ENGS26 Final Project Report: The Inverted Pendulum Cart

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Abstract

In this report, we discuss the process of characterizing the inverse pendulum mounted to a car from first principles. We then propose a control system governed by a PID controller to keep the pendulum upright and balanced.

Contents

1 Introduction

The inverted pendulum is a four-wheeled vehicle with a belt driven rear axle. The vehicle has an inverted pendulum mounted to it with a pin (Fig. 1). Our objective is to stabilize the pendulum by creating a control system that drives the car to keep the pendulum upright.

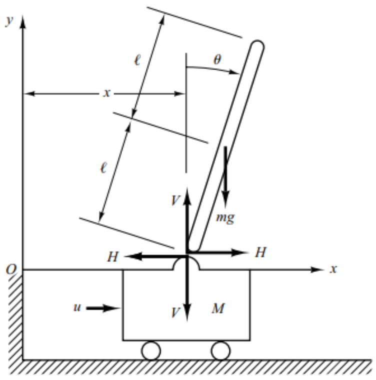

Figure 1: General model of an inverted pendulum on a car [1].

2 System Modeling

2.1 Physics Model

First, let's model the car and pendulum as a transfer function. Where we take the force applied by the motor, $F(s)$, as the input, and the output is the resulting angle of the pendulum, $\Theta(s)$. For convenience we can use example 3-5 from our textbook as our starting point [1]. By doing this we have our starting point as

$$F(s) = (M+m)X(s)s^2 + m\frac{L}{/}2\Theta(s)s^2, \tag{1}$$

where M and m are the weight of the car and pendulum respectively and L is the length of the pendulum from the pivot point to the center of mass. We also define $X(s)$ as the horizontal translation of the car. We can also use the example to get a transfer function relating horizontal translation of the car to the pendulum angle

$$X(s) = \Theta(s)\frac{Bs^2 - mgL}{mLs^2}, \tag{2}$$

where we define $A = (M+m)$, $B = (I + m(\frac{L}{2})^2)$, and I as the moment of inertia of the rod such that $I = \frac{1}{3}mL^2$. Plugging Eq. 2 into Eq. 1 and solving for the transfer function $\frac{\Theta(s)}{F(s)}$:

$$\frac{\Theta(s)}{F(s)} = \frac{\frac{mL}{AB+(mL)^2}}{s^2 - \frac{AmgL}{AB+(mL)^2}}. \tag{3}$$

Thus concluding our model of the physical system relating a force applied on the system to the resulting angle of the inverted pendulum.

2.2 Electronic Components Modelling and Calibration

Now we need to model the electrical components. This model will relate a reference voltage to the force being applied. Our electrical components are comprised of an IR sensor that detects the angle of the pendulum and outputs a voltage, and a DC motor that drives the rear wheels via an axle.

Starting with the IR sensor we need to determine the gain that translates radians to a voltage by measuring and plotting the voltage output at different angles. We see these results in Fig. 2. Using polyfit we determined the gain to be

$$K_{IR} = 218\frac{V}{rad}. \tag{4}$$

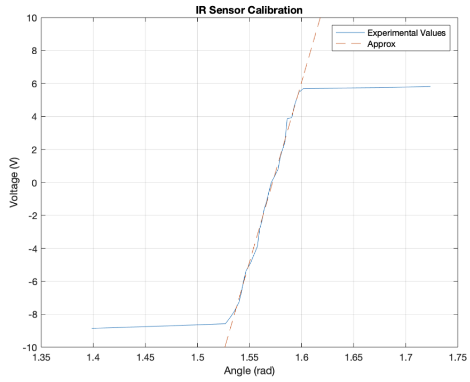

Figure 2: Calibration plot for K_{IR}.

Continuing our analysis to the DC motor, we model the motor system as seen in Fig. 3.

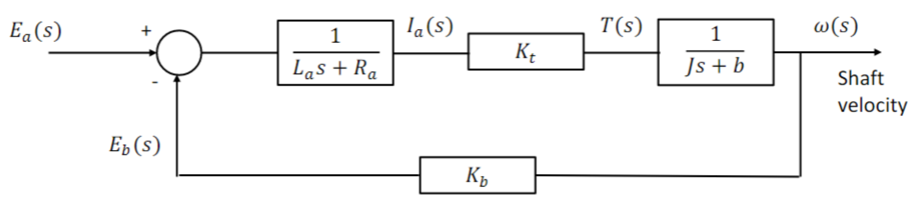

Figure 3: Block diagram for a DC motor.

