Ternary self-distributive operations and quantum invariants of knots

Emanuele Zappala

Yale University

December 4, 2021

General information

- Article: Quantum invariants of framed links from ternary self-distributive cohomology arXiv:2102.10776, to appear in Osaka J. Math.
- This research was funded by the Estonian Research Council (grant: MOBJD679), while I was at the University of Tartu.

In a nutshell

- The cocycle invariant introduced by Carter, Jelsovsky, Kamada, Langford and Saito admits a ternary generalization that uses ternary cohomology.
- A ribbon category can be constructed from ternary structures, twisted by cohomology. This gives a “quantum” version of the cocycle invariant.
- This paradigm generalizes to symmetric monoidal categories, where now we have self-distributive objects.
- There are several examples from Hopf algebras and Lie algebras.

Recall quandles

Definition

A quandle is a set X togehter with a binary operation $* : X \times X \longrightarrow X$ satisfying the following three axioms

- $x * x = x$, for all $x \in X$,
- the right multiplicaiton map $- * x : X \longrightarrow X$ is a bijection for all $x \in X$, where $-$ is a placeholder,
- $(x * y) * z = (x * z) * (y * z)$, for all $x, y, z \in X$.

Recall quandles

Definition

A quandle is a set X togehter with a binary operation $* : X \times X \longrightarrow X$ satisfying the following three axioms

- $x * x = x$, for all $x \in X$,
- the right multiplicaiton map $- * x : X \longrightarrow X$ is a bijection for all $x \in X$, where $-$ is a placeholder,
- $(x * y) * z = (x * z) * (y * z)$, for all $x, y, z \in X$.

Remark

The three axioms in the definition of quandle correspond to Reidmeister moves of type I, II and III.

Examples of quandles

- Any group G with operation given by conjugation: $x * y = y^{-1}xy$.

Examples of quandles

- Any group G with operation given by conjugation: $x * y = y^{-1}xy$.
- $\mathbb{Z}/n\mathbb{Z}$ with operation given by $x * y = 2y - x$.

Examples of quandles

- Any group G with operation given by conjugation: $x * y = y^{-1}xy$.
- $\mathbb{Z}/n\mathbb{Z}$ with operation given by $x * y = 2y - x$.
- Any $\Lambda(= \mathbb{Z}[t, t^{-1}])$-module M is a quandle with $a * b := ta + (1 - t)b$, for $a, b \in M$, and is called an *Alexander quandle*.

Examples of quandles

- Any group G with operation given by conjugation: $x * y = y^{-1}xy$.
- $\mathbb{Z}/n\mathbb{Z}$ with operation given by $x * y = 2y - x$.
- Any $\Lambda(= \mathbb{Z}[t, t^{-1}])$-module M is a quandle with $a * b := ta + (1 - t)b$, for $a, b \in M$, and is called an *Alexander quandle*.
- Given a group G and an automorpism $f \in \mathrm{Aut}(G)$, it is easy to show that $x * y := f(xy^{-1})y$ defines a quandle structure. This is called a *generalized Alexander quandle*.

Ternary racks/quandles (TSD)

- A set X together with a ternary operation $T : X \times X \times X \longrightarrow X$ satisfying the properties:
  - $T(T(x, y, z), u, v) = T(T(x, u, v), T(y, u, v), T(z, u, v))$ for all $x, y, z, u, v \in X$.
  - The map $T(-, y, z) : X \longrightarrow X$ is a bijection for all $y, z \in X$.
  - $T(x, x, x) = x$ for all $x \in X$. .

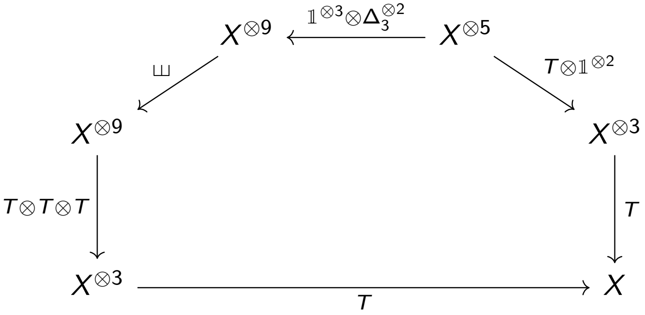
Ternary racks/quandles (TSD)

- A set X together with a ternary operation $T : X \times X \times X \longrightarrow X$ satisfying the properties:
  - $T(T(x,y,z),u,v) = T(T(x,u,v),T(y,u,v),T(z,u,v))$ for all $x,y,z,u,v \in X$.
  - The map $T(-,y,z) : X \longrightarrow X$ is a bijection for all $y,z \in X$.
  - $T(x,x,x) = x$ for all $x \in X$. .
- Examples:
  - Iteration of binary self-distriutive operation: $T(x,y,z) = (x * y) * z$.
  - Heap of a group: $T(x,y,z) = xy^{-1}z$.

