A guide to 400 level Math classes

Applied classes:

MATH 411/412: Complex functions/Contour integration

From calculus, you will be familiar with the idea of a differentiable function, and know that some functions are smoother than others. For example, the function

\[ f(x) = \begin{cases} 
  x^2 & \text{if } x \geq 0 \\
  0 & \text{if } x \leq 0 
\end{cases} \]

is differentiable everywhere, but its derivative is not differentiable at \( x = 0 \). Surprisingly, the situation for functions \( f(z) \) of a complex variable \( z \in \mathbb{C} \) is completely different: a complex function that is one time differentiable is automatically infinitely differentiable. Such functions are called holomorphic, and include many special functions like trig functions and the exponential function. This is also the right context to properly understand the identity \( e^{iz} = \cos z + i \sin z \) for a complex number \( z \).

The first part of this sequence, MATH 411, gives an introduction to complex functions and the remarkable Cauchy-Riemann equations, certain differential equations which capture what it is to be holomorphic. Holomorphic functions are also analytic, meaning that they are given locally by formal power series. This point of view leads to a revolution in the methods of calculus when you work with complex rather than real functions.

The second part of the course, MATH 412, is based around Cauchy’s Theorem and the powerful method of contour integration. This can be used to compute many difficult integrals involving real functions by embedding the real line into the complex plane—shockingly, certain seemingly impossible calculations become rather easy from the new viewpoint!

The theory of complex functions and contour integration developed in these two classes is a true gem of mathematics which every math major should see. As this an applied sequence, the emphasis is on expanding your computational toolkit rather than insisting on rigorous proofs as the theory is developed. The course is also of particular relevance to physics majors.
MATH 421M/422: Fourier analysis

Almost everything in our universe that changes over time can be described in terms of appropriate partial differential equations (PDE). These two applied classes provide a general introduction to PDE and the mathematical methods for solving them, with emphasis on applications in physics. The teaching of these classes alternates annually between the Mathematics Department and a science department (usually Physics).

MATH 421M begins by introducing classical Fourier series on the circle, Bessel functions, and Legendre functions. These tools are applied to find explicit solutions to Laplace and Poisson equations subject to appropriate boundary conditions, and to solve the wave equation over certain domains. Then MATH 422 covers more of the general theory of PDE: the Fourier transform, Green’s functions, fundamental solutions for the heat equation, the mean value theorem, and the maximum principle.

This sequence provides important background about PDE for all mathematicians, but is especially recommended if you are interested in applied mathematics relevant to physical sciences and engineering.

MATH 433: Curves and surfaces

This course, centered around curves and surfaces, is an introduction to a broad and deep branch of mathematics known as differential geometry. One of the most important concepts in the course is the notion of curvature, which leads naturally to the discovery of Riemannian geometry, and plays an essential role in Einstein’s theory of general relativity. The theory of curves and surfaces has many applications including in computer graphics, medical imagining, Computer-Aided Design (CAD), industrial design and animation.

This class begins by introducing the notion of curvature and torsion for curves, with some global results. Then we move on to surfaces, developing some local theory. We discuss principle curvatures, Gaussian curvature and mean curvature, culminating in the Gauss-Bonnet theorem which links the curvature of a surface to its underlying topology. The course ends with discussion of the calculus of variations.

The only pre-requisites for this class are MATH 281/282, MATH 342 and the bridge requirement—the class is independent of the other 400 level analysis and topology classes.

MATH 456: Networks and combinatorics

The first part of this class focuses on different ways of counting the number of elements in a set, such as the number of permutations of 1, . . . , n with no fixed points (an example where the number e actually comes into play as part of the answer). Amazingly, one can often solve such enumeration problems without listing out all the possibilities, often in quite sneaky ways. The techniques covered are useful for figuring out things like how many steps it will take for a computer program to run, or how many times you need to shuffle a deck of cards before all orderings are about equally likely.
We will also develop the important technique of generating functions, applying them to understand the sequence of Catalan numbers. These count many related things, for example, the 4th Catalan number, 14, is the number of triangulations of a regular hexagon, which appear as the vertex labels of the following remarkable 3-dimensional polyhedron:

What is the 5th Catalan number (it counts triangulations of a regular heptagon)?

In the later part of the quarter, we turn to graph theory, looking at questions such as how many ways we can color the vertices of a graph with $r$ colors so no two neighbors are the same color, and how to find a perfect matching in a graph. The theory of perfect matchings can be used to prove that if you deal a standard deck of cards into 13 piles of 4 cards each, it is always possible to select one card from each pile so that you have exactly one card of each rank. Perfect matchings are useful tools for solving scheduling problems, e.g. assigning classes to classrooms.