After some analysis, we can simplify the transfer function of the DC motor as

$$\frac{I(s)}{V_{in}} = \frac{K}{R} \frac{s + b/J}{s + \frac{K_b K_t}{RJ} + \frac{b}{J}} \tag{5}$$

$$= K_a \frac{s+z}{s+p}. \tag{6}$$

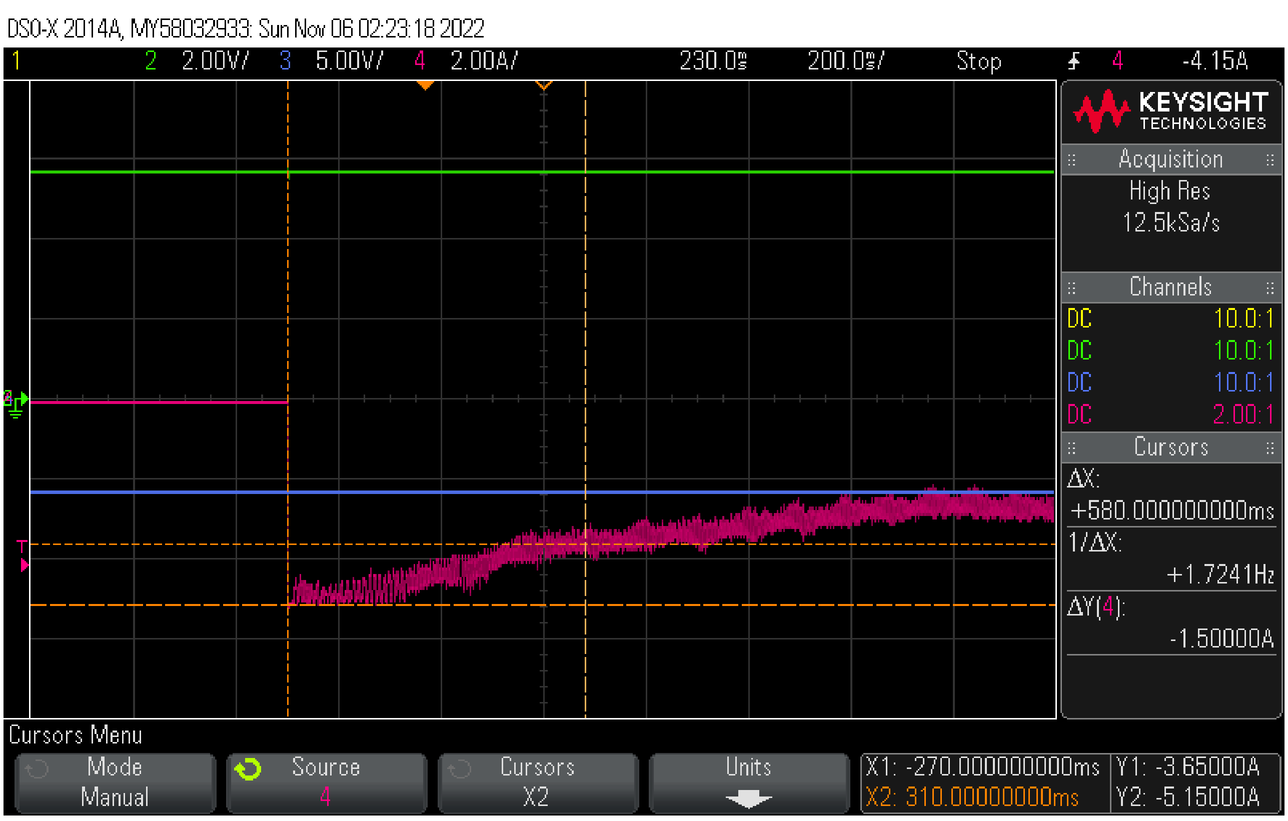

Figure 4: DC motor current draw from 5.62V (max pendulum) step input. CH.2: pendulum sensor, CH.3: E(s), CH.4: Current to the motor.

