QUALIFYING EXAM, Winter 2024

Algebraic Topology

NAME ____________________________

(STUDENT NUMBER ___________  SIGNATURE ___________)

Do all 10 problems. Please write clearly.

Problem 1 Let \( f : S^n \times S^n \to S^{2n} \) be the quotient map collapsing \( S^n \lor S^n \) to a point. Show that \( f \) induces the zero map on all homotopy groups but \( f \) is not nullhomotopic.

Problem 2 Define the Hopf invariant. Assume the Hopf invariant is a homomorphism. Prove that \( h([\iota_{2n}, \iota_{2n}]) \) is non-zero, and use this to prove that \( \pi_{4n-1}(S^{2n}) \) contains \( \mathbb{Z} \).

Problem 3 State the Freudenthal Theorem. Prove that the group \( \pi_4(S^3) \) has order two.

Problem 4 Give a construction of an Eilenberg-McLane space \( K(\pi, n) \). Prove that
\[
H_{n+1}(K(\pi, n); \mathbb{Z}) = 0
\]
if \( n \geq 2 \) and \( \pi \) is an arbitrary abelian group.

Problem 5 Let \( f : S^{2n} \to S^{2n} \) be a map of degree zero. Prove that there exist two points \( x, y \in S^{2n} \) such that \( f(x) = x \) and \( f(y) = -y \).

Problem 6 State the Lefschetz Fixed Point Theorem. Prove that any map
\[
f : \mathbb{H}P^{4k} \times \mathbb{R}P^{2n} \to \mathbb{H}P^{4k} \times \mathbb{R}P^{2n}
\]
always has a fixed point.

Problem 7 Let \( h : S^3 \to S^2 \) be the Hopf map. If \( c : T^3 \to S^3 \) is the map which collapses the complement of a ball to a point, prove that \( h \circ c : T^3 \to S^2 \) induces the trivial map on homology and homotopy, but is not homotopic to a constant map.

Problem 8 Show that a closed simply-connected 3-manifold \( M \) is homotopy equivalent to \( S^3 \).

Problem 9 Compute the homotopy groups \( \pi_q(\mathbb{C}P^n) \) for \( q \leq 2n + 1 \).

Problem 10 Let \( M \) be a closed, simply-connected manifold of dimension \( 4k + 2 \). Show that the Euler characteristic of \( M \) is even.
1. Let \( f : S^n \times S^n \to S^{2n} \) be the quotient map collapsing \( S^n \vee S^n \) to a point. Show that \( f \) induces the zero map on all homotopy groups but \( f \) is not nullhomotopic.

**Solution.** (This solution does not use homology, there is much easier argument with homology groups.)
Consider the embedding \( i : S^n \vee S^k \to S^n \times S^k \).

**Lemma 1** The embedding \( i : S^n \vee S^k \to S^n \times S^k \) induces a surjection
\[
i_* : \pi_q(S^n \vee S^k) \longrightarrow \pi_q(S^n \times S^k)
\]
for all \( q \geq 1 \).

**Proof.** Let \( x \) be a base point of \( S^n \vee S^k \). By construction, the compositions
\[
i_n : S^n \vee \{x\} \xrightarrow{i(n)} S^n \vee S^k \xrightarrow{i} S^n \times S^k \xrightarrow{pr(n)} S^n
\]
\[
i_k : \{x\} \vee S^k \xrightarrow{i(k)} S^n \vee S^k \xrightarrow{i} S^n \times S^k \xrightarrow{pr(k)} S^k
\]
(where \( i(n) \) and \( i(k) \) are canonical embeddings and \( pr(n) \) and \( pr(k) \) are the projections of the first and the second factor), coincide with the identity maps. We have an isomorphism:
\[
\pi_q(S^n \times S^k) \cong \pi_q(S^n) \oplus \pi_q(S^k).
\]
Thus the homomorphism \( i_* : \pi_q(S^n \vee S^k) \longrightarrow \pi_q(S^n \times S^k) \) is surjective. \( \square \)

**Lemma 2** The element \( w \in \pi_{n+k-1}(S^n \vee S^k) \) has infinite order, i.e. the homotopy group \( \pi_{n+k-1}(S^n \vee S^k) \) is infinite.

