Problem 1.  (i) Define a degree of a map $f : M \to N$ between oriented manifolds of the same dimension.

(ii) Let $f : \mathbb{H}P^n \to \mathbb{H}P^n$ be a map of degree 128. Determine $n$.

Solution. (i) Let $\dim M = \dim N = k$. Then a map $f : M \to N$ induces a homomorphism

$$f_* : H_k(M; \mathbb{Z}) \to H_k(N; \mathbb{Z})$$

Since the groups $H_k(M; \mathbb{Z}) \cong \mathbb{Z}$, $H_k(N; \mathbb{Z}) \cong \mathbb{Z}$ (the manifolds $M$ and $N$ are oriented). Then the homomorphism $f_* : \mathbb{Z} \to \mathbb{Z}$ is given as $1 \to \lambda$. Then $\deg f = \lambda$.

(ii) Let $f : \mathbb{H}P^n \to \mathbb{H}P^n$ be a map of degree 128. We have homomorphisms

$$f_* : H_4(\mathbb{H}P^n; \mathbb{Z}) \to H_4(\mathbb{H}P^n; \mathbb{Z}), \quad f^* : H^*(\mathbb{H}P^n; \mathbb{Z}) \to H^*(\mathbb{H}P^n; \mathbb{Z}).$$

Since $H^*(\mathbb{H}P^n; \mathbb{Z}) \cong \mathbb{Z}[x]/x^{n+1}$, where $x \in H^4(\mathbb{H}P^n; \mathbb{Z})$ is a generator, we have that $f^*(x) = \alpha x$ for some integer $\alpha$. Then $f^*(x^n) = \alpha^n x^n$. We also have the isomorphism

$$H^{4n}(\mathbb{H}P^n; \mathbb{Z}) \cong \text{Hom}(H_{4n}(\mathbb{H}P^n; \mathbb{Z}), \mathbb{Z}).$$

It follows from naturality that

$$f_* : H_{4n}(\mathbb{H}P^n; \mathbb{Z}) \to H_{4n}(\mathbb{H}P^n; \mathbb{Z})$$

is also multiplication by $\alpha^n$. By assumptions, we have that $\alpha^n = 128 = 2^7$. We obtain that $n = 7$.

Problem 2. (i) Prove that $S^{2n} \times S^{2n}$ is not homotopy equivalent to $S^{2n} \lor S^{2n} \lor S^{4n}$.

(ii) Prove that $\Sigma(S^{2n} \times S^{2n})$ is homotopy equivalent to $\Sigma(S^{2n} \lor S^{2n} \lor S^{4n})$.

Solution. (i) We have that

$$H^*(S^{2n} \times S^{2n}; \mathbb{Z}) = \mathbb{Z}[x, y]/(x^2, y^2), \quad \deg x = 2n, \quad \deg y = 2n.$$

$$H^*(S^{2n} \lor S^{2n} \lor S^{4n}; \mathbb{Z}) = \mathbb{Z}[x, y, z]/(x^2, y^2, z^2, xy, xz, yz), \quad \deg x = 2n, \quad \deg y = 2n, \quad \deg z = 4n.$$

Thus the cohomology rings $R_1 := H^*(S^{2n} \lor S^{2n} \lor S^{4n}; \mathbb{Z})$ and $R_2 := H^*(S^{2n} \lor S^{2n} \lor S^{4n}; \mathbb{Z})$ are not isomorphic. Indeed, assume the rings $R_1$ and $R_2$ are isomorphic and $\varphi : R_1 \to R_2$. Let $x, y \in R_1$ be generators. Then

$$x \mapsto ax + by, \quad y \mapsto cx + dy,$$

where the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has determinant $\pm 1$. Then we have

$$0 \neq xy \mapsto (ax + by)(cx + dy) = accx^2 + (ad + be)xy + y^2 = 0 \quad \text{(in } R_2)$$

Thus $R_1$ has to be mapped to zero in $R_2$. Contradiction. This implies that $S^{2n} \times S^{2n}$ is not homotopy equivalent to $S^{2n} \lor S^{2n} \lor S^{4n}$.

