\begin{matrix} \downarrow \\ \mathbf{i} \\ \mathbf{j} \end{matrix} \rightarrow \mathbf{R}$$

$$(\underbrace{\mathbf{i}-\mathbf{j}}_{\sqrt{2}})(\underbrace{\mathbf{i}-\mathbf{j}}_{\sqrt{2}}) = \frac{\mathbf{i}^2 + \mathbf{i}\mathbf{j} - \mathbf{i}\mathbf{j} + \mathbf{j}^2}{2} = -1$$

Define a subgroup $C \subseteq \tilde{\Sigma}_4$

$$C = \left\langle -1, \frac{i-j}{\sqrt{2}}, 1, \frac{j-i}{\sqrt{2}} \right\rangle$$

Then

$$\alpha C = \left\langle -\alpha, \frac{i-k}{\sqrt{2}}, \alpha, \frac{k-i}{\sqrt{2}} \right\rangle$$

$$\alpha^2 C = \left\langle -\alpha^2, j-k, \alpha^2, k-j \right\rangle$$

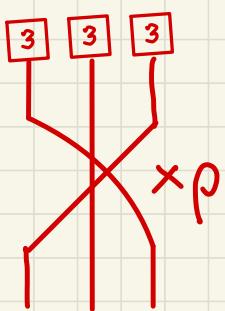
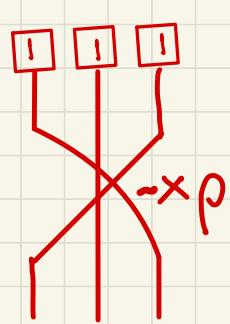
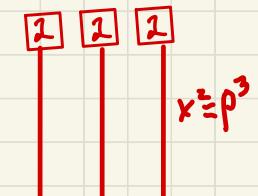
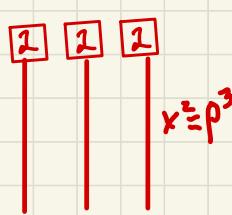
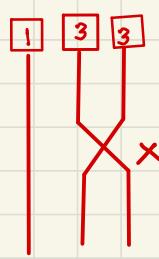
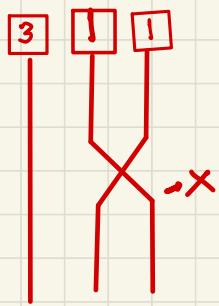
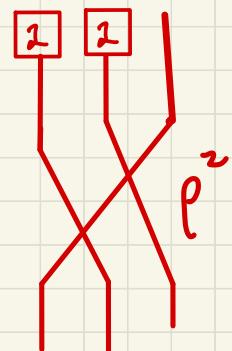
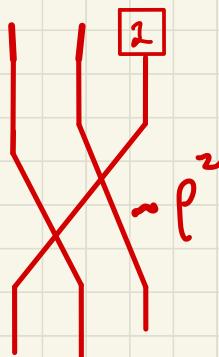
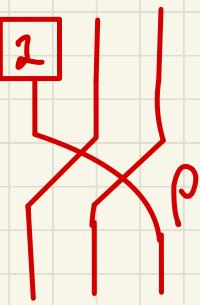
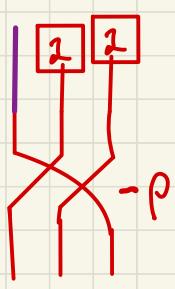
$D = [C, \alpha C, \alpha^2 C]$ is isom. to

the dicyclic group Dic_3

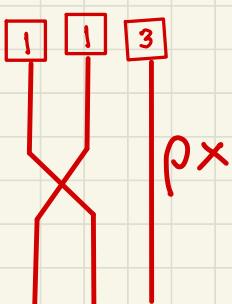
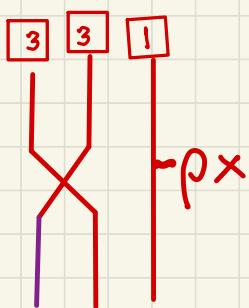
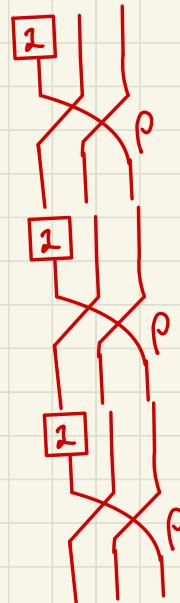
$\begin{matrix} -1 \\ ij \end{matrix}$

