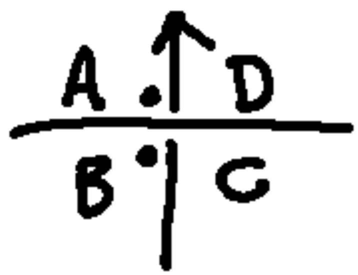
A State Summation Invariant For Knotoids

L Kauffman, UIC
joint work with
N Gügümcü

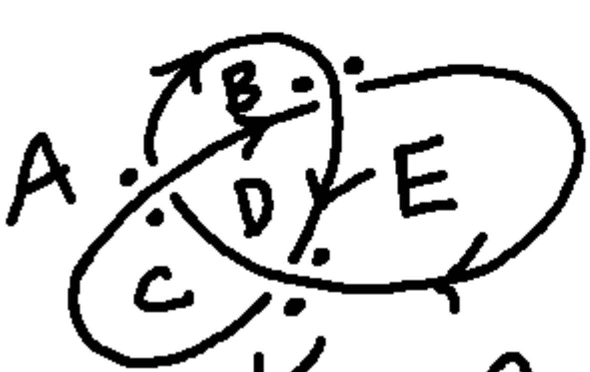
1° Recall state sum for Conway–Alexander Polynomial
(LHK, Formal Knot Theory, PUP 1981)

J.W. Alexander
TAMS (1928)



$\rightsquigarrow Ax + Bx + C + D$
modulo relation

(Alex used $Ax - Bx + C - D$.
Signs can be eliminated.)



K

$$\left.\begin{matrix} Ax + Cx + D + B \\ Ax + Bx + D + E \\ Ax + Ex + D + C \end{matrix}\right\} \qquad \left(\begin{matrix} R \doteq S \\ \text{means} \\ R = \pm x^N S \end{matrix} \right)$$

$$\begin{array}{ccccc} A & B & C & D & E \\ x & 1 & x & 1 & 0 \\ x & x & 0 & 1 & 1 \\ x & 0 & 1 & 1 & x \end{array} \qquad M(K) = \begin{bmatrix} x & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & x \end{bmatrix}$$

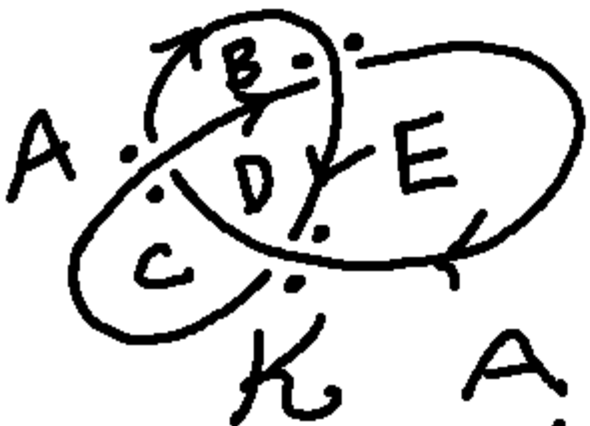
(remove two adjacent region columns)

$$\Delta_K(x) \doteq \mathrm{Det}\,(M(K))$$

$$= x\begin{vmatrix} 1 & 1 \\ 1 & x \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 1 & x \end{vmatrix}$$

$$= x(x-1) + 1 = x^2 - x + 1$$

$$\Delta_K(x) \doteq x^2 - x + 1$$



$$\left.\begin{array}{l} Ax + Cx + D + B \\ Ax + Bx + D + E \\ Ax + Ex + D + C \end{array}\right\}$$

$\left(R \doteq S \text{ means } R = \pm x^N S \right)$

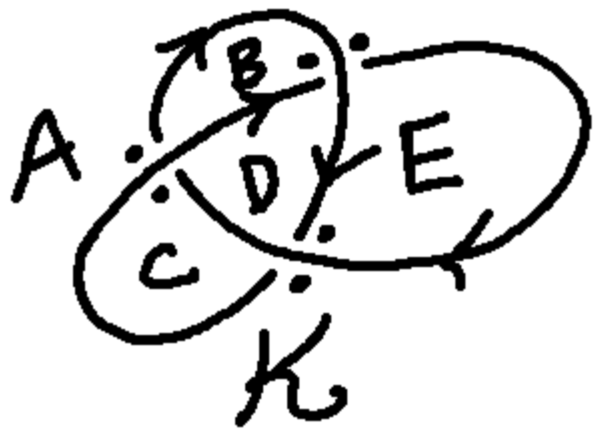
$$\begin{array}{c|cc|ccc|} & A & B & C & D & E \\ 1. & x & 1 & x & 1 & 0 \\ 2. & x & x & 0 & 1 & 1 \\ 3. & x & 0 & 1 & 1 & x \end{array}$$

$M(K)$

(remove two adjacent region columns)

{ ≠0 terms in Det(M) } $\longleftrightarrow$ { Regions Choose Crossings. Each crossing chosen once. }

Marker States $\mathcal{S}(K)$



$$\mathrm{sgn}(S) \overset{\mathrm{def}}{=} (-1)^{b(S)}$$

$b(S) = \#\{$  $\}$

Permutation Sign is Replaced by the "black hole count"

$\underline{1}$

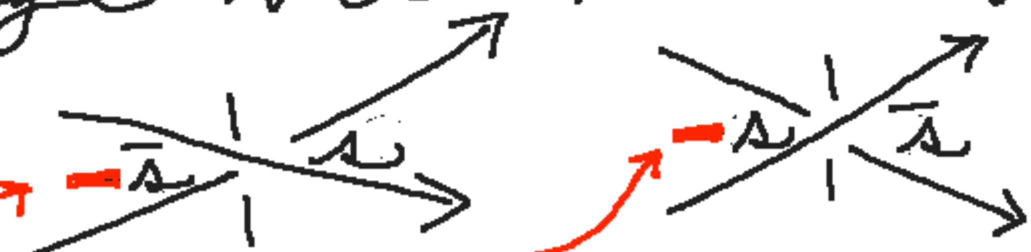
$-x$

$(-1)^2 x^2$

$$\Delta_K(x) \doteq \sum_{S \in \mathcal{S}} \mathrm{sgn}(S) \underbrace{\langle K | S \rangle}_{\underbrace{\Pi \text{ of } x\text{'s } \& \text{ 1's}}}$$

For Conway Alexander change vertex weights to

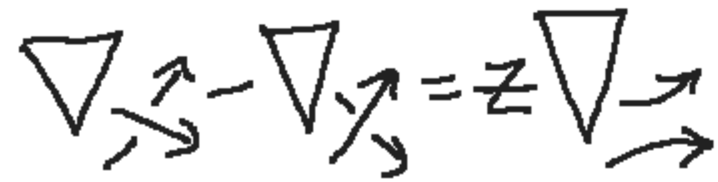
$\bar{s} = 1/s$



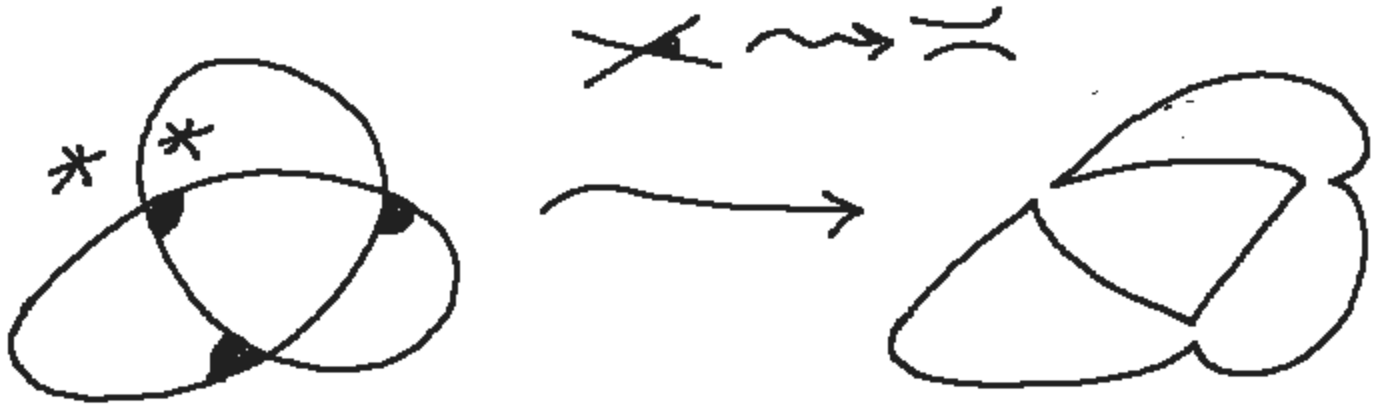
(sgn of state now absorbed in vertex weights)

$$z = s - \bar{s}$$

$$\nabla_K = \sum_{S \in \Delta} \langle K | S \rangle$$



$K \sim K' \Longrightarrow \nabla_K = \nabla_{K'}$



States are in 1-1 corres
with single cycle
smoothings of the diagram.