**Proof.** Now recall that \( S^n \times S^k = (S^n \vee S^k) \cup_w D^{n+k} \), where \( w : S^{n+k-1} \to S^n \vee S^n \) is the Whitehead map. It gives us the commutative diagram:
\[
\begin{array}{c}
S^{n+k-1} \xrightarrow{w} S^n \vee S^k \\
\downarrow{\partial} \quad \downarrow{i} \\
D^{n+k} \xrightarrow{\Phi} S^n \times S^k
\end{array}
\]
Clearly the map \( \Phi : D^{n+k} \to S^n \times S^k \) determines an element \( \iota \in \pi_{n+k}(S^n \times S^k, S^n \vee S^k) \). Consider the map
\[
\tilde{j} : (S^n \times S^k, S^n \vee S^k) \to (S^{n+k}, s_0)
\]
which maps \( S^n \vee S^k \) to the base point \( s_0 \in S^{n+k} \). The composition \( \tilde{j} \circ \Phi : D^{n+k} \to S^{n+k} \) is a representative of a generator of the group \( \pi_{n+k}(S^{n+k}) \cong \mathbb{Z} \). Thus we conclude that the element \( \iota \in \pi_{n+k}(S^n \times S^k, S^n \vee S^k) \) is nontrivial and has infinite order.

Next we consider the long exact sequence in homotopy for the pair \( (S^n \times S^k, S^n \vee S^k) \):
\[
\pi_{n+k}(S^n \times S^k) \xrightarrow{i_*} \pi_{n+k}(S^n \times S^k) \xrightarrow{j_*} \pi_{n+k}(S^n \times S^k, S^n \vee S^k) \xrightarrow{\partial} \pi_{n+k-1}(S^n \vee S^k)
\]
According to Lemma 1, \( i_* \) is epimorphic since \( \pi_{n+k}(S^n \times S^k) = \pi_{n+k}(S^n) \oplus \pi_{n+k}(S^k) \). Thus the homomorphism \( j_* \) is zero, and \( \partial \) is monomorphisms. Since \( w = \partial(i) \) it follows that the group \( \pi_{n+k-1}(S^n \vee S^k) \) is infinite and \( w \) has infinite order. \( \square \)

Now consider the quotient map \( f : S^n \times S^n \to S^{2n} \) collapsing \( S^n \vee S^n \) to a point. It fits into the following commutative diagram:
\[
\begin{array}{c}
S^{2n-1} \xrightarrow{w} S^n \vee S^n \\
\downarrow{\partial} \quad \downarrow{i} \\
D^{2n} \xrightarrow{\Phi} S^n \times S^n \xrightarrow{f} S^{2n}
\end{array}
\]
where \( c : S^n \vee S^n \to * \) is a collapsing map. As we have seen in the proof of Lemma 2, the composition \( f \circ \Phi : (D^{2n}, S^{2n-1}) \to (S^{2n}, *) \) represents a generator in the homotopy group \( \pi_{2n}(S^{2n}) \). Thus the map \( f \)
cannot be homotopic to a constant map. On the other hand, according to Lemma 1, the homomorphism $i_* : \pi_q(S^n \vee S^n) \to \pi_q(S^n \times S^n)$ is epimorphism. Hence we have a commutative diagram

\[
\begin{array}{ccc}
\pi_q(S^n \vee S^n) & \xrightarrow{c_*} & 0 \\
\downarrow {i_*} & & \downarrow i \\
\pi_q(S^n \times S^n) & \xrightarrow{f_*} & \pi_q(S^{2n})
\end{array}
\]

where $i_*$ is epimorphic. Thus $f_*$ is a trivial homomorphism for each $q$. \hfill \Box

2. Define the Hopf invariant. Assume the Hopf invariant is a homomorphism. Prove that $h([t_{2n}, t_{2n}])$ is non-zero, and use this to prove that $\pi_{4n-1}(S^{2n})$ contains $\mathbb{Z}$.

