algebraic concordance ————and———— almost classical knots

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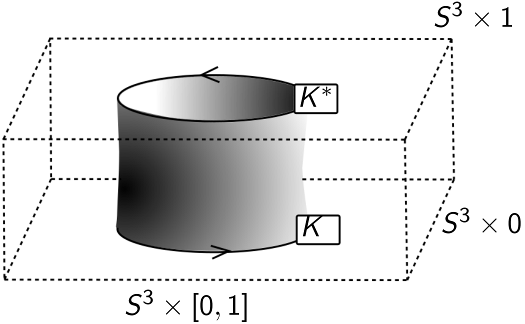
[CM] Algebraic concordance and almost classical knots (https://arxiv.org/abs/2002.01505)

motivation

Knots K, K^* in S^3 are **concordant** if:



Concordance group

$\mathcal{C}$:=abelian group of concordance classes of knots

- Amphicheiral knots have order 2 in $\mathcal{C}$.
- Other torsion in $\mathcal{C}$? Unknown!
- Classical obstructions are:
  1. .... the Arf invariant, and
  2. .... the algebraic concordance group.

Algebraic concordance group

- A is a $2n \times 2n$ dim. matrix over a field $\mathbb{F}$, $\chi(F) \neq 2$, and

$$\det((A - A^\intercal)(A + A^\intercal)) \neq 0$$

.

- A is *metabolic* if $\exists$ P such that $\det(P) \neq 0$ and $PAP^\intercal$:

$$PAP^\intercal = \left[\begin{array}{c|c} 0 & B \\ \hline C & D \end{array}\right].$$

If $A \oplus -B$ is metabolic, A is *(algebraically) concordant* to B.

- $\mathcal{G}^{\mathbb{F}}$:=abelian group of these algebraic concordance classes.

Theorem (J. Levine)

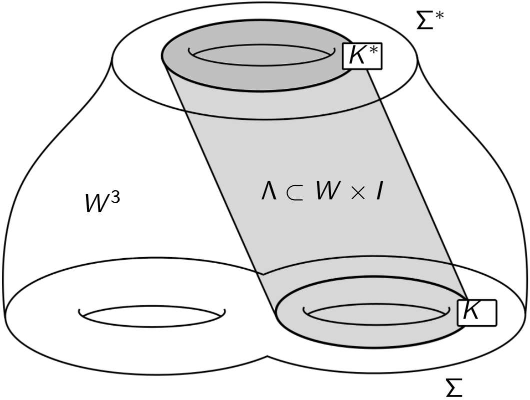
$$\mathcal{G}^{\mathbb{Q}} \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$$

.

Virtual concordance

Σ, Σ^* closed oriented surfaces.

Knots $K \subset \Sigma \times I$, $K^* \subset \Sigma^* \times I$ are **virtually concordant** if:



This is equivalent to concordance of virtual knots, à la Kauffman.

Virtual knot concordance group

$\mathcal{VC}$:= concordance group of long virtual knots.

$$\mathcal{C} \rightarrowtail \mathcal{VC} \longrightarrow \substack{\text{2 component} \\ \text{ribbon tubes}} \subset B^3 \times I$$

The structure of $\mathcal{VC}$ is mysterious.

- (C, 2020) $\mathcal{VC}$ is not abelian.
- (C, 2016) Every $[K] \in \mathcal{VC}$ contains a long virtual knot that is not band-pass equivalent to either the trefoil or the unknot.
- (C, 2019) Every virtual concordance class contains a prime hyperbolic representative and a prime satellite representative.

Questions & Today's goals

Question

Is there any non-classical torsion in $\mathcal{VC}$?