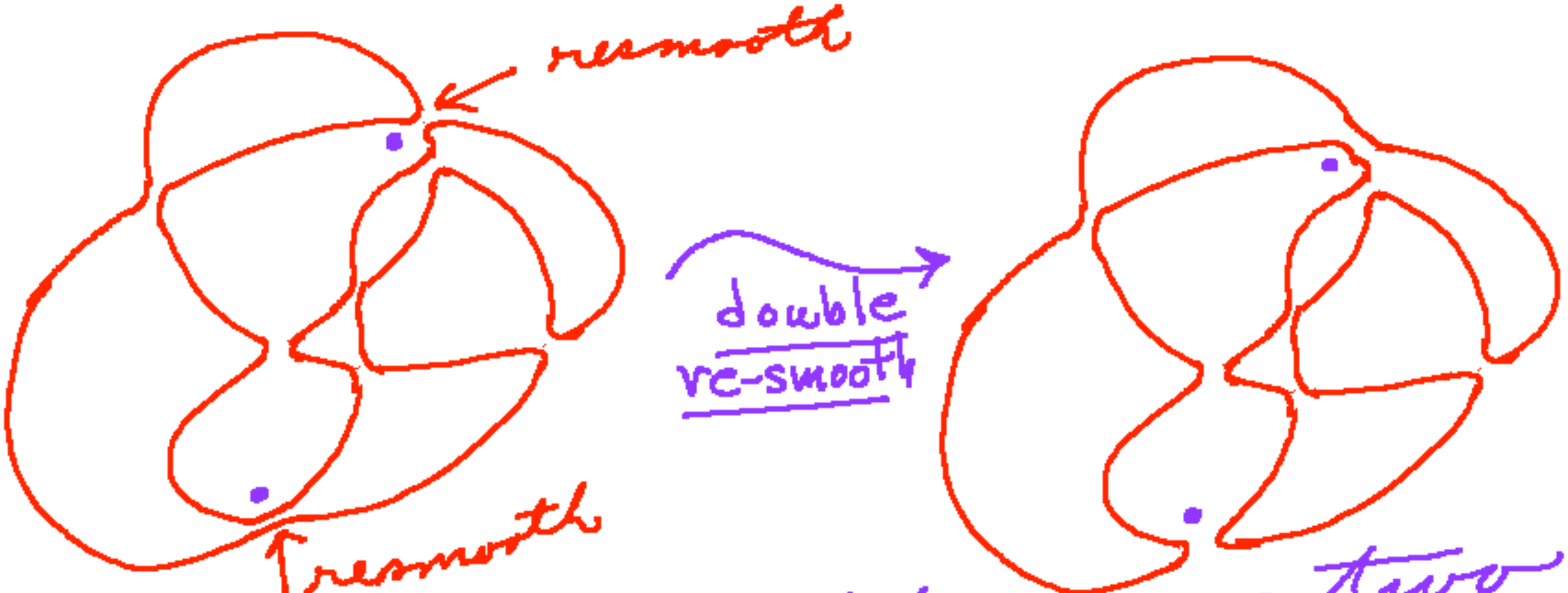
States are in
1-1 correspondence
with maximal
trees in the
checkerboard
graph.

$\boxed{CT}$

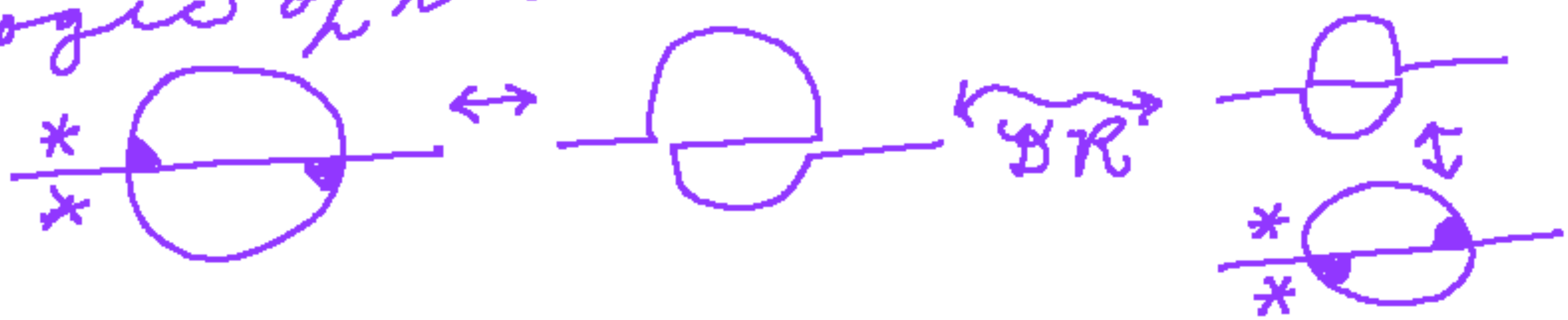
Clock Theorem. The states of a diagram form a lattice with a unique clocked state and a unique counter-clocked state. Clocking moves connect any two states.

Individual clocking moves change the sign of the state.

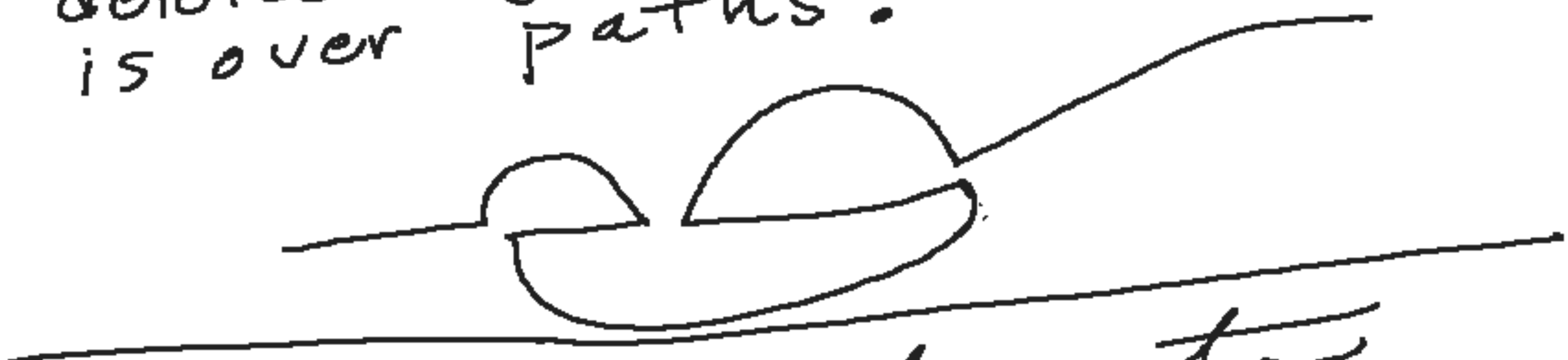
(It is a consequence of the CT that the state sum computes the Det(M).)



It is the case that any two states can be connected by sequences of double-resmoothings. This is the underlying logic of the Clock Theorem.



For long knots + links, we have a standard choice of deleted regions. The sum is over paths.



All of this applies to the bracket poly / Jones poly with different vertex weights.

A State Sum Invariant for Knotoids

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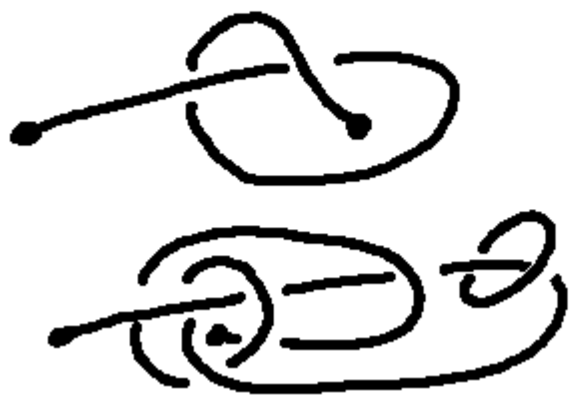
Joint work with Neslihan Gügümcü.