**Solution.** Let $\varphi \in \pi_{4n-1}(S^{2n})$, and let $f : S^{4n-1} \to S^{2n}$ be a representative of $\varphi$. Let $X_\varphi = S^{2n} \cup_f D^{4n}$. We compute the cohomology groups of $X_\varphi$:

\[ H^n(X_\varphi; \mathbb{Z}) = \begin{cases} 
\mathbb{Z}, & q = 0, 2n, 4n, \\
\mathbb{Z} \oplus \mathbb{Z}, & q = 2n, \\
0, & \text{otherwise.}
\end{cases} \]

Let $a \in H^2n(X_\varphi; \mathbb{Z})$, $b \in H^4n(X_\varphi; \mathbb{Z})$ be generators. Since $a^2 = a \cup a \in H^4n(X_\varphi; \mathbb{Z})$, then $a^2 = hb$, where $h \in \mathbb{Z}$. The number $h(\varphi) = h$ is the **Hopf invariant** of the element $\varphi \in \pi_{4n-1}(S^{2n})$. The following result holds:

**Lemma 3** $h(\varphi_1) + h(\varphi_2) = h(\varphi_1 + \varphi_2)$. We use Lemma 3 to prove

**Theorem 4** *The Hopf invariant is not trivial, in particular,*

\[ h([t_{2n}, t_{2n}]) = 2. \]

**Proof.** First we compute the cohomology groups:

\[ H^q(S^{2n} \times S^{2n}; \mathbb{Z}) = \begin{cases} 
\mathbb{Z}, & q = 0, 4n, \\
\mathbb{Z} \oplus \mathbb{Z}, & q = 2n, \\
0, & \text{otherwise.}
\end{cases} \]

Let $c_1, c_2 \in H^2n(S^{2n} \times S^{2n})$ be such generators that the homomorphisms

\[ p_1^*: H^2n(S^2_{1n} \times S^2_{2n}) \to H^2n(S^2_{1n} \times S^2_{2n}), \]
\[ p_2^*: H^2n(S^2_{2n} \times S^2_{2n}) \to H^2n(S^2_{1n} \times S^2_{2n}), \]

induced by the projections

\[ S^2_{1n} \times S^2_{2n} \xrightarrow{p_1} S^2_{1n}, \quad S^2_{1n} \times S^2_{2n} \xrightarrow{p_2} S^2_{2n}, \]

send the generators $c_1$ and $c_2$ to the generators of the groups $H^2n(S^2_{1n}), H^2n(S^2_{2n})$. Let $d \in H^4n(S^{2n} \times S^{2n})$ be a generator. It is known that

\[ c_1c_2 = d. \]

We also note that $c_1^2 = 0$ and $c_2^2 = 0$ since by naturality $p_1^*(c_1)^2 = 0$ and $p_2^*(c_2)^2 = 0$. So we have that the ring $H^*(S^2_{1n} \times S^2_{2n})$ is generated over $\mathbb{Z}$ by the elements $1, c_1, c_2$ with the relations $c_1^2 = 0, c_2^2 = 0$. In particular, we have:

\[ (c_1 + c_2)^2 = c_1^2 + 2c_1c_2 + c_2^2 = 2d. \]

We consider the factor space

\[ X = S^{2n} \times S^{2n} / \sim, \]

where we identify the points $(x, x_0) = (x_0, x)$, where $x_0$ is the base point of $S^{2n}$. Next, we need the following fact:

**Lemma 5** *The space $X = S^{2n} \times S^{2n} / \sim$ is homeomorphic to the space $S^{2n} \cup_f D^{4n}$, where $f$ is the map defining the Whitehead product $[t_{2n}, t_{2n}]$.***
Proof of Lemma 5] Recall that \( S^{2n} \times S^{2n} = (S^{2n} \vee S^{2n}) \cup_w D^{4n} \), where \( w \) is the map we described above. The generator \( \iota_{2n} \) is represented by the identical map \( S^{2n} \to S^{2n} \). The composition
\[
S^{4n-1} \xrightarrow{w} S^{2n} \vee S^{2n} \xrightarrow{Id} S^{2n}
\]
represents the element \([\iota_{2n}, \iota_{2n}]\). It means that the identification \((S^{2n}, x_0) = (x_0, S^{2n})\) we just did in the space \( S^{2n} \times S^{2n} \) is the same as to attach \( D^{4n} \) with the attaching map \((Id \vee Id) \circ w\).