Let's first determine the value of K_t, the motor's torque constant. Looking at the motor spec sheet we have two points of torque output to current draw. By finding slope between these data points we can determine $K_t = 7.53\frac{oz-in}{A}$ which we can convert to

$$K_t = 0.0532\frac{Nm}{A}. \tag{7}$$

Applying IVT to Eq. 13 we can isolate K_a. Then we can apply FVT to isolate $\frac{z}{p}$. We accomplished this coefficient identification by applying a voltage to our car and measuring the current draw from the motor (Figure 4). From this data we can see that K_a, our initial value, is

$$K_a = \frac{5.15A}{5.62V} = 0.92\frac{A}{V}. \tag{8}$$

Where 5.62 V is the amplitude of our step function. Furthermore, we can determine p by analyzing the step response.

$$p = \frac{1}{\tau} = \frac{1}{580ms} = 1.724sec^{-1} \tag{9}$$

Applying FVT we can now find our last missing variable, z,

$$K_aU(s)\frac{z}{p} = 2.275A. \tag{10}$$

$$z = \frac{2.275A * 1.724sec^{-1}}{0.92\frac{A}{V} * 5.62V} = 0.76sec^{-1}. \tag{11}$$

We can now finally write our final transfer function for the DC motor, $G(s)$

$$G(s) = 0.92\frac{s + 0.76}{s + 1.724}. \tag{12}$$

2.3 Putting all together

Combining the motor and pendulum transfer functions we can now relate output torque to a reference voltage, V_{in}, by writing it as follows

$$\frac{T(s)}{V_{in}(s)} = \frac{I(s)}{V_{in}} * K_t. \tag{13}$$

Additionally, we can convert this torque on the motor shaft to force at the wheel by multiplying Eq. 13 by the gear ratio from the motor gear, or *driving* gear, to the wheel axle wheel, *driving* gear and dividing by the radius of the wheel to convert from torque to force. The gear ratio is calculated by $K_{gear} = \frac{driving}{driven} = 15/60 = 0.25$.

$$\frac{F(s)}{V_{in}(s)} = \frac{T(s)}{V_{in}(s)} * K_{Gear} * \frac{1}{r} = 0.2576\frac{s + 0.76}{s + 1.724}. \tag{14}$$

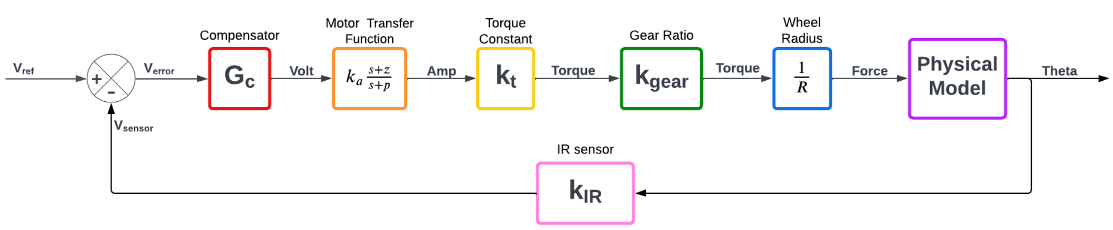

Figure 5: Complete block diagram for the inverted pendulum system.

Taking a step back now to see how all of these components fit together by drawing a high level block diagram (Figure 5) we find that by multiplying the physical model block by our motor block we can relate a reference voltage to the angle of the pendulum. We also place K_{IR} as the sensor gain in the feedback. Putting it all together, our final transfer function relating the pendulum angle to a reference voltage (corresponding to the pendulum at 0 rad) is

$$Open - Loop = \frac{\Theta(s))}{V_{in}(s)} = 0.07\frac{s + 0.76}{s^3 + 1.724s^2 - 7.998s - 13.79} \tag{15}$$

3 Analysis of The Uncompensated System

Following our discussion from Section 2, we can draw the root locus as seen in Figure. 6. From the root locus we can see that the system is, for the most part, unstable. We would need a proportional gain $K \gtrsim 260$ for the step response to be stable; affording a maximum damping ratio, ζ, of ~ 0.25. However, with this proportional gain we would immediately saturate the op-amp. Therefore, we conclude that we would need more advanced compensator designs to most effectively control the system with our current hardware as discussed in the following sections.

