Problem 1
Let $(X, \mathcal{B}, \mu)$ be a measure space. For any $S \subset X$, define
\[
\mu^*(S) = \inf \{ \mu(G) : S \subset G, G \in \mathcal{B} \}.
\]
Let $S_n \subset X$ for all $n \in \mathbb{N}$. Show that
\[
\mu^*(\bigcup_{n=1}^{\infty} S_n) \leq \sum_{n=1}^{\infty} \mu^*(S_n).
\]

Problem 2
Let $f$ be a Borel measurable function on $[-1, 1]$. Prove or disprove the following statement:
There exists a continuous function $g$ such that $f = g \ a.e.m$, where $m$ is the Lebesgue measure on $\mathbb{R}$.

Problem 3 Suppose that $\{f_n\}$ is a sequence of Lebesgue integrable function on $[0, 1]$ such that $|f_n(x)| \leq \pi$ for all $n$ and $x \in [0, 1]$. Suppose that
\[
\lim_{n \to \infty} f_n(x) = 0 \ a.e.m \text{ and } \left|\frac{f_n(x)}{x^2}\right| \leq x^{-1/2} \text{ for all } x \in (0, 1], \ a.e.m \text{ and } n \in \mathbb{N}.
\]
Compute the
\[
\lim_{n \to \infty} \int_{(0,1]} \frac{f_n(x)}{1 - \cos(x)} \, dm,
\]
where $m$ is the Lebesgue measure. Justify your computation.

Problem 4
Let $\mu$ be a (positive) Borel measure on $[a, b]$ such that
\[
L(f) = \int_{[a,b]} f \, d\mu \text{ for all } f \in L^2([a, b], m)
\]
gives a bounded linear functional on $L^2([a, b], m)$, where $m$ is the Lebesgue measure. Show that $\mu$ has the following property: for any $\epsilon > 0$, there exists $\delta > 0$ such that, if $E \subset [a, b]$ is a Borel measurable set with $0 < m(E) < \delta$, then $\mu(E) < \epsilon$.

Problem 5
Let $D$ be the open unit disk of the plane. Show that there is a bounded linear functional $F$ on $L^2(D, m)$, where $m$ is the Lebesgue (area) measure on $D$, such that
\[
F(z^k) = (1/2)^k \text{ for all } k = 1, 2, \ldots, 2023.
\]

Problem 6
Let $X$ and $Y$ be Banach spaces and $\varphi : X \times Y \to \mathbb{C}$ be a function. Suppose that, for each fixed $y \in Y$, $\varphi_y : X \to \mathbb{C}$ defined by $\varphi_y(x) = \varphi(x, y)$ is a continuous linear functional, and for each $x \in X$, $\varphi_x : Y \to \mathbb{C}$ defined by $\varphi_x(y) = \varphi(x, y)$ is also a continuous linear functional. Prove that there exists $C > 0$ such that

$$|\varphi(x, y)| \leq C\|x\|\|y\|$$

for all $x \in X, y \in Y$.

**Problem 7**

Let $D$ be the open unit disk of the plane and $f$ be a continuous function on the closure of $D$ which is analytic in $D$. Suppose that there is a constant $k$ such that $|f(z)| = k$ for all $z \in \partial D$. Prove that either $f$ has a zero in $D$ or $f$ is a constant.

**Problem 8**

Let $D$ be the open unit disk and $\bar{D}$ be the closure. Let $A$ be the set of continuous functions on $\bar{D}$ which are analytic on $D$. It is known that $A$ is an algebra over $\mathbb{C}$. Put

$$\|f\| = \sup\{|f(z)| : z \in D\}.$$

Then it is a norm on $A$. Show that $A$ with this norm is a Banach space.

**Problem 9**

Compute

$$\int_{|z|=1} z^2 \sin(1/z)dz.$$