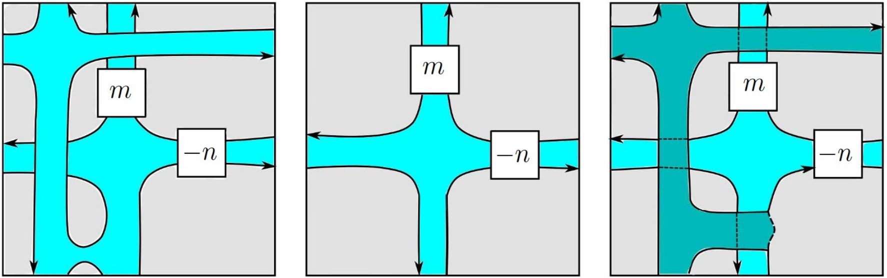
Today's goals

- Generalize Arf invariant, algebraic concordance group to homologically trivial knots in $\Sigma \times I$.
- Classify the uncoupled concordance group.
- Find a geometric realization of these groups.
- Give a potential example of non-classical torsion in $\mathcal{VC}$.

almost classical knots

Definition (Almost classical)

$K \subset \Sigma \times I$ is *almost classical* if $[K] = 0 \in H_1(\Sigma \times I; \mathbb{Z})$. In other words, iff it bounds a Seifert surface in $\Sigma \times I$.



$F_1(m,n)$ $F_0(m,n)$ $F_{-1}(m,n)$

Directed Seifert forms

- $F \subset \Sigma \times I$ a Seifert surface of an AC knot $K \subset \Sigma \times I$.
- Directed Seifert pairing: $\theta^{\pm}_{K,F} : H_1(F) \times H_1(F) \to \mathbb{Z}$,

$$\theta^{\pm}_{K,F}(x,y) = \mathrm{lk}_{\Sigma}(x^{\pm}, y).$$

- Directed Seifert matrix: $A^{\pm} := (\mathrm{lk}_{\Sigma}(a_i^{\pm}, a_j))$.
- Directed Alexander poly:

$$\Delta^{\pm}_{K,F}(t) = \det(A^{\pm} - t(A^{\pm})^{\intercal}).$$

- Quadratic form of (K,F): $q_{K,F} : H_1(F;\mathbb{Z}_2) \to \mathbb{Z}_2$

$$q_{K,F}(x) \equiv \theta^{\pm}_{K,F}(x,x) \equiv \mathrm{lk}_{\Sigma}(x^{\pm}, x) \pmod 2$$

$\boxed{\underline{\text{NOTE:}}\ A^+ \neq (A^-)^\intercal \text{ (in general)}}$

Lemma

$$A^+ + (A^+)^\intercal = A^- + (A^-)^\intercal, \textit{ or}$$

$$A^- - A^+ = -(A^- - A^+)^\intercal$$

algebraic concordance

Coupled concordance group $(\mathcal{VG}, \mathcal{VG})^{\mathbb{F}}$

- An $\mathbb{F}$-Seifert couple is a pair $\mathbf{A} = (A^+, A^-)$ of $2n \times 2n$ matrices over $\mathbb{F}$ such that:

$$A^- - A^+ \text{ is skew-symmetric \& } \det(A^- - A^+) \neq 0.$$

- $\mathbf{A} = (A^+, A^-)$ is called *metabolic (or null-concordant)* if $A^{\pm}$ are *simultaneously congruent* over $\mathbb{F}$ to matrices in block form:

$$\left[\begin{array}{c|c} 0 & P^{\pm} \\ \hline Q^{\pm} & R^{\pm} \end{array}\right].$$

- $\mathbf{A} = (A^+, A^-)$ is admissible if $\det(A^+ + (A^+)^{\intercal}) \neq 0$.
- Concordance classes of admissible Seifert couples form a group $(\mathcal{VG}, \mathcal{VG})^{\mathbb{F}}$

Uncoupled concordance group $\mathcal{VG}^{\mathbb{F}}$

- An $\mathbb{F}$-*directed matrix* is a $2n \times 2n$ dimensional matrix A with coefficients in $\mathbb{F}$ such that $\det(A + A^{\intercal}) \neq 0$.

- A is metabolic if congruent over $\mathbb{F}$ to a matrix having a half dimensional block of zeros.

- Concordances classes of directed Seifert matrices form a group $\mathcal{VG}^{\mathbb{F}}$, called the uncoupled concordance group.