Categorical TSD

In a symmetric monoidal category:

- Comonoid object (X, Δ);
- Morphism $T : X \otimes X \otimes X \longrightarrow X$ such that



Examples

- Linearize any set-theoretic TSD operation given above, with $\Delta(x) = x \otimes x$.

Examples

- Linearize any set-theoretic TSD operation given above, with $\Delta(x) = x \otimes x$.
- Quantum heap: Involutory Hopf algebra H with operation $x \otimes y \otimes z \longrightarrow xS(y)z$.

Examples

- Linearize any set-theoretic TSD operation given above, with $\Delta(x) = x \otimes x$.
- Quantum heap: Involutory Hopf algebra H with operation $x \otimes y \otimes z \longrightarrow xS(y)z$.
- Actually, any involutory Hopf monoid with same operation as above.

Examples

- Linearize any set-theoretic TSD operation given above, with $\Delta(x) = x \otimes x$.
- Quantum heap: Involutory Hopf algebra H with operation $x \otimes y \otimes z \longrightarrow xS(y)z$.
- Actually, any involutory Hopf monoid with same operation as above.
- Lie lagebra $\mathfrak{g}$. Define $X = \mathbb{C} \oplus \mathfrak{g}$, TSD operation

$$T(a,x)\otimes(b,y)\otimes(c,z) = (abc, bcx+b[x,z]+c[x,y]+[[x,y],z].),$$

and $\Delta(a,x) = (a,x) \otimes (1,0) + (1,0) \otimes (0,x)$.

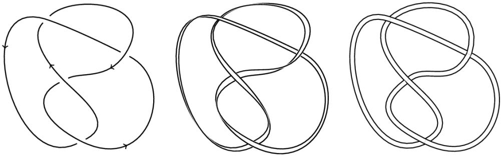
Recall some cohomology

- Define $C_n(X)$ to be the free abelian group generated by $(2n+1)$-tuples $(x_0, x_1, \cdots, x_{2n})$ of elements of a ternary rack X.
- Define differentials $\partial_n C_n(X) \longrightarrow C_{n-1}(X)$ as:

$$\begin{aligned}\partial_n(x_0, x_1, \cdots, x_{2n}) \\ = \sum_{i=1}^{2n-1} (-1)^i [(x_1, \cdots, \hat{x}_i, \hat{x}_{i+1}, \cdots, x_n) \\ -(T(x_0, x_i, x_{i+1}), \cdots, T(x_{i-1}, x_i, x_{i+1}), \hat{x}_i, \hat{x}_{i+1}, \cdots, x_n)].\end{aligned}$$

- Dualize to get cohomology.

Recall (Framed) Knot Diagrams:



(A) Knot Diagram (B) Framed Knot (C) Blackboard Framing

Figure: Taken from Even-Zohar, Chaim. The writhe of permutations and random framed knots. Random Struct. Algorithms 51 (2017): 121-142.

- Define colorings of framed diagrams.
- Define Boltzmann weights using diagrammatic interpretation of ternary quandles.

Theorem

The Boltzmann sum

$$\Theta(\mathcal{D}) = \sum_{\mathcal{C}} \prod_{\tau} \mathcal{B}(\phi, \tau, \mathcal{C})$$

is an invariant of framed links.

Quantum (linearized) version

Construct a category $\mathcal{R}_\alpha(X)$, from a ternary TSD set (X, T) , and endow it with a braiding c^α and a nontrivial twist θ^α, where α is a TSD 2-cocycle: $c^\alpha x \otimes y \otimes z \otimes w =$
$\alpha(x, z, w)\alpha(y, z, w)z \otimes w \otimes T(x, z, w) \otimes T(y, z, w)$,
$\theta^\alpha x \otimes y = \alpha(x, x, y)\alpha(y, x, y)T(x, x, y) \otimes T(y, x, y)$.

Quantum (linearized) version

Construct a category $\mathcal{R}_\alpha(X)$, from a ternary TSD set (X, T) , and endow it with a braiding c^α and a nontrivial twist θ^α, where α is a TSD 2-cocycle: $c^\alpha x \otimes y \otimes z \otimes w =$
$\alpha(x, z, w)\alpha(y, z, w) z \otimes w \otimes T(x, z, w) \otimes T(y, z, w)$,
$\theta^\alpha x \otimes y = \alpha(x, x, y)\alpha(y, x, y) T(x, x, y) \otimes T(y, x, y)$.

