QUALIFYING EXAM, Winter 2024

Algebraic Topology

NAME ____________________________________________

(STUDENT NUMBER _____________ SIGNATURE _____________)

Do all 10 problems. Please write clearly.

**Problem 1** Let $f : S^n \times S^n \to S^{2n}$ be the quotient map collapsing $S^n \vee S^n$ to a point. Show that $f$ induces the zero map on all homotopy groups but $f$ is not nullhomotopic.

**Problem 2** Define the Hopf invariant. Assume the Hopf invariant is a homomorphism. Prove that $h([\iota_{2n}, \iota_{2n}])$ is non-zero, and use this to prove that $\pi_{4n-1}(S^{2n})$ contains \mathbb{Z}.

**Problem 3** State the Freudenthal Theorem. Assuming that the group $\pi_4(S^3)$ is non-trivial, prove that it has order two.

**Problem 4** Give a construction of an Eilenberg-McLane space $K(\pi, n)$. Prove that

$$H_{n+1}(K(\pi, n); \mathbb{Z}) = 0$$

if $n \geq 2$ and $\pi$ is an arbitrary abelian group.

**Problem 5** Let $f : S^{2n} \to S^{2n}$ be a map of degree zero. Prove that there exist two points $x, y \in S^{2n}$ such that $f(x) = x$ and $f(y) = -y$.

**Problem 6** State the Lefschetz Fixed Point Theorem. Prove that any map

$$f : \text{HP}^{4k} \times \text{RP}^{2n} \to \text{HP}^{4k} \times \text{RP}^{2n}$$

always has a fixed point.

**Problem 7** Let $h : S^3 \to S^2$ be the Hopf map. If $c : T^3 \to S^3$ is the map which collapses the complement of a ball to a point, prove that $h \circ c : T^3 \to S^2$ induces the trivial map on homology and homotopy, but is not homotopic to a constant map.

**Problem 8** Show that a closed simply-connected 3-manifold $M$ is homotopy equivalent to $S^3$.

**Problem 9** Compute the homotopy groups $\pi_q(\text{CP}^n)$ for $q \leq 2n + 1$.

**Problem 10** Let $M$ be a closed, simply-connected manifold of dimension $4k + 2$. Show that the Euler characteristic of $M$ is even.