Relating the algebraic concordance groups

There are surjections:

$$\pi^{\pm} : (\mathcal{VG}, \mathcal{VG})^{\mathbb{F}} \longrightarrow \mathcal{VG}^{\mathbb{F}}, \pi^{\pm}(A^{+}, A^{-}) = A^{\pm}.$$

...and an injection:

$$\iota : \mathcal{G}^{\mathbb{F}} \to (\mathcal{VG}, \mathcal{VG})^{\mathbb{F}}, \iota(A) = (A, A^{\intercal})$$

Theorem (C-Mukherjee)

The classical knot concordance group $\mathcal{G}^{\mathbb{Z}}$ embeds as a subgroup of $(\mathcal{VG}, \mathcal{VG})^{\mathbb{Q}}$ into the equalizer of π^{+} and π^{-}.

Directed isometric structures

$$A \in \mathcal{VG}^{\mathbb{F}} \longrightarrow (A + A^{\intercal}, A^{-1}A^{\intercal})$$

- $A^{-1}A^{\intercal}$ is an isometry of $A + A^{\intercal}$.

- Even though $A^{+} + (A^{+})^{\intercal} = A^{-} + (A^{-})^{\intercal}$, isometric structures can be different.

Classification of $\mathcal{VG}^{\mathbb{F}}$

Theorem (C-Mukherjee)

Let $\mathcal{I}(\mathbb{F})$ denote the fundamental ideal of the Witt ring over $\mathbb{F}$. Then:

$$\mathcal{VG}^{\mathbb{F}} \cong \mathcal{I}(\mathbb{F}) \oplus \mathcal{G}^{\mathbb{F}}.$$

Idea of proof:

- For $A \in \mathcal{VG}^{\mathbb{F}}$, define an isometric structure $(A + A^{\intercal}, A^{-1}A^{\intercal})$.
- Decompose $B = A + A^{\intercal}$ into the primary components of the irreducible factors of the characteristic polynomial $\lambda(t)$ of $A^{-1}A^{\intercal}$.
- If $t - 1$ divides $\lambda(t)$, B has a component in $\mathcal{I}(\mathbb{F})$. All other components are in $\mathcal{G}^{\mathbb{F}}$.

$\square$

Corollaries

Useful facts

1. If 1 is not a root of the directed Alexander polynomial, the concordance class is in $\mathcal{G}^{\mathbb{F}}$.
2. If 1 is a root of the Alexander polynomial, you get additional obstructions coming from the $\mathcal{I}(\mathbb{F})$ summand.

Theorem (C-Mukherjee)

For $\mathbb{F}$ a global field of characteristic 0, the only possible finite order of elements in $\mathcal{VG}^{\mathbb{F}}$ and $(\mathcal{VG}, \mathcal{VG})^{\mathbb{F}}$ are 1, 2, and 4.

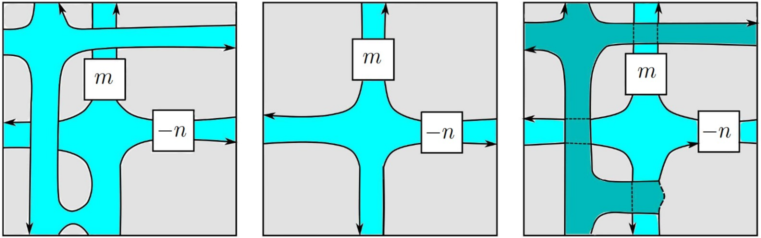
Examples

Theorem (C-Mukherjee)

There are infinitely many knots $K \subset \Sigma \times I$ such that for each $k \in \{1, 2, 4\}$, K bounds a Seifert surface having (uncoupled) algebraic concordance order k.

Proof.

Set $m_k = 3 + 19^2 \cdot 4k$, $n = 11$. Choose k so that m_k is prime. Then $\text{order}(A_0^+) = 4$, $\text{order}(A_{\pm 1}^{\pm}) = 1$, $\text{order}(A_+^{-1}) = 2$. □



$F_{-1}(m, n)$

geometric realization

virtual concordance of seifert surfaces

Definition

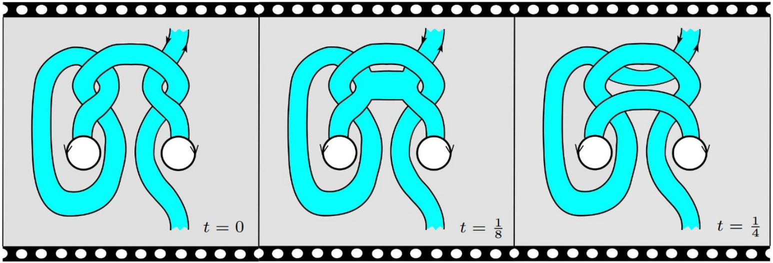
Two Seifert surfaces $F_0 \subset \Sigma_0 \times I, F_1 \subset \Sigma_1 \times I$ of knots K_0, K_1 will be called *virtually concordant* if there is:

1. a compact oriented 3-manifold W,
2. a properly embedded annulus A in $W \times I$, and
3. a compact, oriented 3-manifold $M \subset W \times I$,

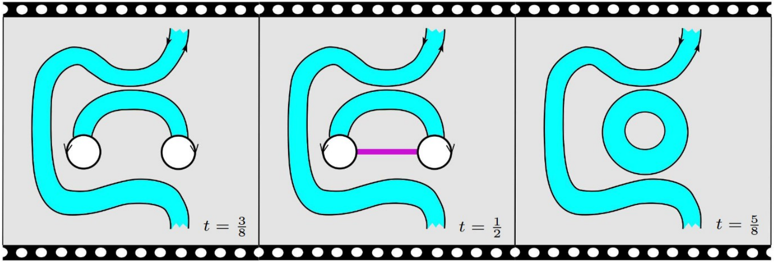
such that the following conditions are satisfied:

1. $\partial W = \Sigma_1 \sqcup -\Sigma_0$,
2. $\partial A = K_1 \sqcup -K_0$,
3. $\partial M = F_1 \cup A \cup -F_0$, and
4. $M \cap (\partial W \times I) = F_1 \sqcup -F_0$.

Example



Example



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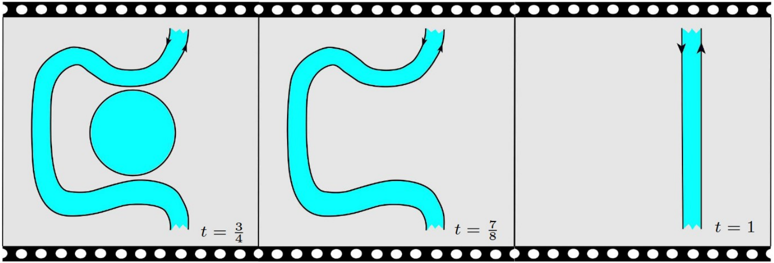


Theorem (C-Mukherjee)

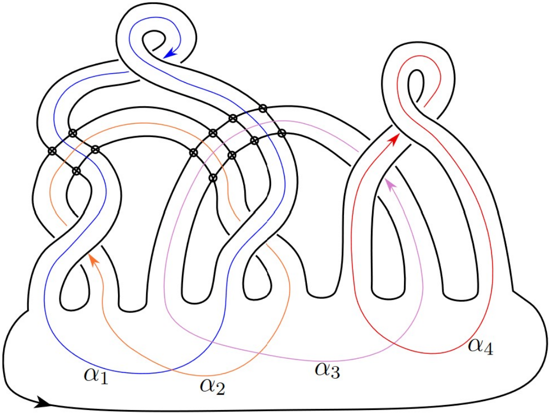
$$\underset{\textit{seifert surfaces}}{\textit{virtually concordant}} \implies \underset{\textit{algebraically concordant}}{\mathbb{Z}-\textit{Seifert couples}}$$

Corollary (C-Mukherjee)

$$\underset{\textit{seifert surfaces}}{\textit{virtually concordant}} \;\&\; \underset{\textit{regular}}{\textit{quadratic forms}} \implies \underset{\textit{are equal}}{\textit{their Arf invariants}}$$

Example: 6.85091

A^- has order 1, A^+ has order 2



Does this have order 2 in $\mathcal{VC}$?

Here are the isometric structures for A^+:

$$A^+ + (A^+)^\intercal = \begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, (A^+)^{-1}(A^+)^\intercal = \begin{bmatrix} -1 & 2 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ -2 & 2 & -2 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Here is the primary decomposition:

$$B' \oplus B'' = \left[\begin{array}{cc|cc} -2 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ \hline 0 & 0 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{array}\right], S' \oplus S'' = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -4 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thank you!