(ii) Recall that $S^{2n} \times S^{2n} = (S^{2n} \lor S^{2n}) \lor w D^{4n}$, where $w : S^{4n-1} \to S^{2n} \times S^{2n}$ is the Whitehead map. Since $\Sigma(w) : \Sigma(S^{4n-1}) \to \Sigma(S^{2n} \times S^{2n})$ is null-homotopic, we obtain that

$$\Sigma(S^{2n} \times S^{2n}) \cong \Sigma(S^{2n} \lor S^{2n} \lor w D^{4n}) \cong \Sigma(S^{2n}) \lor \Sigma(S^{2n}) \lor \Sigma(w) \Sigma(D^{4n})$$

$$\cong S^{2n+1} \lor S^{2n+1} \lor S^{4n+1}.$$  

Clearly, $\Sigma(S^{2n} \lor S^{2n} \lor S^{4n}) \cong S^{2n+1} \lor S^{2n+1} \lor S^{4n+1}$.  

□
Problem 3. Prove that $\mathbb{RP}^n \times S^k$ is homotopy equivalent to $S^n \times \mathbb{RP}^k$ if and only if $n = k$.

Solution. First, if $k = n$, then $\mathbb{RP}^n \times S^k$ is homeomorphic to $S^n \times \mathbb{RP}^k$. Now assume $\mathbb{RP}^n \times S^k$ is homotopy equivalent to $S^n \times \mathbb{RP}^k$. We compare the cohomology rings

$$H^*(\mathbb{RP}^n \times S^k; \mathbb{Z}_2) \text{ and } H^*(S^n \times \mathbb{RP}^k; \mathbb{Z}_2).$$

Recall that

$$H^*(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/x^{n+1}, \quad H^*(\mathbb{RP}^k; \mathbb{Z}_2) \cong \mathbb{Z}_2[u]/u^{k+1}.$$ 

Denote by $y \in H^k(S^k; \mathbb{Z}_2)$, $w \in H^n(S^n; \mathbb{Z}_2)$ generators. Then we have the isomorphisms:

$$H^*(\mathbb{RP}^n \times S^k; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y]/(x^{n+1}, y^2),$$

$$H^*(S^n \times \mathbb{RP}^k; \mathbb{Z}_2) \cong \mathbb{Z}_2[w, u]/(u^{k+1}, w^2).$$

Consider two cases: (1) $n > k \geq 2$; (2) $n > k = 1$.

(1) Assume $\mathbb{RP}^n \times S^k$ and $S^n \times \mathbb{RP}^k$ are homotopy equivalent and $f : \mathbb{RP}^n \times S^k \to S^n \times \mathbb{RP}^k$ is a homotopy equivalence. Then $f$ induces isomorphism

$$f^* : H^*(S^n \times \mathbb{RP}^k; \mathbb{Z}_2) \to H^*(\mathbb{RP}^n \times S^k; \mathbb{Z}_2).$$

Consider $H^1(S^n \times \mathbb{RP}^k; \mathbb{Z}_2) = \mathbb{Z}_2$ generated by $1 \otimes u$. Then we have that $f^*(1 \otimes u) = (x \otimes 1)$. However, we have that

$$0 = f^*((1 \otimes u)^{k+1}) = (x \otimes 1)^{k+1} = x^{k+1} \otimes 1 \neq 0$$

since $n > k$. Contradiction.

(2) Again, assume $\mathbb{RP}^n \times S^1$ and $S^n \times \mathbb{RP}^1$ are homotopy equivalent. However $\pi_1(\mathbb{RP}^n \times S^1) = \mathbb{Z}_2 \oplus \mathbb{Z}$ and $\pi_1(S^n \times \mathbb{RP}^1) = \mathbb{Z}$. Contradiction. \qed

Problem 4. Compute homotopy groups $\pi_q(\mathbb{RP}^2 \vee S^2)$ for $q = 1, 2, 3$.