Compute the cohomology of \( X \):
\[
H^q(X; \mathbb{Z}) = \begin{cases} 
\mathbb{Z}, & q = 0, 2n, 4n, \\
0, & \text{otherwise}.
\end{cases}
\]

We note that the projection \( S^{2n} \times S^{2n} \to X \) sends the generator \( c \in H^{2n}(X) \) to \( c_1 + c_2 \). Besides the generator \( d \) maps to a generator of \( H^{4n}(X; \mathbb{Z}) \) (we denote it also by \( d \)). So we have: \( c^2 = 2d \), or \( h([\iota_{2n}, \iota_{2n}]) = 2 \).

3. State the Freudenthal Theorem. Assuming that the group \( \pi_4(S^3) \) is non-trivial, prove that it has order two.

Solution. Let \( X \) be a space with a base point \( x_0 \). We construct the homomorphism
\[
\Sigma : \pi_q(X) \to \pi_{q+1}(\Sigma X)
\]
as follows. Let \( \alpha \in \pi_q(X) \), and a map \( f : S^q \to X \) be a representative of \( \alpha \). The map
\[
\Sigma f : \Sigma S^q = S^{q+1} \to \Sigma X
\]
defined by the formula \( \Sigma f(y,t) = (f(y), t) \in \Sigma X \) gives a representative for \( \Sigma(\alpha) \in \pi_{q+1}(\Sigma X) \). It is not hard to check that
1. \( f \sim g \) implies that \( \Sigma f \sim \Sigma g \);
2. \( \Sigma f + \Sigma g \sim \Sigma(f + g) \).

The homomorphism \( \Sigma \) is the suspension homomorphism.

Theorem 6 (Freudenthal Theorem) The suspension homomorphism
\[
\Sigma : \pi_q(S^n) \to \pi_{q+1}(S^{n+1})
\]
is isomorphism for \( q \leq 2n - 1 \) and epimorphism for \( q = 2n - 1 \).

If \( X = S^2 \), and \( \iota \in \pi_2(S^2) \) is a generator given by the identity map \( S^2 \to S^2 \). We use the fact that the Hopf map \( H : S^2 \to S^2 \) gives a generator \( z \in \pi_3(S^2) \cong \mathbb{Z} \). Also we use the fact that the Hopf invariant \( h(z) = 1 \). Indeed, the space \( X_H = S^2 \cup_H e^4 \) coincides (up to homotopy) with \( \mathbb{C}P^2 \), and \( H^*(\mathbb{C}P^2; \mathbb{Z}) = \mathbb{Z}[x]/x^3 \), where \( x \in H^2(\mathbb{C}P^2; \mathbb{Z}) \cong \mathbb{Z} \) is a generator.

Consider the Whitehead product \([\iota, \iota] \in \pi_3S^2 \). We use the facts that the Hopf invariant \( h : \pi_3(S^2) \to \mathbb{Z} \) is a homomorphism and \( h([\iota, \iota]) = 2 \), i.e. \( [\iota, \iota] = 2z \). According to Freudenthal Theorem, the suspension \( \Sigma : \pi_3(S^2) \to \pi_4(S^3) \) is onto (since \( 3 = 2 \cdot 2 - 1 \)). On the other hand, \( \Sigma[\iota, \iota] = 0 \). Thus \( \pi_4(S^3) \cong \mathbb{Z}_2 \) or \( \pi_4(S^3) \cong 0 \). To show that \( \pi_4(S^3) \neq 0 \) one needs more technology.

4. Give a construction of an Eilenberg-McLane space \( K(\pi, n) \). Prove that
\[
H_{n+1}(K(\pi, n); \mathbb{Z}) = 0
\]
if \( n \geq 2 \) and \( \pi \) is an arbitrary abelian group.