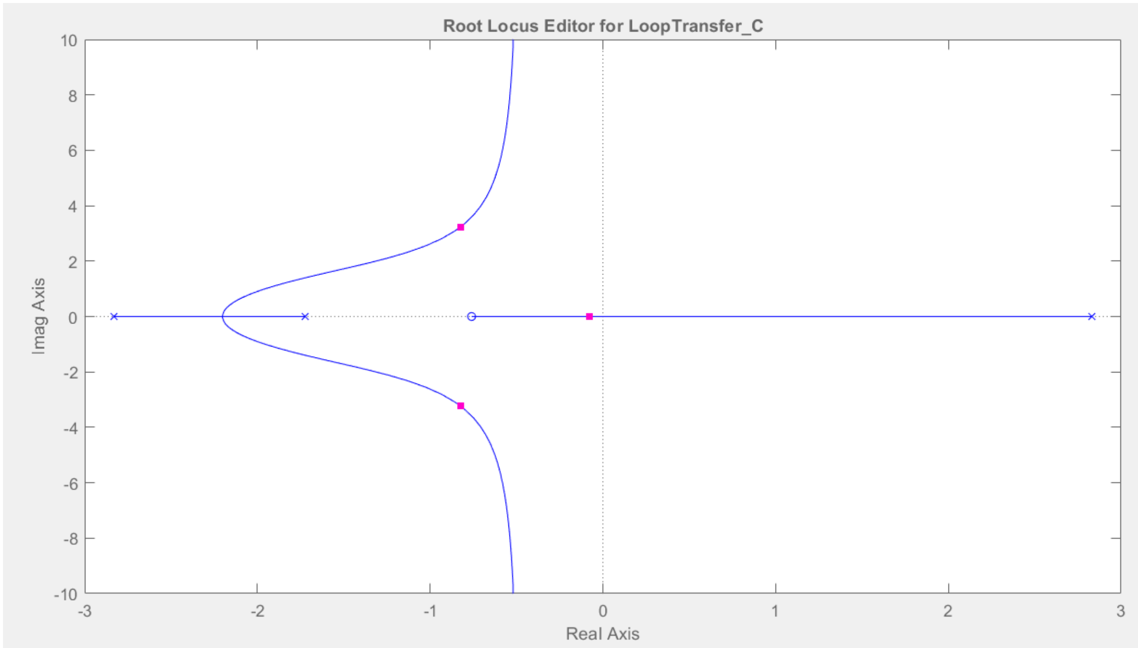

Figure 6: Uncompensated system root locus.

4 Design Specifications

In this section we will go through each of the listed specs in the "ENGS26 Final Project Handout" and define each one within the scope of our project. Looking at the aforementioned spec list, we are tasked with designing a system that can meet as many of the following specification:

4.1 Stability

We'll therefore declare that a properly stable system will exhibit a non-infinite steady-state error and a gain value that falls on the left-half plane

4.2 Steady-State Error

The steady-state error of the closed-loop system in response to a step reference input must be zero

4.3 Transient Response

We are looking for fast and well-damped transient response. Since the movement of the pendulum as it falls is on the scale of less than a second we will define a fast response as a settling time of less than 0.5secs.

Additionally, in order to reduce overshoot in the system we will define a well-damped response as one with less than 25% overshoot. Using the percent overshoot vs dampening ratio we can see that this relates to a dampening coefficient of roughly 0.4.

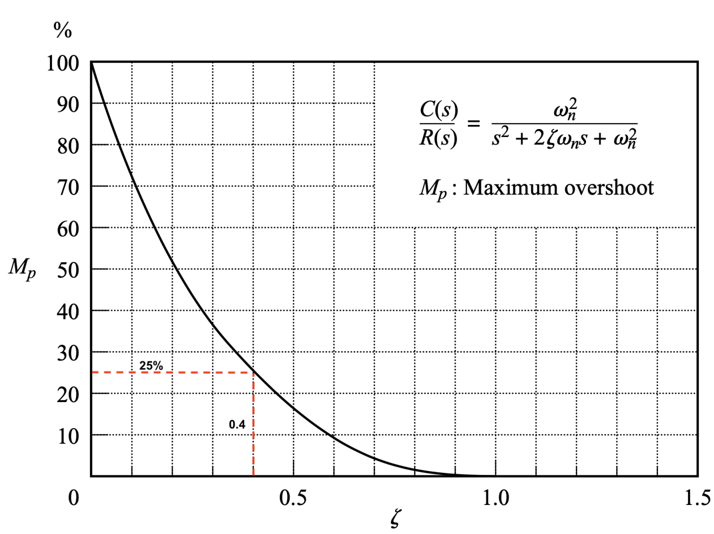

Figure 7: Overshoot vs Damping Coefficient Graph. With a percent overshoot of 25%

4.4 Closed-Loop Frequency Response

We will attempt to keep the maximum magnitude of the closed-loop response below 1dB of the DC value of the system. This value is directly tied to the dampening of the system, so by ensuring we have a low maximum magnitude (relative to the DC value) we know we'll have a relatively well-damped system.