Solution. We consider a covering $S^2 \vee (S^2 \vee S^2) \to \mathbb{RP}^2 \vee S^2$. Then we have that $\pi_1(\mathbb{RP}^2 \vee S^2) = \mathbb{Z}_2$. Then we have:

$$\pi_2(\mathbb{RP}^2 \vee S^2) = \pi_2(S^2 \vee S^2 \vee S^2) = \mathbb{Z}^3,$$

$$\pi_3(\mathbb{RP}^2 \vee S^2) = \pi_3(S^2 \vee S^2 \vee S^3) = \mathbb{Z}^1 \oplus \mathbb{Z}^3$$

Here we have $\pi_3S^2_i = \mathbb{Z}$, $i = 1, 2, 3$, and for each pair $i < j$, we have the Whitehead map

$$w_{ij} : S^3 \to S^2 \vee S^2 \text{ generating the group } \pi_3(S^2_i \vee S^2_j) = \mathbb{Z}.$$ 

Recall that $w_{ij}$ is the attaching map of the 4-cell $e^4$, where $S^2_i \times S^2_j = (S^2_i \vee S^2_j) \cup w_{ij} D^4$. \qed

Problem 5. (i) Define the Homotopy Lifting Property.

(ii) Let $n \geq 1$. Consider the map

$$g : S^{4n-2} \times S^5 \xrightarrow{\text{proj}} (S^{4n-2} \times S^5)/(S^{4n-2} \vee S^5) \xrightarrow{\text{Hof}} \mathbb{HP}^n.$$ 

Prove that $g$ induces trivial homomorphism in homology and homotopy groups. Prove, however, that $g$ is not homotopic to a constant map.
Solution. (i) We say that a homotopy lifting property holds for a map \( p : E \to B \), if for any CW-complex \( Z \) and commutative diagram

\[
\begin{array}{ccc}
E & \xrightarrow{p} & B \\
\downarrow{g} & & \downarrow{g} \\
Z & \xrightarrow{\tilde{g}} & B
\end{array}
\]

and a homotopy \( G : Z \times I \to B \), such that \( G|_{Z \times \{0\}} = g \) there exists a homotopy \( \tilde{G} : Z \times I \to E \) such that \( \tilde{G}|_{Z \times \{0\}} = \tilde{g} \) and the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{p} & B \\
\downarrow{\tilde{g}} & & \downarrow{G} \\
Z \times I & \xrightarrow{\tilde{G}} & B
\end{array}
\]

commutes.

(ii) We examine the homomorphism \( g_* \) in homotopy:

\[
g_* : \pi_q(S^{4n-2} \times S^5) \xrightarrow{p_*} \pi_q(S^{4n+3}) \xrightarrow{H_*} \pi_q(\mathbb{H}P^n).
\]

We notice that \( \pi_q(S^{4n-2} \times S^5) = \pi_q(S^{4n-2}) \oplus \pi_q(S^5) \). Maps \( S^{4n-2} \to S^{4n+3} \) and \( S^5 \to S^{4n+3} \) are null-homotopic, so the homomorphism \( \pi_q(S^{4n-2} \times S^5) \xrightarrow{p_*} \pi_q(S^{4n+3}) \) is trivial, thus the composition

\[
g_* : \pi_q(S^{4n-2} \times S^5) \xrightarrow{p_*} \pi_q(S^{4n+3}) \xrightarrow{H_*} \pi_q(\mathbb{H}P^n)
\]

is trivial as well. We examine the homomorphism

\[
g_* : H_q(S^{4n-2} \times S^5) \xrightarrow{p_*} H_q(S^{4n+3}) \xrightarrow{H_*} H_q(\mathbb{H}P^n), \quad q > 0.
\]

We notice that the homomorphism \( H_* : H_q(S^{4n+3}) \to H_q(\mathbb{H}P^n) \) is trivial for \( q > 0 \). Indeed, \( H_{4n+3}(S^{4n+3}) = \mathbb{Z} \) and \( H_{4n+3}(\mathbb{H}P^n) = 0 \). Thus \( g_* : H_q(S^{4n-2} \times S^5) \to H_q(\mathbb{H}P^n) \) is trivial if \( q > 0 \).