Solution. Let \( n \) be a positive integer and \( \pi \) be a group (abelian) if \( n \geq 2 \). A space \( X \) is called an Eilenberg-McLane space of the type \( K(\pi, n) \) if
\[
\pi_q(X) = \begin{cases} 
\pi & \text{if } q = n \\
0 & \text{else}.
\end{cases}
\]

Theorem 7 Let \( n \) be a positive integer and \( \pi \) be a group (abelian) if \( n \geq 2 \). Then the Eilenberg-McLane space of the type \( K(\pi, n) \) exists and unique up to weak homotopy equivalence.
We recall the construction. Let \( \{g_\alpha\} \) be generators of the group \( \pi \), and \( \{r_\beta\} \) be relations (if \( n > 1 \) we mean relations in the abelian group). Let \( X = \bigvee_{\alpha} S^n_\alpha \). Then \( \pi_q(X) = 0 \) if \( q \leq n - 1 \) and \( \pi_n(X) = \bigoplus_\alpha \mathbb{Z} \) (or free group with generators \( \{g_\alpha\} \) if \( n = 1 \)). Each relation \( r_\beta \) defines a unique element \( r_\beta \in \pi_n(X) \). We choose maps \( r_\beta : S^n_\beta \rightarrow X \) representing the above relations and attach cells \( e^{n+1}_\beta \) using \( r_\beta \) as the attaching maps. Let \( Y \) be the resulting space. Then \( \pi_q(Y) = 0 \) if \( q \leq n - 1 \) and \( \pi_n(Y) = \pi \).

We notice that the cellular chain complex of \( Y \) is given as

\[
\mathcal{E}_q(Y) = \begin{cases} \bigoplus_\alpha \mathbb{Z}(g_\alpha) & \text{if } q = n, \\ \bigoplus_\beta \mathbb{Z}(r_\beta) & \text{if } q = n + 1, \\ 0 & \text{else} \end{cases}
\]

Since the boundary homomorphism \( \partial_{n+1} : \mathcal{E}_{n+1}(Y) \rightarrow \mathcal{E}_{n+1}(Y) \) is given by the inclusions \( r_\beta \in \pi_n(X) \), it is obvious that \( \ker \partial_{n+1} = 0 \), which gives that

\[
H_q(X; \mathbb{Z}) = \begin{cases} \pi & \text{if } q = n, \\ 0 & \text{else} \end{cases}
\]

In order to get the Eilenberg-McLane space \( K(\pi, n) \) out of the space \( Y \), we have to attach more cells. The first step here is to kill the homotopy group \( \pi_{n+1}(Y) \) and construct \( Y_{n+2} \): it requires to attach \( (n+2) \)-cells; however, it does not effect the \((n+1)\)-homology. We obtain that

\[
H_{n+1}(Y_{n+2}; \mathbb{Z}) = H_{n+1}(Y; \mathbb{Z}) = 0.
\]

The same holds when we attach more cells to kill the homotopy group \( \pi_{n+2}(Y_{n+1}) \) and construct \( Y_{n+3} \): we attach \((n+3)\)-cells. We obtain that

\[
H_{n+1}(K(\pi, n); \mathbb{Z}) = \cdots = H_{n+1}(Y_{n+3}; \mathbb{Z}) = H_{n+2}(Y; \mathbb{Z}) = 0.
\]

This proves that \( H_{n+1}(K(\pi, n); \mathbb{Z}) = 0 \)

5. Let \( f : S^{2n} \rightarrow S^{2n} \) be a map of degree zero. Prove that there exist two points \( x, y \in S^{2n} \) such that \( f(x) = x \) and \( f(y) = -y \).

**Solution.** Since \( \deg f = 0 \), there exists \( x \in S^{2n} \) with \( f(x) = x \). Indeed, since the trace of the homomorphism \( f_* : H_2(S^{2n}; \mathbb{Z}) \rightarrow H_2(S^{2n}; \mathbb{Z}) \) is trivial, we have that the Lefschetz number \( \text{Lef}(f) \) is the same as the trace of the homomorphism \( f_* : H_0(S^{2n}; \mathbb{Z}) \rightarrow H_0(S^{2n}; \mathbb{Z}) \), i.e. \( \text{Lef}(f) = \pm 1 \).

We also need the following

**Claim A.** Let \( f, g : S^{2n} \rightarrow S^{2n} \) be two maps such that \( f(x) \neq -g(x) \) for any \( x \in S^{2n} \). Then \( f \sim g \).