4.5 Output Magnitude

In order to evaluate the output magnitude of the op-amp, we'll measure the control effort across the compensator. To not saturate the op-amp, we will need to keep the output voltage below 12V. In turn, this means we will try to keep the gain of the compensator below 5 to fight against saturation

4.6 Closed-Loop Bandwidth

The bandwidth of a system is defined as the frequency at which the magnitude of the closed-loop frequency response decreases by 3 dB below the value at DC. Since bandwidth size and response time are proportional to each other, we need to make sure that our system has a rather large bandwidth.

That being said, with a larger bandwidth we introduce higher frequency noise into our system and a less robust system overall. Therefore, when deciding our bandwidth specs o the system we decided that we need to have a system with a bandwidth of at least $30\frac{rad}{sec}$ and no more than $60\frac{rad}{sec}$.

4.7 System Robustness

The robustness of a system is defined as the capabilities of a system to withstand physical disturbances and uncertainties in our calculations. For our system we will try to design a system with enough of a buffer to withstand an external wind which we will model by blowing on the pendulum.

4.8 Gain & Phase Margin

As mentioned in the previous spec, we should have a buffers in our system in case of any unforeseen disturbances in our system. To quantify this spec in relation to the gain and phase margin we will try to achieve a phase margin of at least 30 degrees and a gain margin of 6dB. This will give us plenty of wiggle room or any external disturbances to the system.

5 Compensator Design

With our specifications laid out we worked with the *sisotool* function in MATLAB to generate a functioning compensator that can meet the requirements.

Looking at our current system, we can see that it is type 0. Therefore, we will need to introduce a pole at the origin to increase the type of the system and thereby meet the $e_{ss} = 0$ requirement for a unit step input.

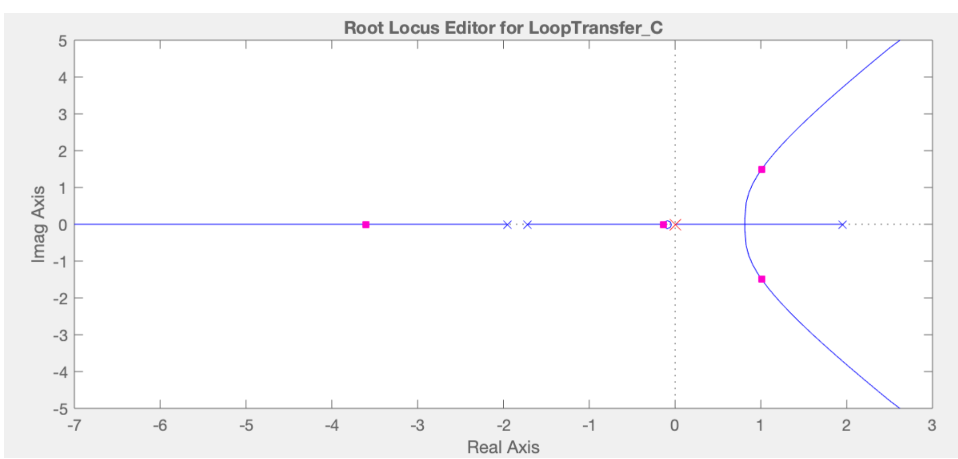

Figure 8: Root locus plot of the open-loop system with an integral control compensator; K = 1

Following suit, we must begin to pull the root locus plot over to the right half plane in order to bring stability to the system (without needing a ridiculous amount of gain). By adding a zero at -1 in the system we can have the root locus loop back into the left-hand side of the s-plane. This will allow us to find a reasonable gain for which we can stabilize the system.