Now assume that \( g : S^{4n-2} \times S^5 \to \mathbb{H}P^n \) is null-homotopic. Let \( G : (S^{4n-2} \times S^5) \times I \to \mathbb{H}P^n \) is a null-homotopy. We have the following diagram:

\[
\begin{array}{ccc}
(S^{4n-2} \times S^5) \times \{1\} & \xrightarrow{\tilde{G}} & S^{4n+3} \\
\downarrow{t_1} & & \downarrow{H} \\
(S^{4n-2} \times S^5) \times I & \xrightarrow{G} & \mathbb{H}P^n \\
\downarrow{t_0} & & \downarrow{g} \\
(S^{4n-2} \times S^5) \times \{0\}
\end{array}
\]

Here the map \( c : (S^{4n-2} \times S^5) \times \{1\} \to \mathbb{H}P^n \) is constant. We obtain a commutative diagram:

\[
\begin{array}{ccc}
(S^{4n-2} \times S^5) \times \{1\} & \xrightarrow{\tilde{c}} & S^{4n+3} \\
\downarrow{t_1} & & \downarrow{H} \\
(S^{4n-2} \times S^5) \times \{1\} & \xrightarrow{c} & S^{4n+3} \\
\downarrow{t_0} & & \downarrow{g} \\
(S^{4n-2} \times S^5) \times \{0\}
\end{array}
\]
In particular, we obtain a commutative diagram in integral homology:

\[
\begin{array}{ccc}
Z = H_{4n+3}(S^{4n-2}) \times (1) & \xrightarrow{e_*} & H_{4n+3}(S^{4n+3}) = Z \\
\end{array}
\]

\[\begin{array}{ccc}
0 = H_{4n+3}(S^3) & \xrightarrow{i_*} & \text{Contradiction.}
\end{array}\]

**Problem 6.** Let \( K, L \subset \mathbb{R}^p \) be two compact finite simplicial complexes, \( K \cap L = \emptyset \). We say that \( K, L \) are not linked if there exist a hyperplane \( \Pi \subset \mathbb{R}^p \), and an isotopy \( F_t : \mathbb{R}^p \to \mathbb{R}^p \), so that \( F_0 = Id \), \( F_t(K) \cap F_t(L) = \emptyset \) for each \( t \in [0, 1] \), and the sets \( F_1(K) \) and \( F_1(L) \) are separated by the hyperplane \( \Pi \).

Assume \( K, L \subset \mathbb{R}^p \) are two disjoint finite simplicial complexes of dimensions \( k, \ell \) respectively. Suppose \( k + \ell + 1 < p \). Prove that the complexes \( K \) and \( L \) are not linked.

**Solution.** First let \( \Pi \subset \mathbb{R}^p \) be a hyperplane such that \( K \cap \Pi = \emptyset \), and \( L \cap \Pi = \emptyset \). If \( K \) and \( L \) are in the different half-spaces, then we are done. Let \( K \) and \( L \) be in \( \mathbb{R}^p_+ \). We want to produce an isotopy \( F_t : \mathbb{R}^p \to \mathbb{R}^p \) such that \( F_0 = Id_{\mathbb{R}^p} \) and \( F_t(K) \) and \( F_t(L) \) are separated by the hyperplane \( \Pi \). We need the following statement.

**Lemma.** There exists a point \( x_0 \in \mathbb{R}^p_+ \) such that any line going through \( x_0 \) does not intersect both \( K \) and \( L \).

**Proof of Lemma.** Let \( W_1, \ldots, W_\nu \subset \mathbb{R}^p \) be planes (of minimal dimensions) containing the simplices \( \Delta_1, \ldots, \Delta_\nu \) of the simplicial complex \( K \), and let \( U_1, \ldots, U_\mu \subset \mathbb{R}^p \) be the corresponding planes containing the simplices of \( L \). Notice that \( \dim W_i \leq k \) and \( \dim U_j \leq \ell \), \( i = 1, \ldots, \nu \), \( j = 1, \ldots, \mu \). Let \( \Pi_{ij} \) be a minimal plane containing \( W_i \) and \( U_j \). Notice that the maximal dimension of \( \Pi_{ij} \) is \( k + \ell + 1 \). Indeed, let \( w \in W_i \), \( u \in U_j \) be any points. Then a basis of \( W_i \), a basis of \( U_j \), and the vector \( w - u \) generate \( \Pi_{ij} \), see Figure below:

Since \( k + \ell + 1 < p \), there exists a point \( x_0 \) of \( \mathbb{R}^p_+ \), such that \( x_0 \notin \cup \Pi_{ij} \).