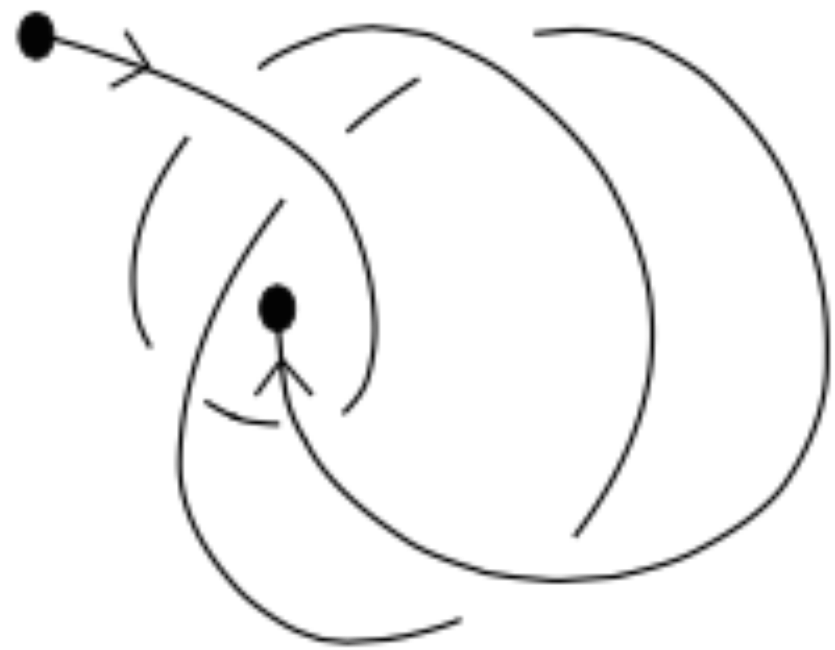
2. A *knotoid* is a diagram with endpoints in possibly different regions, taken up to Reidemeister moves that avoid endpts.

Thus knotoids generalize (1-1) tangles.



Term due to Turaev.

A knotoid (V. Turaev) is a knot diagram with two ends.
The ends can be in different regions of the diagram.
We study knotoids up to Reidemeister moves.
The moves do not move arcs across the ends of
the diagrams.



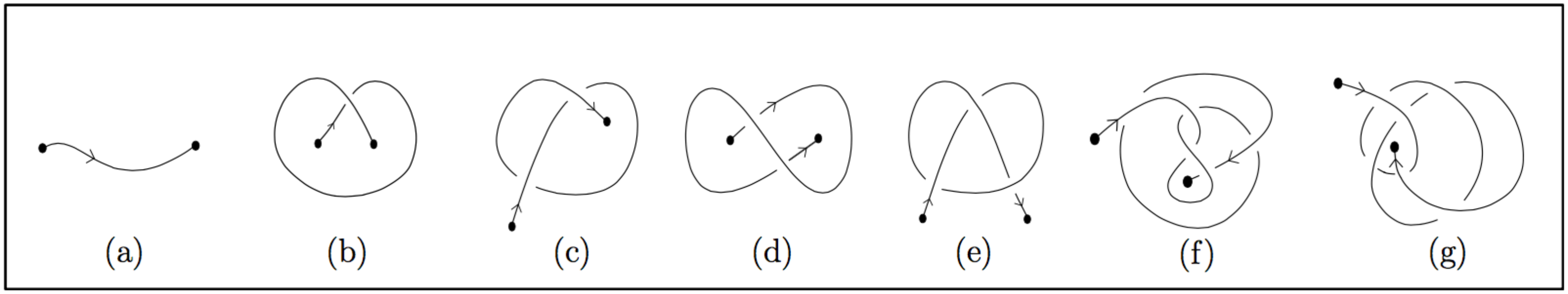
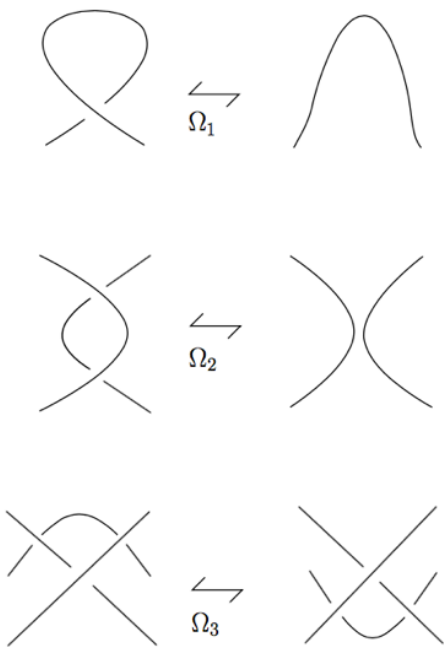
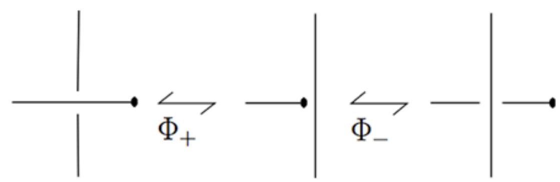


FIGURE 1. **Knotoid diagrams**



(a) $\Omega_{i=1,2,3}$- **moves**



(b) **Forbidden knotoid moves**

2.1. **An Interpretation of Classical Knotoids in 3-Dimensional Space.** Let K be a knotoid diagram in $\mathbb{R}^2$. The plane of the diagram is identified with $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. K can be embedded into $\mathbb{R}^3$ by pushing the overpasses of the diagram into the upper half-space and the underpasses into the lower half-space in the vertical direction. The tail and the head of the diagram are attached to the two lines, $\mathrm{t} \times \mathbb{R}$ and $\mathrm{h} \times \mathbb{R}$ that pass through the tail and the head, respectively and is perpendicular to the plane of the diagram. Moving the endpoints of K along these special lines gives rise to embedded open oriented curves in $\mathbb{R}^3$ with two endpoints of each on these lines.