Indeed, here is the homotopy:

\[
H(x, t) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}, \quad 0 \leq t \leq 1.
\]

Since \( f(x) \neq -g(x) \), the line \( (1-t)f(x) + tg(x) \) never passes through the origin.

Thus we already found a point \( x \in S^{2n} \) such that \( f(x) = x \). Now we assume that there is no \( y \in S^{2n} \) such that \( f(y) = -y \). Then by Claim A, \( f \) is homotopic to the identity map. Then, of course, \( \deg f \neq 0 \).

6. State the Lefschetz Fixed Point Theorem. Prove that any map

\[
f : \mathbb{H}P^{4k} \times \mathbb{R}P^{2n} \rightarrow \mathbb{H}P^{4k} \times \mathbb{R}P^{2n}
\]

always has a fixed point.

**Solution:** Let \( A \) be a finitely generated abelian group. Denote \( F(A) \) the free part of \( A \), so that \( A = F(A) \oplus T(A) \), where \( T(A) \) is a maximum torsion subgroup of \( A \). Let \( \varphi : A \rightarrow A \) be an endomorphism of \( A \). We define \( F(\varphi) : F(A) \rightarrow F(A) \) by composition:

\[
F(\varphi) : F(A) \xrightarrow{\text{inclusion}} A \xrightarrow{\varphi} A \xrightarrow{\text{projection}} F(A).
\]
We define $\text{Tr}(\varphi) = \text{Tr}(F(\varphi))$. Now let $\mathcal{A} = \{A_q\}_{q \geq 0}$ be a finitely generated graded abelian group, i.e. each group $A_q$ is finitely generated. A homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ of two graded abelian groups is a collection of homomorphisms $\{\varphi_q : A_q \to B_{q-k}\}$ (the number $k$ is the degree of $\Phi$).

Now let $\mathcal{A} = \{A_q\}_{q \geq 0}$ be a finitely generated graded abelian group, and let $\Phi = \{\varphi_q : \mathcal{A} \to \mathcal{A}\}$ be an endomorphism of degree zero. We assume that $F(A_q) = 0$ for $q \geq n$ (for some $n$). We define the Lefschetz number $\text{Lef}(\Phi)$ of the endomorphism $\Phi$ by the formula:

$$\text{Lef}(\Phi) = \sum_{q \geq 0} (-1)^q \text{Tr}(\varphi_q).$$

**Lefschetz Fixed Point Theorem.** Let $X$ be a finite CW-complex, $f : X \to X$ be a map such that $\text{Lef}(f) \neq 0$. Then $f$ has a fixed point, i.e. such point $x_0 \in X$ that $f(x_0) = x_0$.

Now we consider a map $f : \mathbb{H}P^4k \times \mathbb{R}P^{2n} \to \mathbb{H}P^4k \times \mathbb{R}P^{2n}$

We recall that $H^*(\mathbb{H}P^4k; \mathbb{Z}) \cong \mathbb{Z}[z]/z^{4k+1}$, where $z \in H^4(\mathbb{H}P^4k; \mathbb{Z})$ is a generator. On the other hand, we have that

$$H^q(\mathbb{R}P^{2n}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ \mathbb{Z}/2 & \text{if } q = 2, 4, \ldots, 2n, \\ 0 & \text{else.} \end{cases}$$

The Künneth formula gives that

$$H^q(\mathbb{H}P^4k \times \mathbb{R}P^{2n}; \mathbb{Z}) \cong H^q(\mathbb{H}P^4k; \mathbb{Z}) \mod torsion$$

Then the map $f : \mathbb{H}P^4k \times \mathbb{R}P^{2n} \to \mathbb{H}P^4k \times \mathbb{R}P^{2n}$ induces a ring homomorphism

$$f^* : H^*(\mathbb{H}P^4k \times \mathbb{R}P^{2n}; \mathbb{Z}) \to H^*(\mathbb{H}P^4k \times \mathbb{R}P^{2n}; \mathbb{Z})$$