Now we continue with Solution. The isotopy \( F_t \) may be costructed as follows. Consider the space of all lines going through the point \( x_0 \in \mathbb{R}^p \). This is the projective space \( \mathbb{RP}^{p-1} \). Choose a continuous nonnegative function \( \varphi : \mathbb{RP}^{p-1} \to \mathbb{R} \) such that \( \varphi(\lambda) = 0 \) if \( \lambda \cap L \neq \emptyset \), and \( \varphi(\lambda) = v_0 > 0 \) if \( \lambda \cap K \neq \emptyset \). Now the isotopy \( F_t : \mathbb{R}^p \to \mathbb{R}^p \) moves a point \( x \in \mathbb{R}^p \) along the line \( \lambda \) (connecting \( x \) and \( x_0 \)) toward \( x_0 \) with the velocity \( \varphi(\lambda) \), where \( \varphi \) is as above.
By assumption, the operation $\theta$ isomorphism. There is a operation $\theta$ in the computation.

The only difference between reduced and nonreduced homology is a $X$. Clearly at some moment the image of $p$ if $a$ in the case $X$. Let $X$. Let $X$.

Solution. Assume $K(Z, n)$ is a finite CW-complex of dimension $k$. Consider two cases: (1) $n = 2m$, (2) $n = 2m + 1$.

(1) $n = 2m$. We choose $q$ such that $mq > k$. Consider $\mathbb{C}P^m$; we have

$$H^*(\mathbb{C}P^m; \mathbb{Z}) = \mathbb{Z}[x]/x^{m+1}, \quad \text{deg} \ x = 2.$$  

There is an operation $\theta$ in $O(\mathbb{Z}, 2m, 2m + 2q)$ given by $\theta(z) = z^{2q}$ for $z \in H^{2m}(X; \mathbb{Z})$. Since we have an isomorphism $O(\mathbb{Z}, 2m, 2m + 2q) \cong H^{2m + 2q}(K(\mathbb{Z}, 2m); \mathbb{Z})$.

By assumption, the operation $\theta$ is trivial, however, there is a non-trivial operation $x^m \to x^{mq}$. Contradiction.

(2) $n = 2m + 1$. The argument is the same, but we use $\mathbb{C}P^m \times S^1$ instead $\mathbb{C}P^m$.

Problem 7. Prove that, if a CW-complex $X$ is an Eilenberg MacLane space of type $K(\mathbb{Z}, n)$ for $n \geq 2$, then $X$ has cells of arbitrarily high dimension.

Solution. Assume $K(\mathbb{Z}, n)$ is a finite CW-complex of dimension $k$. Consider two cases: (1) $n = 2m$, (2) $n = 2m + 1$.

(1) $n = 2m$. We choose $q$ such that $mq > k$. Consider $\mathbb{C}P^m$; we have

$$H^*(\mathbb{C}P^m; \mathbb{Z}) = \mathbb{Z}[x]/x^{m+1}, \quad \text{deg} \ x = 2.$$  

There is an operation $\theta$ in $O(\mathbb{Z}, 2m; 2m + 2q)$ given by $\theta(z) = z^{2q}$ for $z \in H^{2m}(X; \mathbb{Z})$. Since we have an isomorphism $O(\mathbb{Z}, 2m; 2m + 2q) \cong H^{2m + 2q}(K(\mathbb{Z}, 2m); \mathbb{Z})$.

By assumption, the operation $\theta$ is trivial, however, there is a non-trivial operation $x^m \to x^{mq}$. Contradiction.

(2) $n = 2m + 1$. The argument is the same, but we use $\mathbb{C}P^m \times S^1$ instead $\mathbb{C}P^m$.

Problem 8. Let $X \subset S^n$ be homeomorphic to $S^p \vee S^q$, $1 \leq p < q \leq n - 1$. Compute the homology groups $\tilde{H}_q(S^n \setminus X)$.