MATH 458: Mathematical cryptography

How to send secret messages via the internet? It turns out that you can be completely open about your enciphering scheme and, at the same time, be quite confident about its security. This class is devoted to mathematical principles behind public key cryptography. The topics include modular arithmetic, crypto systems based on discrete logarithm (Diffie-Hellman and El Gamal), the RSA cryptographic scheme and related mathematics (generation of large primes and factorization of large integers). Additional topics vary and might include quadratic residues and Goldwasser-Micali crypto system, finite field arithmetic, elliptic curves, quantum cryptography, cryptographic schemes based on lattices and more. This is a stand-alone one term class treating one of the most essential applications of pure mathematics in the modern world. The only pre-requisite is MATH 341, but if you
have taken any parts of MATH 346/347 or MATH 444/445/446 you will see plenty more connections to the mathematics there.

**MATH 497M: Deterministic dynamical modeling in biology**

This is a new applied class multilisted with BI 497M (see also MATH 499M below in the section on probability/statistics classes). It is expected to be taught for the first time in 2025/26! Please be aware that at present credit in this class cannot be counted towards the math major, although exceptions may be made in consultation with a mathematics advisor.

The course covers deterministic dynamical models in biology, i.e., mathematical models that describe the behavior of a system over time as a result of internal feedback loops and external forcings. Such models tend to be more realistic and powerful than static regression models, especially when a mechanistic understanding is sought for a biological phenomenon. The course’s focus will be on deterministic continuous-time models (differential equations) and deterministic discrete-time models (iterative maps), including stability analysis, periodicity, numerical simulations and fitting models to multivariate data. Examples will cover a broad spectrum of topics, such as neuroscience, physiology, cell biology, epidemiology, population dynamics, ecosystems, conservation and species invasion. Exercises will include mathematical calculations as well as scientific programming. As most concepts covered are rather fundamental to mathematical modeling, students from other quantitative fields such as physics or data sciences will likely also find great use in them.

**Pure classes:**

**MATH 413: Metric spaces and topology**

This class (new in 2023/24) will give an introduction to analysis in metric spaces (roughly, sets with a well-behaved notion of distance between elements) building up to the general definition of a topological space. The first half of the course will consider familiar notions like limits of sequences and continuity, and the second half will treat concepts of connectedness, completeness and compactness from point-set topology.

The powerful and versatile language introduced in this class is essential for almost all more advanced mathematics, including analysis, topology and geometry, but also more surprisingly in parts of algebra, applied math, probability theory and theoretical physics. This class is the entry point to the subsequent analysis classes MATH 414/415 and the topology class MATH 432. It is a proof-based class. The main pre-requisite is reasonable understanding of the material in MATH 316 (B- or above in that, or both MATH 316 and MATH 317). See the end of this document for a more detailed syllabus.
MATH 414/415: Functions and series/Calculus on manifolds

This course is intended as a two-term sequence in analysis building on the ideas about continuity and metric spaces introduced in MATH 413. MATH 414 starts with some reminders about differentiation and integration as seen in the 200 level calculus sequence (and probably again in MATH 316/317). Key topics like the Mean Value Theorem, Riemann integrals and Fundamental Theorem of Calculus will be reviewed and formalized. After this preparation, the main concept in the class is the notion of uniform convergence, which is the key to understanding properties of continuity, differentiability and integrability for series of functions. This then gets applied first to properly treat known special functions like exponential, logarithmic and trigonometric functions seen before, then to introduce remarkable new functions like the gamma function, which appears in the following identity:

\[ n! = \int_0^\infty x^n e^{-x} \, dx. \]

Students who have taken MATH 413 and MATH 414 are strongly encouraged to continue on with MATH 415, since this combines the language for working in higher dimensions from MATH 413 with the general principles of analysis from MATH 414 to set up the modern approach to calculus on manifolds. In more detail, MATH 415 starts by considering derivatives of functions \( f : \mathbb{R}^n \to \mathbb{R}^m \), the fundamental inverse function theorem and the implicit function theorem (with proofs). Then we develop integration of multi-variable functions, Fubini’s Theorem and the idea of a partition of unity. The last part of the course is an introduction to smooth manifolds, differential forms, and integration over manifolds. The big goal at the end of the term is the general form of Stokes’ theorem, which you can think of as the analog of the Fundamental Theorem of Calculus in higher dimensions.

MATH 432: Topics in topology

Take a basketball and let half of the air out: you have changed the geometry of the object (it is no longer spherical), but you have not changed the object’s topology. In the field of topology we pretend that all objects are made of elastic material—which can be arbitrarily bent, shrunk, or stretched—and we look for properties of objects that are invariant (unchanged) under these kinds of transformations. In the case of the sphere the property of “having a hole in the middle” is this kind of invariant, though it is challenging to capture this property with a precise mathematical definition! As another example, think about a basketball versus the inner tube of a tire: the latter has two kinds of holes, the hole where the air goes and the “doughnut hole” in the middle. Are these two kinds of holes the same or different? How do they compare to the basketball hole?