We notice that $f^*(z) = \lambda z \in H^2(\mathbb{H}P^4k \times \mathbb{R}P^{2n}; \mathbb{Z}) \cong \mathbb{Z}$ modulo torsion. Then we have: $f^*(z^j) = \lambda^j z^j$. We notice that the Universal coefficient formula gives that the homomorphism

$$f_* : H_{2j}(\mathbb{H}P^4k \times \mathbb{R}P^{2n}; \mathbb{Z}) \to H_{2j}(\mathbb{H}P^{2k} \times \mathbb{R}P^{2n}; \mathbb{Z})$$

is also the multiplication by $\lambda^j$ modulo torsion. In particular, we have that

$$F_q(\mathbb{H}P^4k \times \mathbb{R}P^{2n}) \cong H_q(\mathbb{H}P^4k; \mathbb{Z})$$

and the Lefschetz number is given as

$$\text{Lef}(f) = 1 + \lambda + \lambda^2 + \ldots + \lambda^{4k} = \frac{1 - \lambda^{4k+1}}{1 - \lambda}$$

Assume $\text{Lef}(f) = 0$, then the only possibility is $\lambda^{4k+1} = 1$, or $\lambda = 1$, however then $\text{Lef}(f) = 4k + 1 \neq 0$. Thus $\text{Lef}(f) \neq 0$ for any map $f$.

**7.** Let $h : S^3 \to S^2$ be the Hopf map. If $c : T^3 \to S^3$ is the map which collapses the complement of a ball to a point, prove that $h \circ c : T^3 \to S^2$ induces the trivial map on homology and homotopy, but is not homotopic to a constant map.

**Solution.** Consider the homomorphism $(h \circ c)_* : \pi_q(T^3) \to \pi_q(S^3)$ trivial for $q > 1$ and is also trivial for $q = 1$ since $\pi_1S^3 = 0$. Thus $(h \circ c)_*$ is trivial in homotopy groups.

The homomorphism $(h \circ c)_* : \tilde{H}_q(T^3) \to \tilde{H}_q(S^3)$ is also trivial. Indeed, the group $\tilde{H}_3S^3 = \mathbb{Z}$ and trivial otherwise and $\tilde{H}_qS^2$ is non-trivial only if $q = 2$, i.e., any homomorphism $\tilde{H}_qS^3 \to \tilde{H}_qS^2$ is trivial.

Let $g = h \circ c$. Now we assume that the map $g : T^3 \to S^2$ is contractible. Let $g_t : T^3 \to S^2$ be a homotopy such that $g_t = g$ and $g_1(T^3) = x_0 \in S^2$. By the Lifting Homotopy Property, there exists a homotopy $\tilde{g}_t : T^3 \to S^3$ such that the diagram

$$\begin{array}{ccc}
T^3 & \overset{g_t}{\longrightarrow} & S^2 \\
\downarrow & & \downarrow \\
\tilde{g}_t & \overset{h}{\longrightarrow} & S^3
\end{array}$$


commutes. Here \( \tilde{g}_0 = c \) as above. Then we have that \( \tilde{g}_1(T^3) \subset h^{-1}(x_0) = S^1 \). We notice that the homomorphism \( c_* : H_3(T^3) \to H_3(S^3) \) is non-trivial (it is an isomorphism). However \((\tilde{g}_1)_* \) must be trivial since the map \( \tilde{g}_1 \) factors through \( S^3 \):

\[
\begin{array}{c}
S^1 \xrightarrow{c} S^3 \\
\xi \xrightarrow{\tilde{g}_1} h \\
T^3 \xrightarrow{g_1} S^2 
\end{array}
\]

Since \((\tilde{g}_1)_* = c_* \), we obtain a contradiction. \(\square\)

8. Show that a closed simply-connected 3-manifold \( M \) is homotopy equivalent to \( S^3 \).

**Solution.** Let \( M \) be a closed simply-connected 3-manifold. In particular, it means that \( M \) is oriented manifold and \( H_3(M; \mathbb{Z}) \cong \mathbb{Z} \). Since \( \pi_1 M = 0 \), then \( H_1(M; \mathbb{Z}) = \pi_1(M)/[\pi_1 M, \pi_1 M] = 0 \) as well. The Poincare duality then implies that \( H^2(M; \mathbb{Z}) = 0 \). Then the universal coefficient formula