Theorem

The category $\mathcal{R}^*_\alpha(X)$ with braiding induced by c^α and twisting morphisms induced by θ^α is a ribbon category. Moreover, if $[\alpha] = [\beta]$ the two categories $\mathcal{R}^*_\alpha(X)$ and $\mathcal{R}^*_\beta(X)$ are equivalent.

Quantum (linearized) version

Construct a category $\mathcal{R}_\alpha(X)$, from a ternary TSD set (X, T) , and endow it with a braiding c^α and a nontrivial twist θ^α, where α is a TSD 2-cocycle: $c^\alpha x \otimes y \otimes z \otimes w =$
$\alpha(x, z, w)\alpha(y, z, w) z \otimes w \otimes T(x, z, w) \otimes T(y, z, w)$,
$\theta^\alpha x \otimes y = \alpha(x, x, y)\alpha(y, x, y) T(x, x, y) \otimes T(y, x, y)$.

Theorem

The category $\mathcal{R}^*_\alpha(X)$ with braiding induced by c^α and twisting morphisms induced by θ^α is a ribbon category. Moreover, if $[\alpha] = [\beta]$ the two categories $\mathcal{R}^*_\alpha(X)$ and $\mathcal{R}^*_\beta(X)$ are equivalent.

Remark: Here the comultiplication is the natural diagonal map.

Invariants

The previous category gives rise to an invariant of framed links, $\Psi_{\mathcal{D}}(X, T, \alpha)$, as the quantum trace of an endomorphism of $\mathcal{R}^*_\alpha(X)$, associated to a framed braid representing the framed link.

Theorem

Fix a diagram $\mathcal{D}$ of L. Then the ribbon cocycle invariant $\Theta_{\mathcal{D}}(X, T, \alpha)$ and the quantum invariant $\Psi_{\mathcal{D}}(X, T, \alpha)$ coincide.

Symmetric monoidal categories

But the examples of TSD objects in set category are just examples of TSD objects in symmetric monoidal categories!

Symmetric monoidal categories

But the examples of TSD objects in set category are just examples of TSD objects in symmetric monoidal categories!

- Take linear symmetric monoidal categories and introduce a notion of TSD 2-cocycles.

Symmetric monoidal categories

But the examples of TSD objects in set category are just examples of TSD objects in symmetric monoidal categories!

- Take linear symmetric monoidal categories and introduce a notion of TSD 2-cocycles.
- Construct braided categories from object X and categorical 2-cocycle α.

Symmetric monoidal categories

But the examples of TSD objects in set category are just examples of TSD objects in symmetric monoidal categories!

- Take linear symmetric monoidal categories and introduce a notion of TSD 2-cocycles.
- Construct braided categories from object X and categorical 2-cocycle α.
- Get invariants when the symmetric monoidal category satisfies some "finiteness" condition.

Need: 2-cocycles

Convolution invertible morphism $\alpha : X \otimes X \otimes X \longrightarrow \mathbb{I}$ is a categorical 2-cocycles if the diagram

$$
\begin{array}{ccccc}
X^{\otimes 5} & \xrightarrow{\sqcup\!\sqcup_1 \circ (\Delta^3 \mathbb{1}^2)} & X^{\otimes 8} & \xrightarrow{\alpha\alpha \circ (\mathbb{1}^3 T \mathbb{1}^2)} & \mathbb{I}^{\otimes 2} \\
{\scriptstyle \sqcup\!\sqcup_2 \circ (\Delta \mathbb{1}^2 \Delta_2^2)} \downarrow & & & & \Vert \\
X^{\otimes 12} & \xrightarrow[\alpha\alpha \circ (\mathbb{1}^3 T^3)]{} & \mathbb{I}^{\otimes 2} & = \!=\!=\!= & \mathbb{I}
\end{array}
$$

commutes.

In modules

$$\begin{aligned}
&\alpha(x^{(1)} \otimes y^{(1)} \otimes z^{(1)}) \cdot \alpha(T(x^{(2)} \otimes y^{(2)} \otimes z^{(2)}) \otimes u \otimes v) \\
&= \alpha(x^{(1)} \otimes u^{(1)} \otimes v^{(1)}) \\
&\quad \cdot \alpha(T(x^{(2)} \otimes u^{(2)} \otimes v^{(2)}) \otimes T(y \otimes u^{(3)} \otimes v^{(3)}) \otimes \\
&\quad \otimes T(z \otimes u^{(4)} \otimes v^{(4)})).
\end{aligned}$$

Observe that if one takes a linearized TSD this coincides with linearizing the 2-cocycle condition for set-theoretic structures given before.

Examples of cat 2-cocy's

- The obvious one: In linearized TSD structure, take "usual" 2-cocycle α and compose it with a group character.