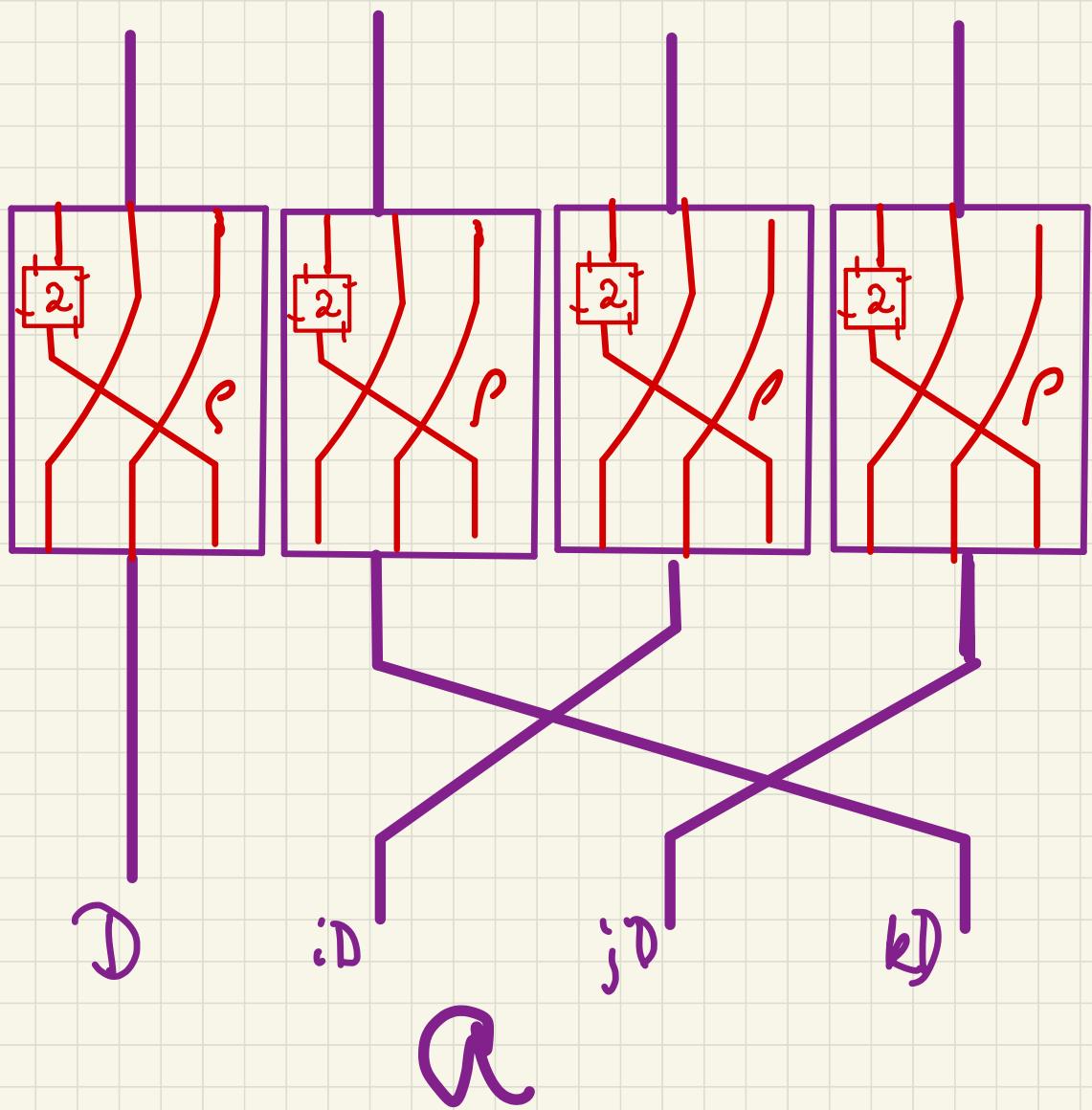
$$\text{Dic}_n = \langle \rho, x : \rho^{2n} = 1, x^2 = \rho^n, \rho x = x\rho^{-1} \rangle.$$