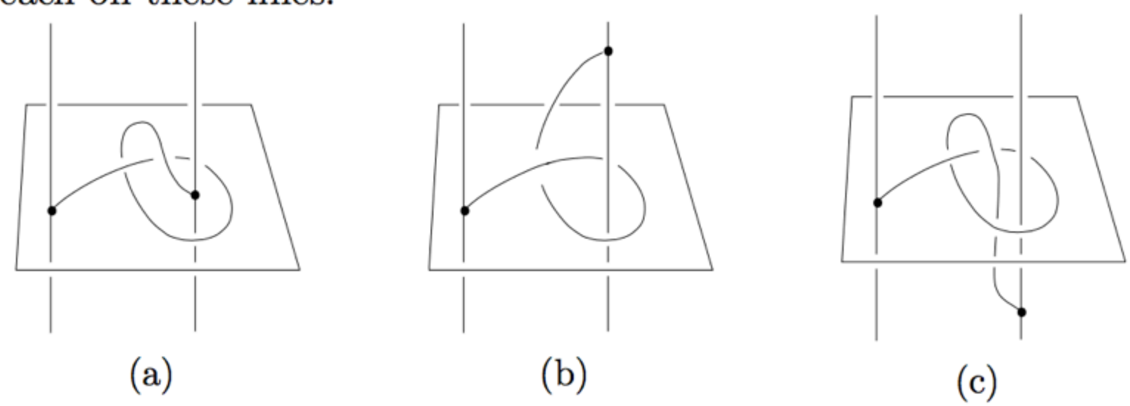
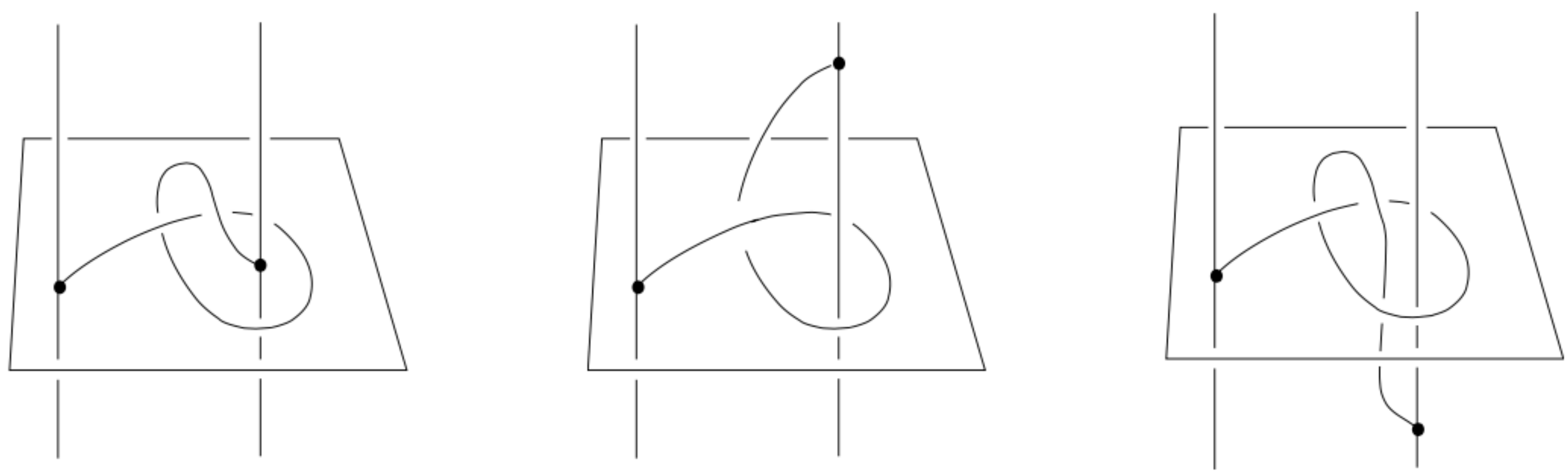
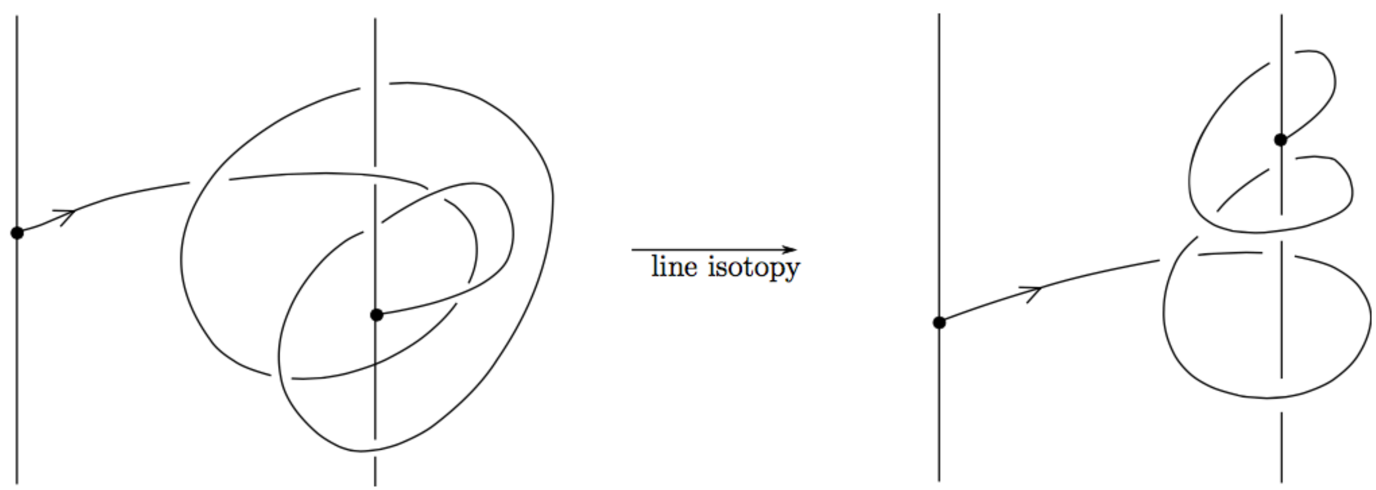


FIGURE 3. **Curves in $\mathbb{R}^3$ obtained by the knotoid diagram in Figure 1(c)**



An embeded arc in R^3 becomes a knotoid on taking a generic projection to a plane.

Restricting isotopies of the arc to endpoint motions on the parallel lines (perpendicular to the plane) and otheswise in the complement of the two lines, preserves the knotoid type of the projection.



Descending Knotoid
diagrams are unknotted.

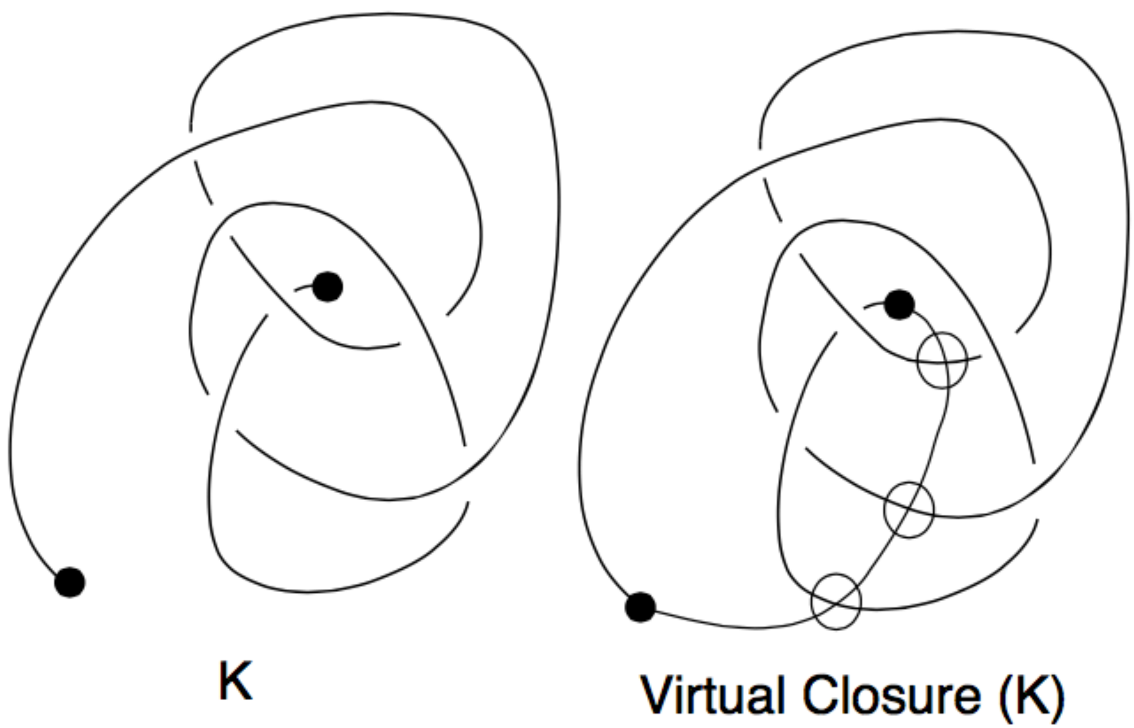
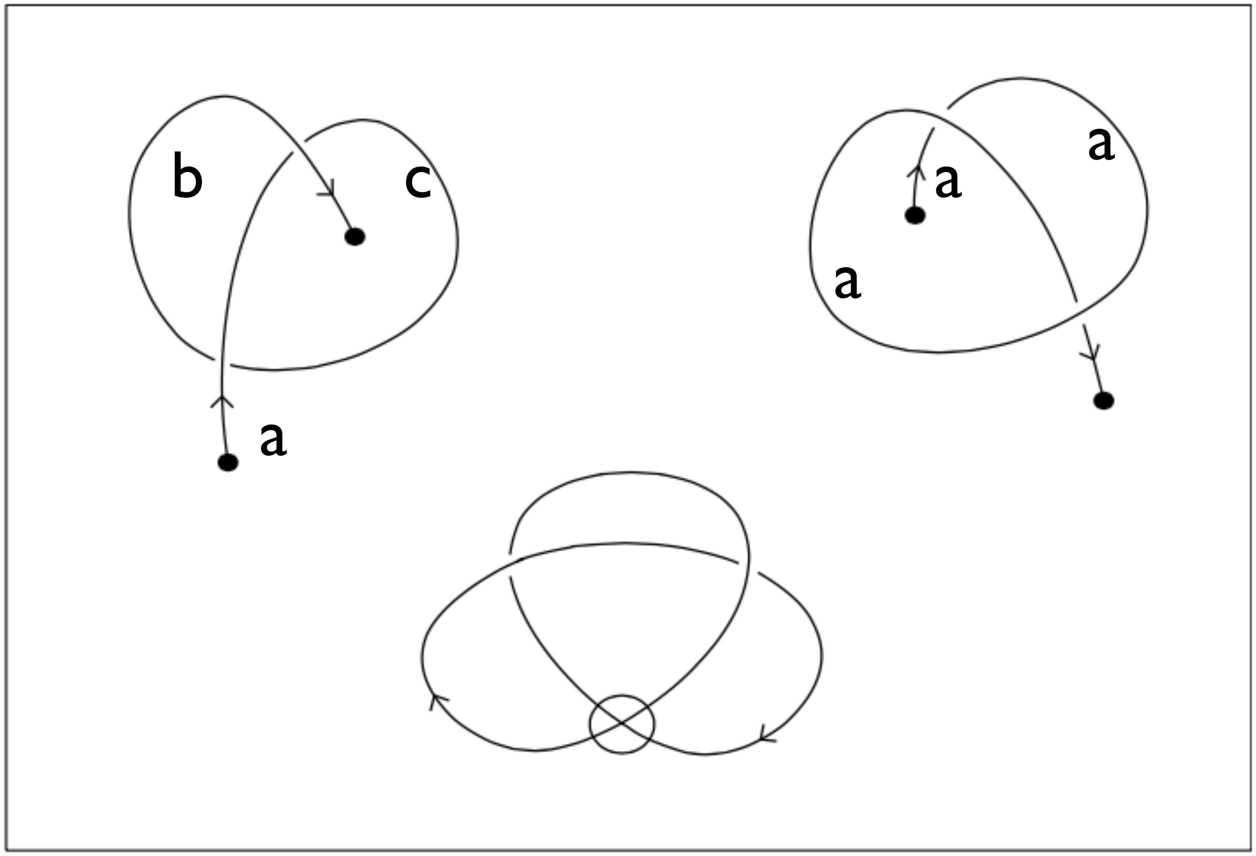