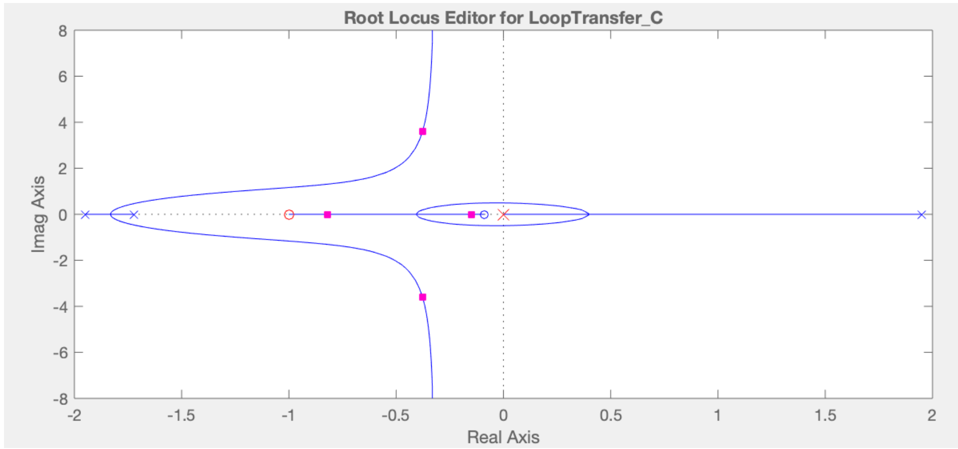

Figure 9: Root locus plot of the open-loop with compensator pole at the origin and zero at s = -1; K =1

This sadly, still doesn't meet the settling time and dampening ratio specifications that we set in the previous section.

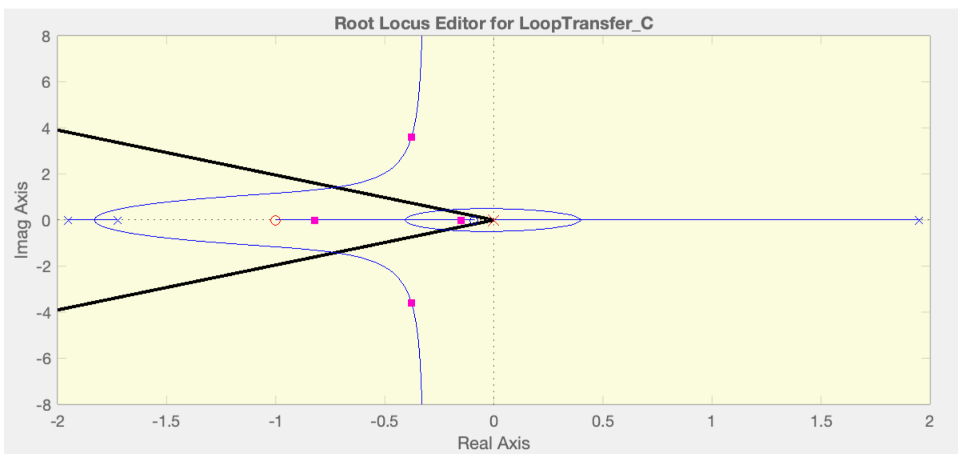

Figure 10: Root locus plot of the open-loop with compensator pole at the origin and zero at s = -1; K =1. Yellowed areas highlight sections that do not meet design specs mention in section 4

In order to meet these specs we'll need the root locus plot to loop back around to beyond our settling time spec. To circumvent this problem we added an additional pole at $s = -20$.

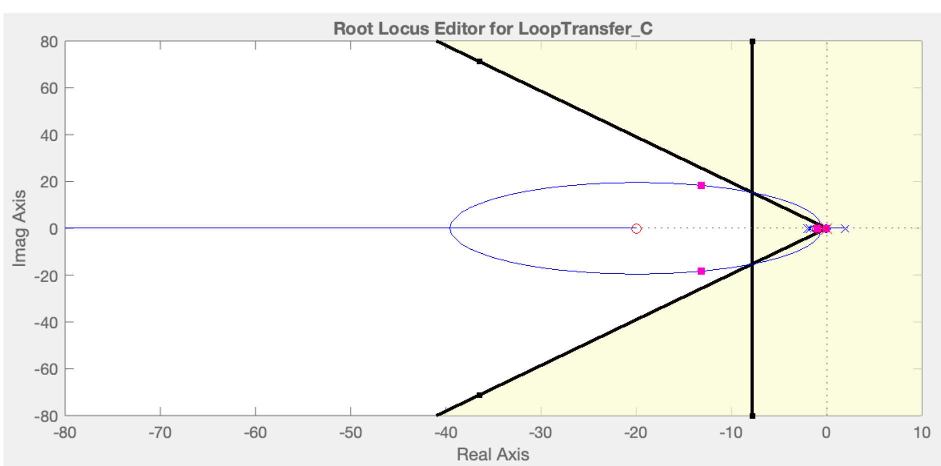

Figure 11: Root locus plot of the open-loop with compensator pole at the origin and zeros at s = -1 and s = -20; K =1. Yellowed areas highlight sections that do not meet design specs mention in section 4

Before moving forward with this compensator we were worried that having the zeros too far apart would induce vibrations in our pendulum and thereby affect our system as a whole. So we chose to iterate over this compensator by moving the zeros slightly closer to each other.