Solution. Let $X = S^n \setminus (S^p \vee S^q)$. To compute the homology groups $H_r(X)$, we will apply Mayer-Vietoris. Let $X_1 = S^n \setminus S^p$ and $X_2 = S^n \setminus S^q$. Then $X = X_1 \cap X_2$, and as both $X_1$ and $X_2$ are open (as they are the complement of the image of a compact set under continuous map) the Mayer-Vietoris theorem applies. Additionally,

$$X_1 \cup X_2 = S^n \setminus \ast \cong \mathbb{R}^n.$$  

Now, consider the Mayer-Vietoris sequence in reduced homology groups (to avoid too many cases).

$$\rightarrow \tilde{H}_{r+1}(\mathbb{R}^n) \rightarrow \tilde{H}_r(X) \rightarrow \tilde{H}_r(S^n \setminus S^p) \oplus \tilde{H}_r(S^n \setminus S^q) \rightarrow \tilde{H}_r(\mathbb{R}^n) \rightarrow .$$  

For any $r$, we have $\tilde{H}_r(\mathbb{R}^n) = 0$, so $\tilde{H}_r(X) \cong \tilde{H}_r(S^n \setminus S^p) \oplus \tilde{H}_r(S^n \setminus S^q)$. By Theorem 16.6, $\tilde{H}_q(S^n \setminus S^k) \cong \mathbb{Z}$ exactly where $q = n - k - 1$ and is zero otherwise. Now we have 2 cases to consider to complete the computation. If $p \neq q$, we have

$$\tilde{H}_r(X) \cong \tilde{H}_r(S^n \setminus S^p) \oplus \tilde{H}_r(S^n \setminus S^q) = \begin{cases} \mathbb{Z} & r = n - p - 1 \\ \mathbb{Z} & r = n - q - 1 \\ 0 & \text{else.} \end{cases}$$  

In the case $p = q$, we have instead

$$\tilde{H}_r(X) \cong \tilde{H}_r(S^n \setminus S^p) \oplus \tilde{H}_r(S^n \setminus S^q) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & r = n - p - 1 \\ 0 & \text{else.} \end{cases}$$  

The only difference between reduced and nonreduced homology is a $\mathbb{Z}$ summand in $H_0(X)$, so this completes the computation.
**Problem 9.** Let $M$ be a compact oriented closed manifold of dimension $\dim M = 2k$. Prove that if $H_{k-1}(M; \mathbb{Z})$ is torsion-free, then the group $H_{k}(M; \mathbb{Z})$ is also torsion-free.

**Solution.** Since $M^{2k}$ is a closed oriented manifold, we have Poincare duality isomorphism

$$H_{k}(M; \mathbb{Z}) \cong H^{k}(M; \mathbb{Z}).$$

Then we use the universal coefficient theorem

$$0 \to \text{Ext}(H_{k-1}(X; \mathbb{Z}); \mathbb{Z}) \to H^{k}(X; \mathbb{Z}) \to \text{Hom}(H_{k}(X; \mathbb{Z}), \mathbb{Z}), \to 0$$

Assume $H_{k-1}(X; \mathbb{Z})$ has torsion, then $\text{Ext}(H_{k-1}(X; \mathbb{Z}); \mathbb{Z})$ is torsion group, and

$$\text{Ext}(H_{k-1}(X; \mathbb{Z}); \mathbb{Z}) \to H^{k}(X; \mathbb{Z})$$

is injection; however $H^{k}(X; \mathbb{Z})$ is torsion free. Contradiction. \qed

**Problem 10.** Prove that any map $f : SO(3) \to T^{3}$ is contractible. Here $SO(3)$ is the special orthogonal group and $T^{3} = S^{1} \times S^{1} \times S^{1}$ is a torus.

**Solution.** The torus $T^{3}$ represents the Eileberg-McLane space $K(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}; 1)$. Then we have the isomorphism

$$H^{1}(X; \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) = [X, K(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}; 1)].$$

If $X = SO(3) \cong \mathbb{RP}^{3}$, then $H^{1}(\mathbb{RP}^{3}; \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) = 0$. Thus any map $f : \mathbb{RP}^{3} \to T^{3}$ is contractible. \qed