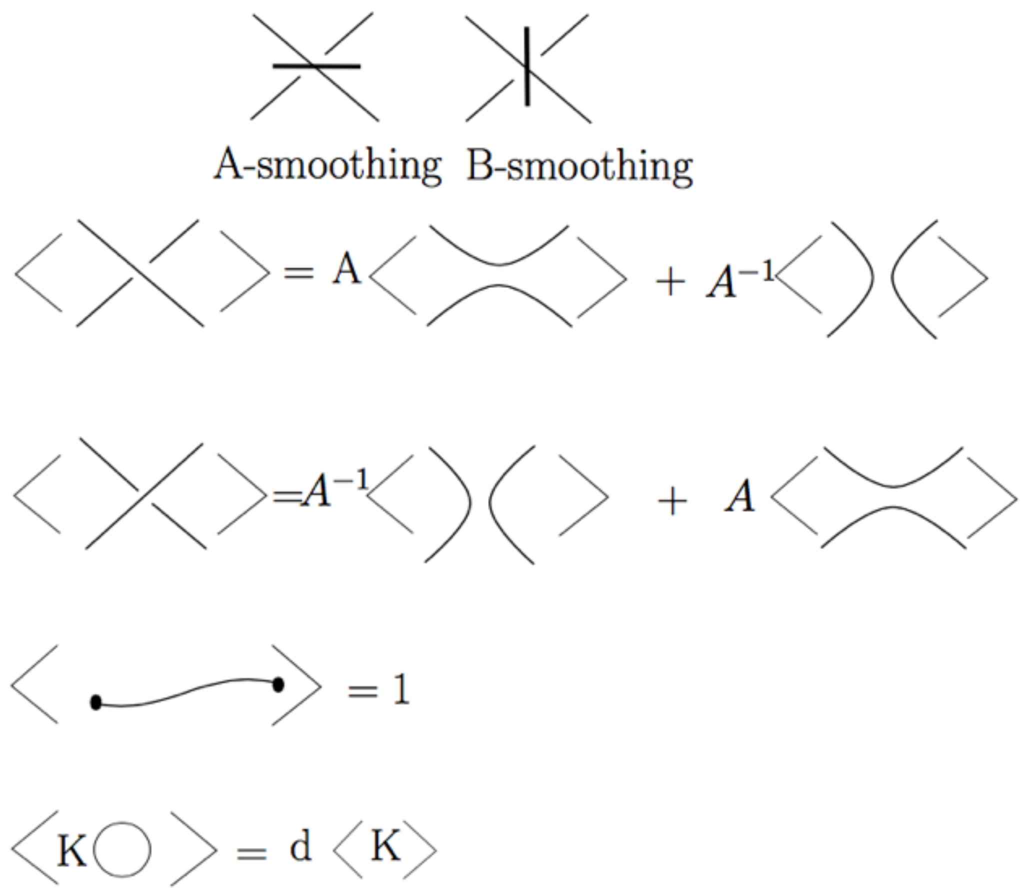
Figure 14: **Knotoid and Its Virtual Closure**

The virtual closure of a knotoid is supported in genus one (add a handle to the 2-sphere).

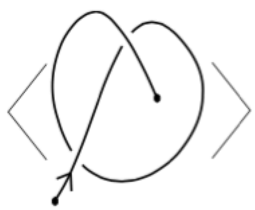
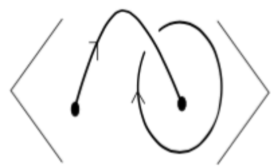
Two Distinct Knotoids with the same Virtual Closure

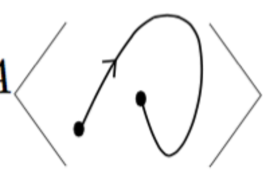
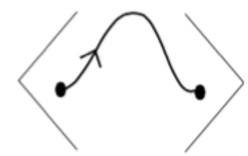


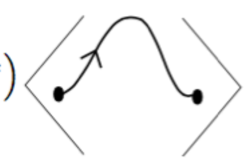
Bracket Polynomial for Knotoids



Bracket Calculation

 $= A$  $+A^{-1}$ 

$= A(A$  $+ A^{-1}$  $) + A^{-4}$ 

$= (A^2 + 1 - A^{-4})$ 

Conjecture: The bracket polynomial detects the unknotted knotoid.

Discussion: This conjecture includes the well-known conjecture that the Jones polynomial detects the unknot.

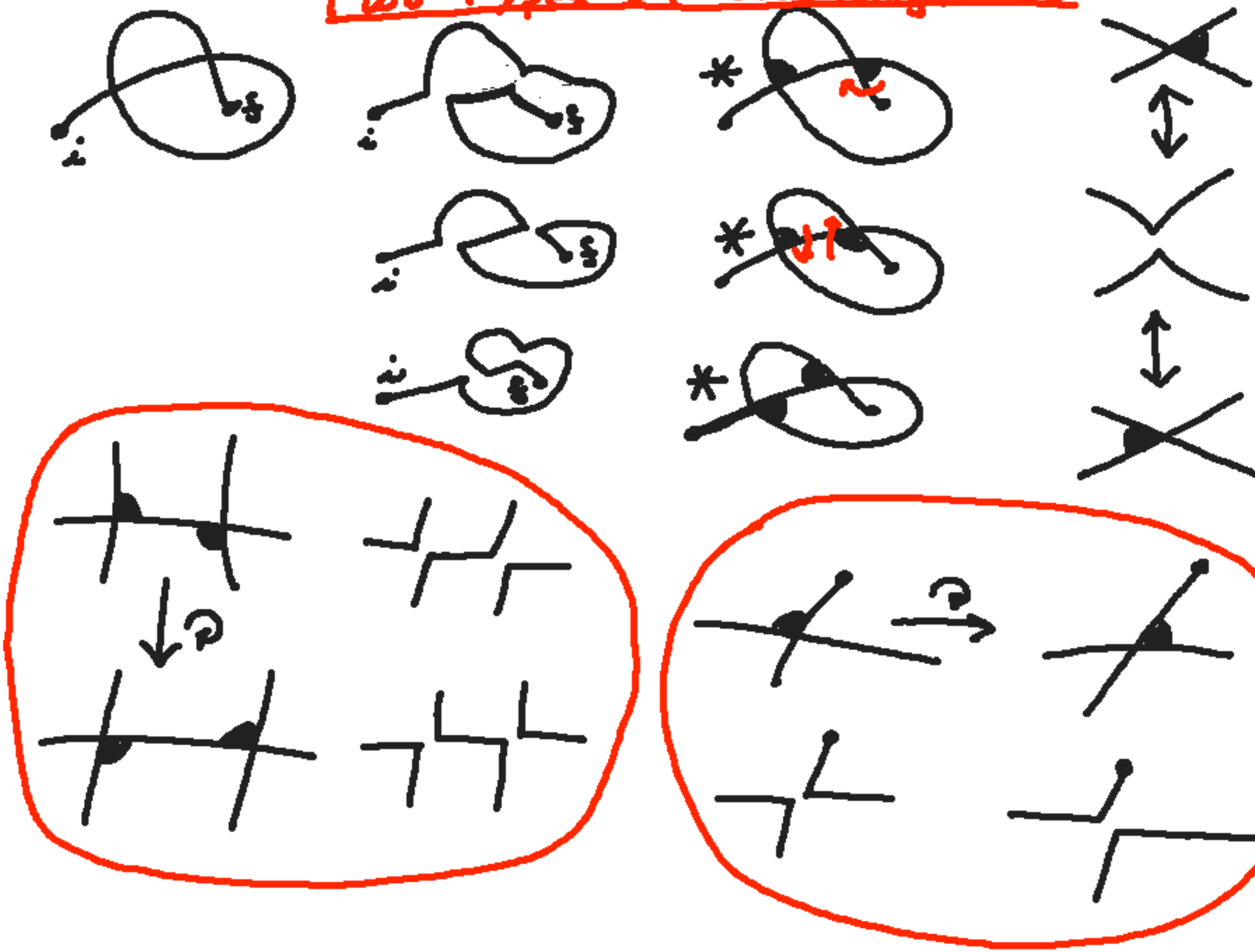
Note that the corresponding conjecture for virtual knots is false. There are non-trivial non-classical virtual knots with unit Jones polynomial. And there are examples of such virtual knots of genus one. This means that we conjecture that such virtual knots are not in the image of the closure map from knotoids.