Why study topology? If one looks at the solution set to some system of equations and starts varying the coefficients somewhat, then (for the most part) the topology of the solution set doesn’t change—think about a circle \( x^2 + y^2 = 1 \) changing into an ellipse when
we change the coefficient of the $x^2$. Topology ends up being an important tool in a huge amount of modern mathematics because it gives us precise ways to understand what “stays the same” when we start varying our objects just a bit.

While MATH 413 introduces students to the basic idea of topological spaces, MATH 432 delves deeper into the subject. We continue to develop the ideas of point set topology, which are the foundational tools needed to talk about nearness, continuity, connectedness, boundedness, and ways of constructing new topological spaces from old ones. We will investigate tons of examples, both in low-dimensions (2 and 3) and in higher ones. But in addition we will also start to develop algebraic topology, which is the fascinating subject of using algebraic methods to measure things about topological spaces (like the presence of “holes”). The main topic here is the so-called fundamental group, but accessing this structure requires us to delve into the theory of covering spaces as well.

**MATH 441: Abstract linear algebra**

Linear algebra is the study of vector spaces over arbitrary fields, linear transformations between them, and bilinear forms. These notions are fundamental to literally all branches of mathematics, from the most abstract number theory to machine learning to modeling in all sorts of applied mathematics. In contrast to MATH 341/342 which are based mostly on column vectors in $\mathbb{R}^n$ and matrices, this course sets up the foundations in an elegant and flexible coordinate-free way which is the gold standard for advanced mathematics.

Topics covered include the abstract definition of a vector space, revisiting the basic ideas of bases and dimension you first saw in MATH 341 (with proofs). Then the rank-nullity theorem for linear transformations is reformulated using the language of kernels, images, and quotient vector spaces. Results about eigenvalues and eigenvectors from MATH 342 are upgraded to full classification theorems for endomorphisms—what happens when a matrix is not diagonalizable? Further topics depending on class preferences could be minimal polynomials and the Cayley-Hamilton Theorem, dual vector spaces and bilinear forms, or tensor products, exterior products and determinants.

The course is also a good one-term introduction to abstract algebra, so that it goes well along with MATH 444. Even if you are more interested in applied mathematics or in probability and statistics, MATH 441 is highly recommended—perhaps it could be your one and only pure mathematics class at the 400 level!

**MATH 444/445: Groups and rings**

These two classes, which should be regarded as a self-contained sequence, give a general introduction to the language and methods of abstract algebra. They are suitable for any math major who has completed 341/342 and the bridge requirement.

One of the most basic but universal notions in modern algebra is the idea of a group, which is used to describe symmetries of pretty much all other structures that arise in
physics, chemistry and mathematics—fundamental particles, crystals, manifolds, and finite geometries, to name but a few. This part of the course focuses mainly on finite groups, which are easy to define but unreasonably difficult to describe completely, although there have been some remarkable accomplishments such as the Classification of Finite Simple Groups completed in 1980. We will work out examples of small finite simple groups like the alternating group $A_5$ (the group of rotational symmetries of the buckminsterfullerene molecule) but can only gape in awe at larger ones such as the Monster, which is of size $2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$.

Groups only have one operation, so as well as being fascinating in their own right, they also serve as a training ground for the next, perhaps more important algebraic structure: rings. These have two operations, addition and multiplication, satisfying familiar axioms like for the integers $\mathbb{Z}$, fields including $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$, and algebras of polynomials or $n \times n$ matrices. The focus in this part of the course is mainly on commutative rings. Amongst other things, we will think about unique factorization in a general setup, putting prime numbers like $13 \in \mathbb{Z}$ and irreducible polynomials like $x^5 + 4x + 2 \in \mathbb{Q}[x]$ on an equal footing.

Although the two algebraic structures of groups and rings play quite different roles in mathematics, the language and tools used to study them have a lot in common. In fact, these techniques and the axiomatic approach you will see in this sequence is the foundation for many subsequent developments in pure mathematics.