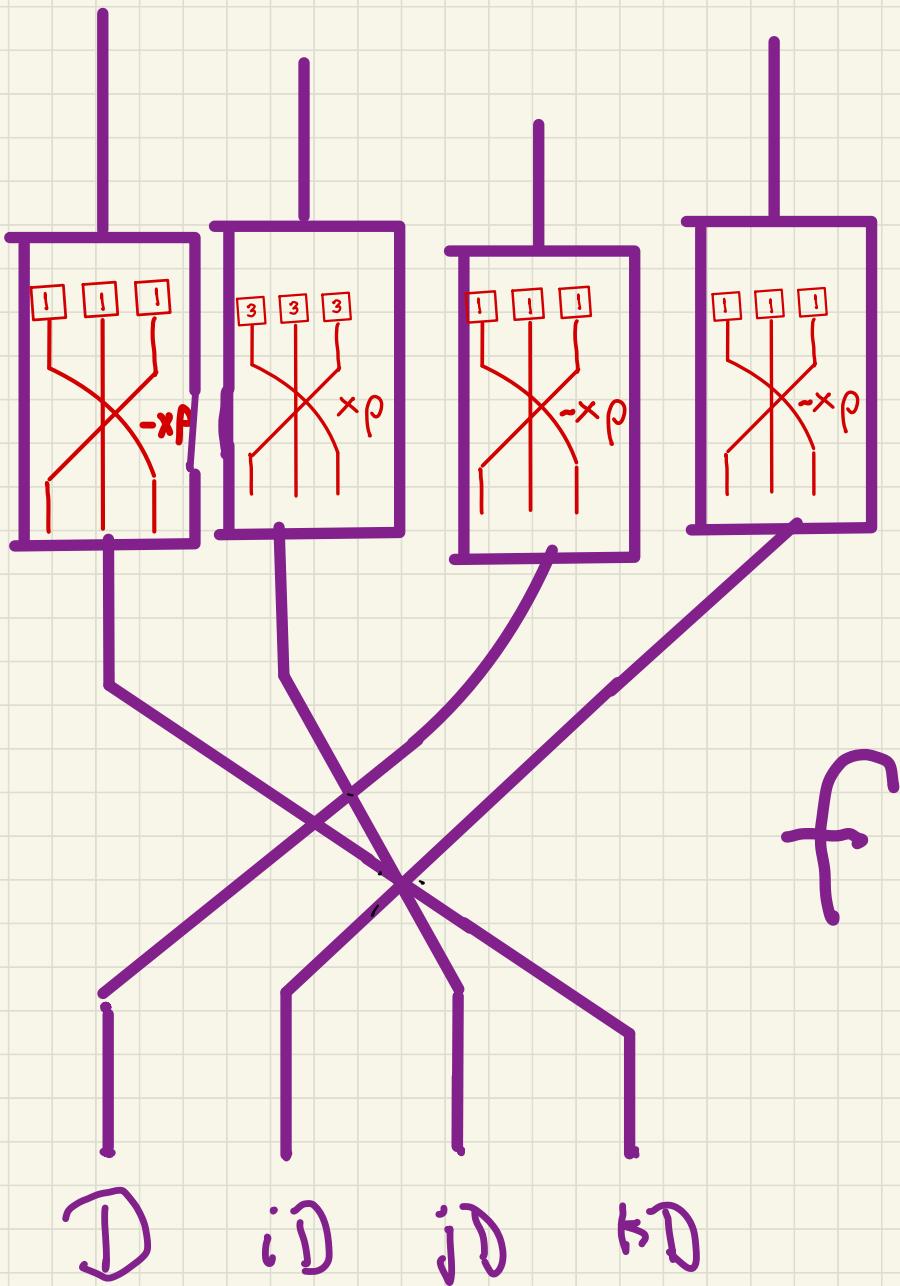
Dic_3
 ρ has order 6



Dic₃

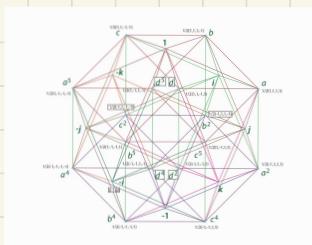






$\mathbb{Z}/6\mathbb{C}$ $\sim \tilde{A}_4 \subset \tilde{A}_5$
 \hookrightarrow \uparrow \uparrow
 Q_8 $\text{binary tetrahedral group}$ $\text{binary Icosahedral group}$
 Order 24 Order 120

So have a 5 string picture for \tilde{A}_5 with "beads"



That's my story, and
I'm sticking to it!

Thank you!