In this talk we show how the FKT (Formal Knot Theory) state summation for Alexander-Conway polynomial generalizes to a <u>potential function</u> $\nabla_K(S)$ for knotoids.

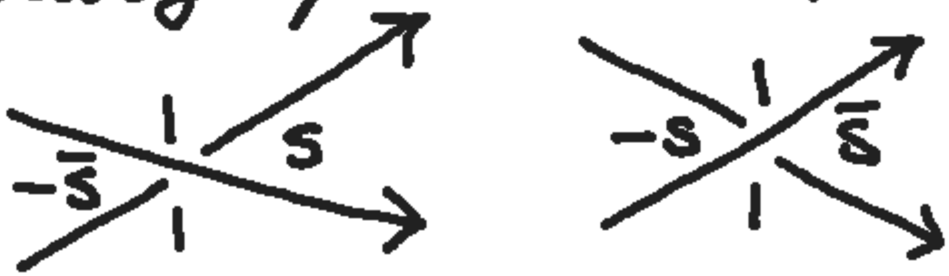
States $= \left\{ \substack{\text{non-crossing} \\ \text{paths from}} \atop i \longrightarrow f \right\}$

$\Longleftrightarrow$ <u>Marker States</u>

Two Types of Cloaking Move



Here we show how to use the path states to define a Conway potential function $\nabla_K(s)$.



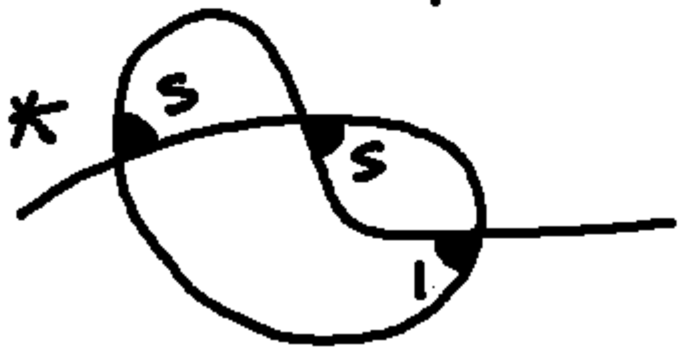
vertex weights.

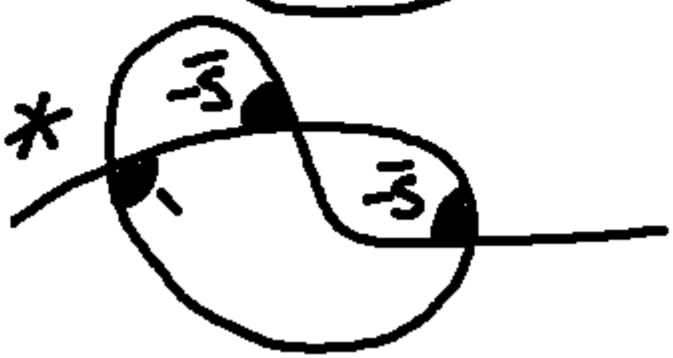
$$\nabla_K(s) = \sum_{\alpha \in Paths(K)} \langle K|\alpha\rangle$$

$\langle K|\alpha\rangle$ = Product of the vertex weights touched by markers in α.

The Classical Case

T





$$\nabla_T = s^2 - 1 + \bar{s}^2$$

$$= (s - \bar{s})^2 + 1$$

$$\nabla_T = z^2 + 1, \quad z = s - \bar{s}$$

$$\left(\begin{aligned} \Delta_K(t) &\doteq \nabla_K(s = \sqrt{t}) \\ &= t - 1 + t^{-1} \\ &\doteq t^2 - t + 1 \end{aligned} \right)$$

Alexander

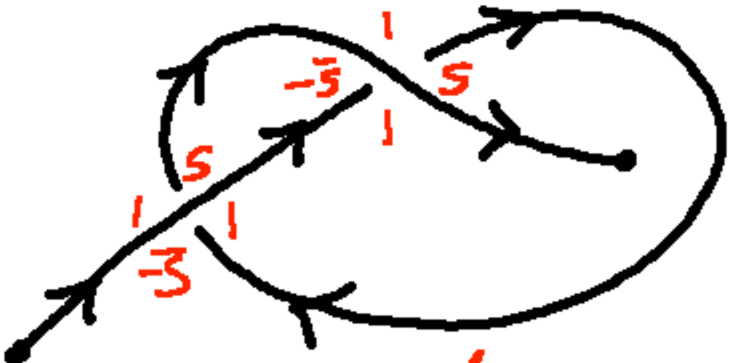
In the classical case we have that for $z = s - s^{-1}$,

$$\nabla_{\searrow\!\!\!\!\!\nearrow} - \nabla_{\nearrow\!\!\!\!\!\searrow} = z\,\nabla_{\rightleftarrows}$$

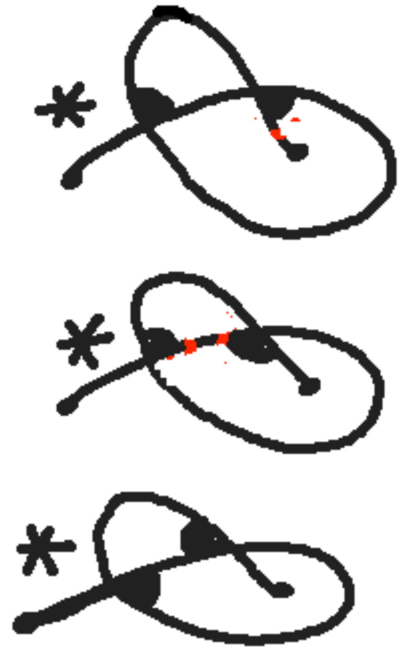
~~f~~ if $\Delta_K(t) =$ alex poly

then $\Delta_K(t) \doteq \nabla_K(s = \sqrt{t})$.