\[
0 \to \text{Ext}(H_1(M; \mathbb{Z}), \mathbb{Z}) \to H^2(M; \mathbb{Z}) \to \text{Hom}(H_2(M; \mathbb{Z}), \mathbb{Z}) \to 0
\]

implies that \( H_2(M; \mathbb{Z}) = 0 \). Thus \( M \) is a simply-connected space with the same homologies as \( S^3 \). Consider a disk \( D^3 \subset M \), then we have a map

\[
f : M \to M/(M \setminus D^3) \cong S^3
\]

which induces isomorphism in homology groups. Since \( M \) is simply-connected, Whitehead Theorem implies that \( f \) induces isomorphism in homotopy groups. Thus \( f \) is weak homotopy equivalence and, consequently, a homotopy equivalence since \( M \) and \( S^3 \) are CW-complexes.

9. Compute the homotopy groups \( \pi_q(\mathbb{C}P^n) \) for \( q \leq 2n + 1 \).

**Solution.** We consider the Hopf bundle \( H : S^{2n+1} \to \mathbb{C}P^n \) with the fiber \( S^1 \). We have a long exact sequence in homotopy groups:

\[
\cdots \to \pi_q(S^1) \to \pi_q(S^{2n+1}) \to \pi_q(\mathbb{C}P^n) \to \pi_{q-1}(S^1) \to \cdots
\]

Consider the case \( q = 1 \). Then we have that \( \pi_1(\mathbb{C}P^n) = 0 \) since \( \pi_1(S^{2n+1}) = 0 \) for \( n \geq 1 \). For \( q = 2 \), we have

\[
\cdots \to \pi_2(S^{2n+1}) \to \pi_2(\mathbb{C}P^n) \to \pi_1(S^1) \to \pi_1(S^{2n+1}) \to \cdots
\]

Thus we have that \( \pi_2(\mathbb{C}P^n) \cong \pi_1(S^1) \cong \mathbb{Z} \) since \( \pi_2(S^{2n+1}) = 0 \) and \( \pi_1(S^{2n+1}) = 0 \).

If \( q \geq 3 \), we obtain:

\[
0 \to \pi_q(S^{2n+1}) \to \pi_q(\mathbb{C}P^n) \to 0
\]

since \( \pi_q(S^1) = 0 \) for \( q \geq 3 \). We obtain:

\[
\pi_q(\mathbb{C}P^n) = \begin{cases} 
0, & \text{if } q = 0, 1 \\
\mathbb{Z}, & \text{if } q = 2 \\
\pi_q(S^{2n+1}), & \text{if } q \geq 3
\end{cases}
\]

Also, we have that \( \pi_q(S^{2n+1}) = 0 \) for all \( q < 2n + 1 \) and \( \pi_{2n+1}(S^{2n+1}) = \mathbb{Z} \). \(\square\)

10. Let \( M \) be a closed, simply-connected manifold of dimension \( 4k + 2 \). Show that the Euler characteristic of \( M \) is even.

**Solution.** Let \( \dim M = n \). According to Poincare duality Theorem, we have that

\[
\text{rank } H_i(M; \mathbb{Z}) = \text{rank } H^{n-i}(M; \mathbb{Z}) = \text{rank } H_{n-i}(M; \mathbb{Z}).
\]

Let \( r_i = \text{rank } H_i(M; \mathbb{Z}) \). Then we have:

\[
\chi(M) = \sum_{i=0}^{4k+2} (-1)^i r_i = -r_{2k+1} + \sum_{i=0}^{2k} \left[ (-1)^i r_i + (-1)^{4k+2-i} r_{4k+2-i} \right].
\]

Now we have that \( r_i = r_{4k+2-i} \) and \( (-1)^{4k+2-i} = (-1)^i \). Hence

\[
\chi(M) = -r_{2k+1} + 2 \sum_{i=1}^{2k} (-1)^i r_i \equiv r_{2k+1} \pmod{2}.
\]

Now, Poincare duality implies that the cup-product on \( H^{2k+1}(M; \mathbb{Z}) \) induces on \( H^{2k+1}(M; \mathbb{Z})/\text{Tor} \) a symplectic form. In particular, it means that rank \( H^{2k+1}(M; \mathbb{Z}) = r_{2k+1} \) is even. \(\square\)