Examples of cat 2-cocy's

- The obvious one: In linearized TSD structure, take "usual" 2-cocycle α and compose it with a group character.
- A less obvious one: Take a (cocommutative) Hopf algebra H and a Hopf 2-cocycle σ. Then composing (twice) the map $\alpha(x \otimes y) := \sigma(x^{(1)} \otimes y^{(1)})\sigma^{-1}(y^{(2)} \otimes S(y^{(3)})x^{(2)}y^{(4)})$ gives a 2-cocycle.

Braiding from TSD objects

Basic assumption: We have a (cocommutative) TSD object in a (linear) symmetric monoidal category, and a categorical 2-cocycle α.

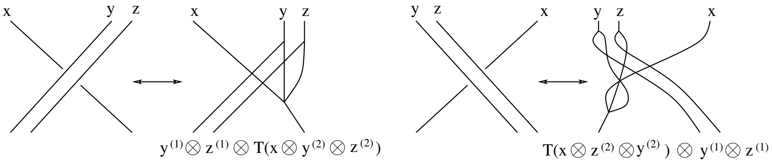
- Define: $c^{\alpha}_{2,2} = (\mathbb{1}^{\otimes 2} \otimes ([\alpha \otimes \alpha] \otimes T \otimes T)) \sqcup_c (\Delta^{\otimes 2}\Delta_4^{\otimes 2})$.
- Define: $\theta^{\alpha}_2 = ([\alpha \otimes \alpha] \otimes T \otimes T) \sqcup_\theta (\Delta_6^{\otimes 2})$.
- Then take all even powers of X, and all combinations of previous two types of morphisms.

In modules

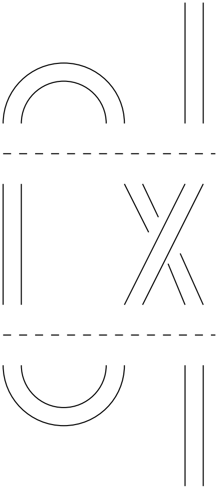
- $$\begin{aligned} & c^{\alpha}_{2,2}(x \otimes y \otimes \otimes z \otimes w) \\ & = \quad z^{(1)} \otimes w^{(1)} \otimes \\ & \quad [\alpha(x^{(1)} \otimes z^{(2)} \otimes w^{(2)}) \cdot \alpha(y^{(1)} \otimes z^{(3)} \otimes w^{(3)})] \cdot \\ & \quad T(x^{(2)} \otimes z^{(4)} \otimes w^{(4)}) \otimes T(y^{(2)} \otimes z^{(5)} \otimes w^{(5)}), \end{aligned}$$

- $$\begin{aligned} \theta^{\alpha}_2(x \otimes y) \quad = \quad & [\alpha(x^{(1)} \otimes x^{(2)} \otimes y^{(2)}) \cdot \alpha(y^{(1)} \otimes x^{(3)} \otimes y^{(3)})] \cdot \\ & T(x^{(4)} \otimes x^{(5)} \otimes y^{(5)}) \otimes T(y^{(4)} \otimes x^{(6)} \otimes y^{(6)}). \end{aligned}$$

String diagrams



String diagrams



Theorem

$\mathcal{R}^*_\alpha(X)$ is a ribbon category. Moreover, if α and β are equivalent, then $\mathcal{R}^*_\alpha(X) \cong \mathcal{R}^*_\beta(X)$ as ribbon categories.

Theorem

$\mathcal{R}^*_\alpha(X)$ is a ribbon category. Moreover, if α and β are equivalent, then $\mathcal{R}^*_\alpha(X) \cong \mathcal{R}^*_\beta(X)$ as ribbon categories.

Under finiteness conditions one naturally gets framed link invariants which give the linearized and set-theoretic versions given above, as subcases.

Lastly...

- The whole construction can be generalized to multiple classes of TSD objects with some coherence conditions.

Lastly...

- The whole construction can be generalized to multiple classes of TSD objects with some coherence conditions.
- The 2-cocycle condition becomes a compatibility condition between 2-cocycles of different TSD objects.

Lastly...

- The whole construction can be generalized to multiple classes of TSD objects with some coherence conditions.
- The 2-cocycle condition becomes a compatibility condition between 2-cocycles of different TSD objects.
- An example of this is a G-family of quandles with Nosaka's 2-cocycles. (This was used by Ishii, Iwakiri, Jand and Oshiro to get handlebody cocycle invariants)

Lastly...

- The whole construction can be generalized to multiple classes of TSD objects with some coherence conditions.
- The 2-cocycle condition becomes a compatibility condition between 2-cocycles of different TSD objects.
- An example of this is a G-family of quandles with Nosaka's 2-cocycles. (This was used by Ishii, Iwakiri, Jand and Oshiro to get handlebody cocycle invariants)
- Unfortunately, I have no examples that do not come from linearized structures.

This is the end

Thank you!