We take leave of these properties for the knotoids.



k



s^2

s

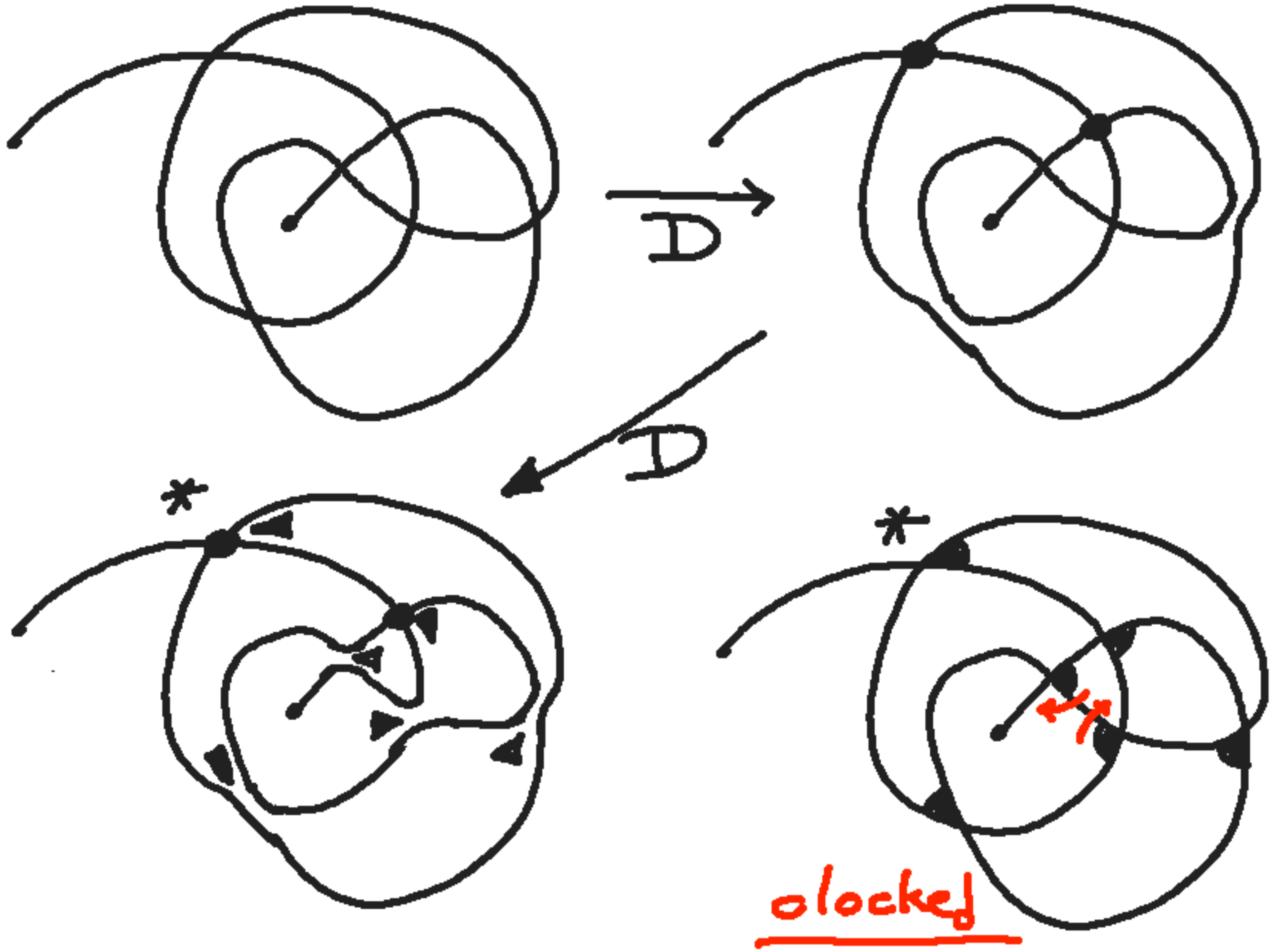
$-\bar{s}$

$$\nabla_k = s^2 + s - \bar{s}$$

The FKT <u>Clock Theorem</u> generalizes so that the states of a knotoid diagram form a lattice with unique clocked state (top) and counterclocked state (bottom).

The clocked state generates all other states via the two types of clocking moves.

The clocked state can be constructed by a "shelling process".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D
*
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blocked

As in the classical case, these path-states for knotoids can be applied to the bracket expansion:

Analogy with trees expansion for Tutte Poly.

So that with appropriate vertex wts

$$\langle K \rangle = \sum_{\alpha \in \text{Paths}} \langle K | \alpha \rangle$$

and this can be examined for Khovanov Homology (Knotoids).

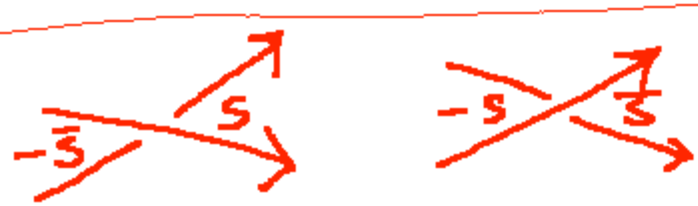
Main Theorem. a) $\nabla_K(s)$ is a knotoid invariant.

b) $\nabla_{K^*}(s) = \nabla_K(\overline{s})$

c) $\nabla_{\overleftarrow{K}}(s) = \nabla_{\overrightarrow{K}}(-\overline{s})$

K^* = Mirror Image of K

$\overrightarrow{K}, \overleftarrow{K}$ Two orientations of K



K

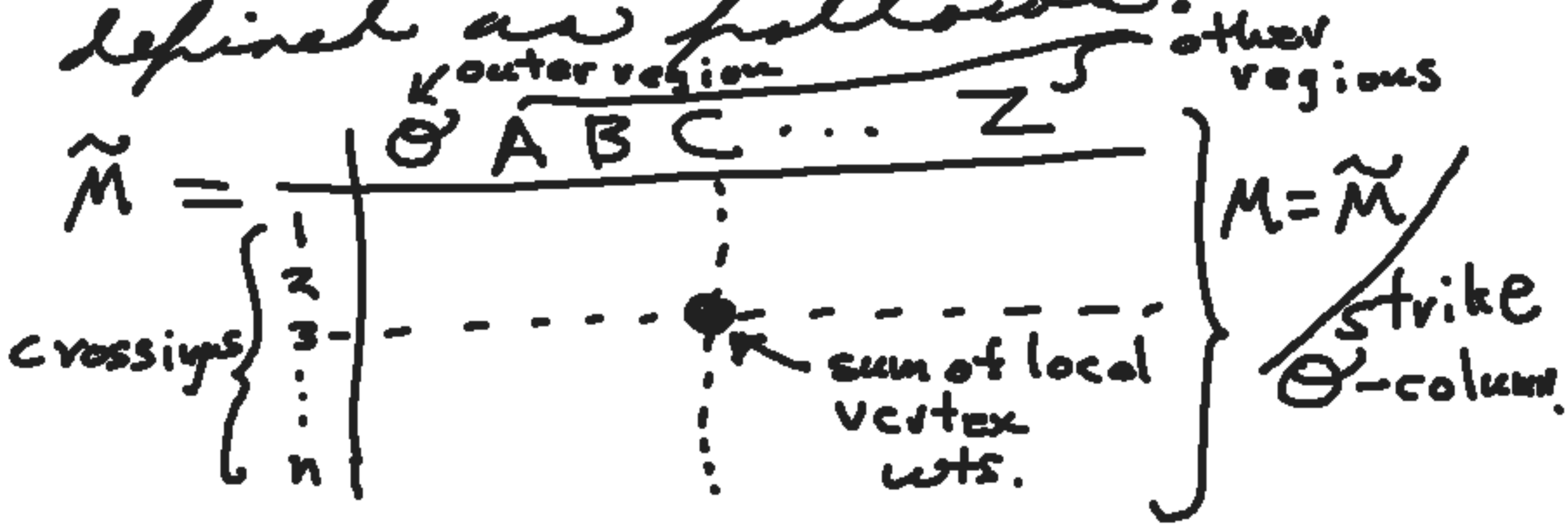
$$\boxed{\nabla_K(s) = s^2 + s - \overline{s}}$$

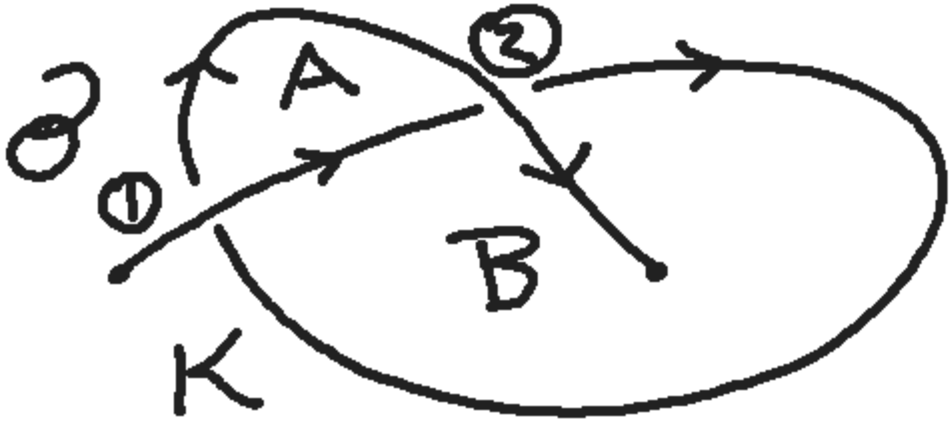
Here we have $K \not\simeq K^*$ and $\overrightarrow{K} \not\simeq \overleftarrow{K}$.