MATH 446: Galois theory

The third term of the year-long abstract algebra sequence focuses on the algebraic structure of a field. Around 1850, Galois realized that the familiar problem of finding zeroes of polynomials (quadratics, cubics, quartics and so on) was a shadow of a much richer structure. In fact, the symmetries between the roots of a polynomial are encoded by a finite group called the Galois group of the polynomial. Then abstract results from finite group theory can be reinterpreted to shed light on the nature of these roots. This leads to Galois’ theorem proving that in general a polynomial of degree 5 or higher cannot be solved by radicals, i.e., there is no general formula unlike for quadratics, cubics and quartics. For example, the irreducible polynomial $x^5 + 4x + 2 \in \mathbb{Q}[x]$ has a real root close to $x = -\frac{1}{2}$ but there is no way to write down the value this root exactly using elementary operations. This is a truly remarkable result which has been a highlight of mathematics for over 150 years. Galois Theory is also the foundation for many other important results, such as the Classification of Finite Fields, which are the basis for algorithms in crytography.

MATH 446 is more demanding than MATH 444/445—it is reasonable just to take the first two terms and not the last—but it is well worth the effort since Galois Theory ties all of the ideas about groups and rings seen earlier in the year together in a beautiful way. MATH 441 is also recommended (although not required) as a pre-requisite for this class since vector spaces play a big role in studying field extensions.
Probability and Statistics classes:

MATH 410: Machine learning and statistics

This class is strongly interdisciplinary in nature and aimed at undergraduate and graduate students from Mathematics, Physics, Biology, Computer Science, Psychology, Economics, Data Science, but open to everybody on campus, including faculty and postdocs. The main goals of this course are for students to gain (a) proficiency with some mathematical tools and theories important in statistics and machine learning; (b) experience with skills and tools of applied work, including modeling, computational methods, programming, and translation/communication. More specifically, students completing this course should learn how to deploy modern methods in statistics and machine learning and develop familiarity with basic concepts of machine learning, statistical learning theory, and statistical inference. Students will learn to design and train neural networks using state-of-the-art software packages and algorithms. The course will also discuss best practices in applying algorithms with particular attention to developing problem-solving ability in the context of classification and regression tasks.

MATH 461/462: Probability theory/Methods of statistics

MATH 461 is a course in probability theory, which is the mathematical foundation for statistics treated in the subsequent MATH 462. These two courses are designed as a back-to-back sequence, and are particularly important for students considering a career as an actuary.

MATH 461 starts by introducing the probability axioms, using them to calculate probabilities, and some brief discussion of computing probabilities via counting. Next is the idea of conditional probability and Bayes' Theorem. Then the course introduces random variables, their distributions, and mean and variance are introduced, along with standard examples such as Binomial, Normal, Poisson, and Geometric. Independence of random variables is covered, along with the Weak Law of Large Numbers, and the Central Limit Theorem. Simulation of random variables via computer is usually also covered, and simulation is shown to be a useful tool for discovering properties of random variables when analytic methods are unavailable or too complicated.

Then MATH 462 gives a general introduction to the theory and methods of parametric statistical inference. Start from some data generated from a probability model belonging to a parametric family of distributions, the main methods of inference for the parameters underlying the model will be developed. The course focuses both on frequentist (e.g. Maximum Likelihood) methods, and Bayesian methods, which are becoming more important. We cover the classical theory of hypothesis testing, but also discuss its limitations and potential for misuse. The use of software (primarily, R, although other choices may be used depending on the instructor) is an integrated part of MATH 462.
MATH 463: Regression and variance

This is a course in applied statistics emphasizing linear models. The class covers fitting linear models (and variants such as logistic and other generalized linear models) to data and using the fitted models to address scientific questions about the data. Critical thinking about the connection between models and data will be emphasized. Computation using statistical software (R) is a core component of the course, and evaluation is based mostly on weekly problem sets.

MATH 467: Stochastic processes

Stochastic processes are mathematical models for systems which evolve probabilistically over time. This course covers the basics of stochastic processes including Markov chains, martingales, Poisson processes, Brownian motion, and their applications. These models are widely utilized in many scientific disciplines, ranging from physics, to biology, to social systems like financial markets (although this is not a course in mathematical finance!).

MATH 499M: Stochastic dynamical modeling in biology

This is a new probability/statistics class multilisted with BI 499M (see also MATH 497M above which is listed in the section on applied classes). It is expected to be taught for the first time in 2024/25. Please be aware that at present credit in this class cannot be counted towards the math major, although exceptions may be made in consultation with a mathematics advisor.

The course covers stochastic dynamical models in biology, i.e., mathematical models that describe the behavior of non-deterministic biological systems over time as a result of internal feedback loops, external forcings and random processes. Such models have become a popular alternative to deterministic models throughout the biological sciences. Topics covered will include stochastic iterative maps, discrete-time Markov chains, continuous-time Markov chains, vector autoregression models, and concepts of time series analysis. Examples will cover a broad spectrum of topics in biology, such as neuroscience, microbiology, population dynamics, genetics, molecular evolution and ecology. As the concepts covered are rather general and fundamental, students from other quantitative fields such as physics, chemistry or data sciences will likely also find great use in them.