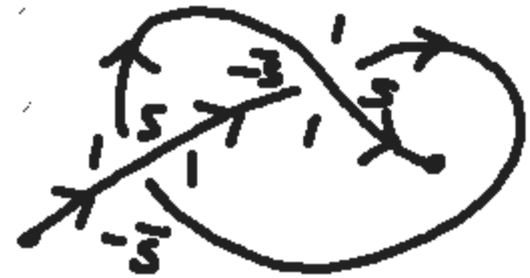
In the classical case, the Conway potential function is equivalent to a <u>determinant</u>. In the knotoid generalization

$$\nabla_K(s) = \mathrm{Permanent}(M(K))$$

where the matrix $M(K)$ is defined as follows:

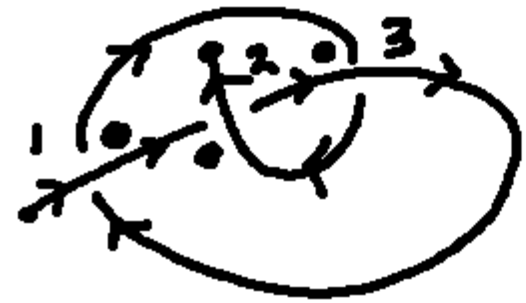


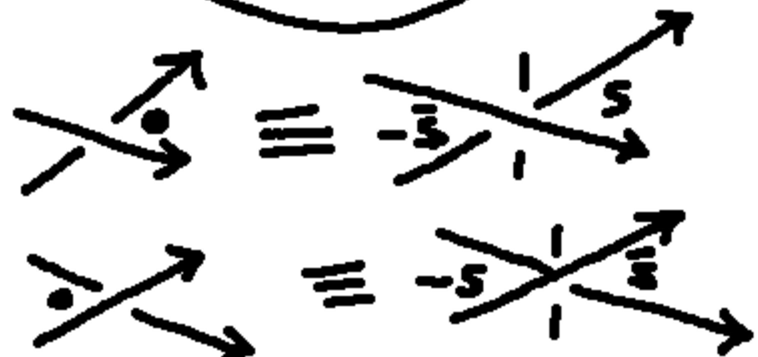




$$\tilde{M} = \begin{array}{c|ccc} & \partial & A & B \\ \hline 1 & 1-\bar{s} & s & 1 \\ 2 & 1 & -\bar{s} & 1+s \end{array}, \qquad M = \underbrace{\begin{pmatrix} s & 1 \\ -\bar{s} & 1+s \end{pmatrix}}_{M}$$

$$\nabla_K(s) = \mathrm{Perm}(M) = s(1+s) - \bar{s} = s^2 + s - \bar{s}.$$





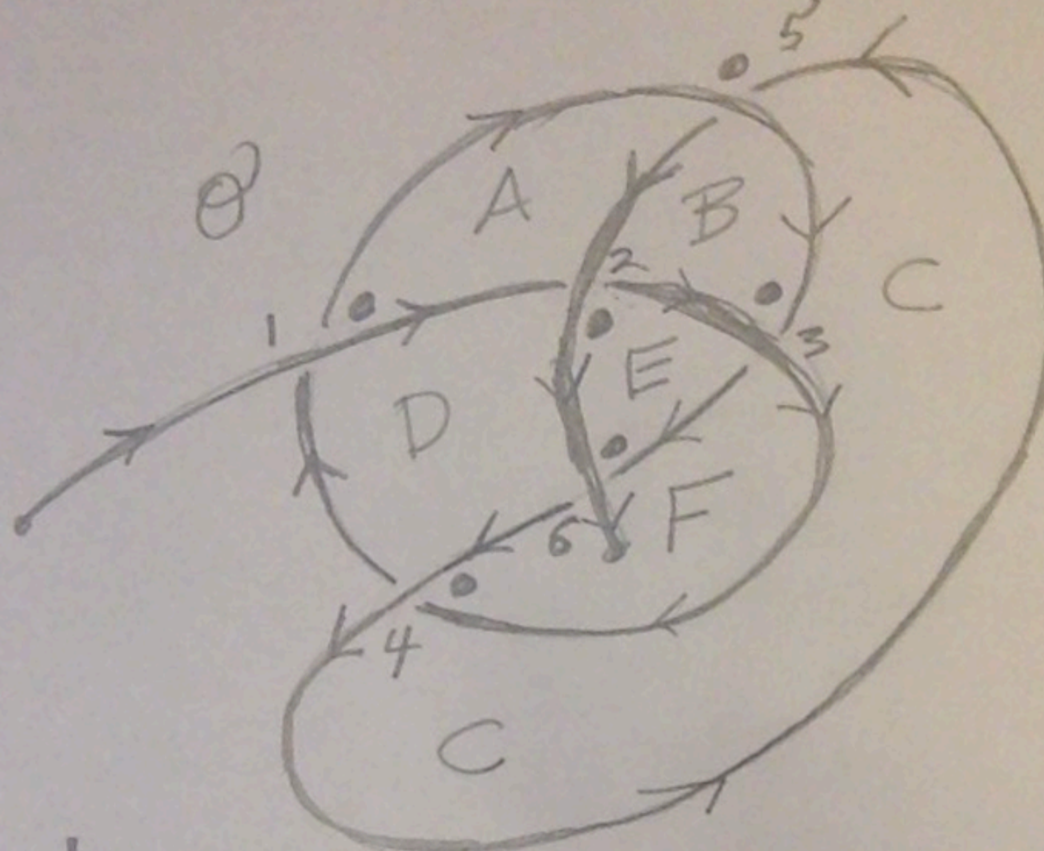
$$\begin{array}{c|c|ccc} & O & A & B & C \\ \hline 1 & 1-\bar{s} & & & \\ 2 & 0 & & \left(\begin{matrix} s & 0 & 1 \\ 1+\bar{s} & 1 & -s \\ -s & 1 & \bar{s} \end{matrix}\right) & \\ 3 & 1 & & & \end{array} = M$$

$$\nabla_k(s) = \mathrm{Perm}(M)$$

In[3]:=

```
MM = {{s, 0, 1}, {1 + 1 / s, 1, -s}, {-s, 1, 1 / s}};
Expand[Simplify[Expand[Permanent[MM]]]]
```

Out[4]= $2 + \frac{1}{s} - s - s^2$



	Ø	A	B	C	D	E	F
1	$1-\bar{s}$	s	0	0	1	0	0
2	0	$-\bar{s}$	1	0	1	s	0
3	0	0	$-s$	1	0	1	$\bar{s}$
4	$\bar{s}$	0	0	1	1	0	$-s$
5	$-s$	1	$\bar{s}$	1	0	0	0
6	0	0	0	0	1	$-s$	$\bar{s}+1$

```
MMM = {{s, 0, 0, 1, 0, 0}, {-1/s, 1, 0, 1, s, 0},
    {0, -s, 1, 0, 1, 1/s}, {0, 0, 1, 1, 0, -s},
    {1, 1/s, 1, 0, 0, 0}, {0, 0, 0, 1, -s, +1/s}};
MatrixForm[MMM]
Expand[Simplify[Expand[Permanent[MMM]]]]
```

atrixForm=

$$\begin{pmatrix} s & 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{s} & 1 & 0 & 1 & s & 0 \\ 0 & -s & 1 & 0 & 1 & \frac{1}{s} \\ 0 & 0 & 1 & 1 & 0 & -s \\ 1 & \frac{1}{s} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -s & \frac{1}{s} \end{pmatrix}$$

)]= $$-\frac{1}{s^3}+\frac{1}{s^2}+\frac{2}{s}-2s$$

This work is new and there are a number of questions. We expect

a) ∇_K is NP hard for knotoids. The transition from det to perm is serious.

b) The path state structure can be used to define a Knotoid Floer Homology with the states generating the chain complex.

Thank you for your attention!