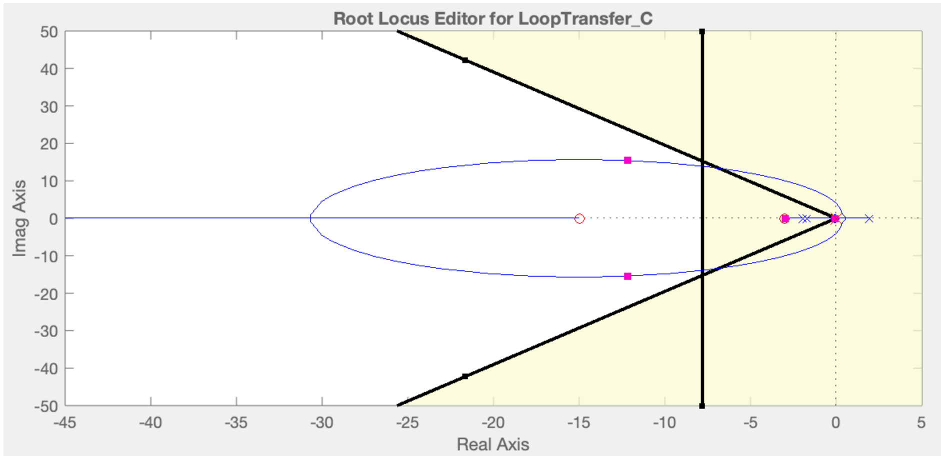

Figure 12: Root locus plot of the open-loop with compensator pole at the origin and zeros at s = -3 and s = -15; K =1.45. Yellowed areas highlight sections that do not meet design specs mention in section 4

This was done by moving the $s = -1$ zero to $s = -3$ and the $s = -20$ zero to $s = -15$. With these zeros (and keeping the integrator pole the same) we were able to closely replicate the specs we had in the earlier iteration without the threat of inducing vibration in our pendulum.

From this point in our compensator we were then able to chose a gain value to specifically meet our specs. As seen in figure 11, we gave some buffer between our our chosen gain value and our dampening ratio and settling time specs. This was to ensure that any error in our prior calculations won't completely break our compensated system.

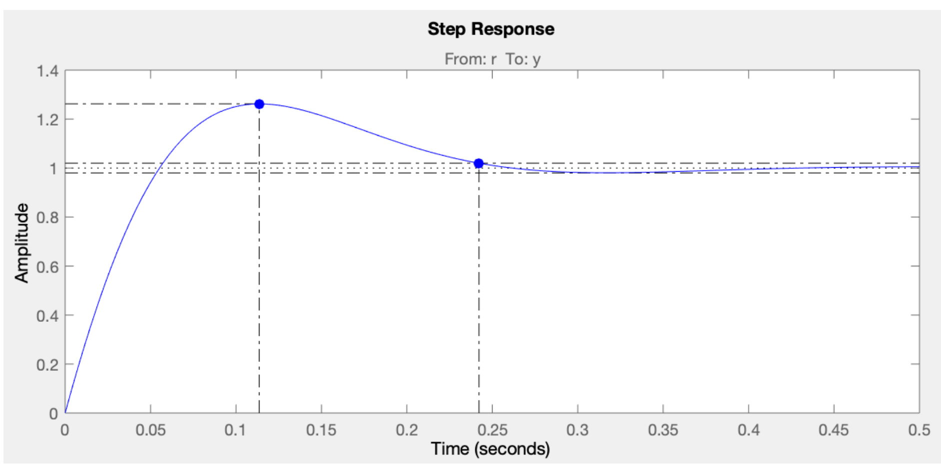

Figure 13: Unit step response of the open-loop compensated system. Settling time: 0.242sec Peak Overshoot: 26%

As we can see in the figure above our final decided gain value of 1.45 allowed us to reach a settling time of roughly 0.25 seconds with a percent overshoot of just over 20%.

Furthermore, when choosing our gain, we made sure to look at the bode plot of the open loop system to make sure that we met the stability characteristics of phase and gain margins as specified in section 4 4.

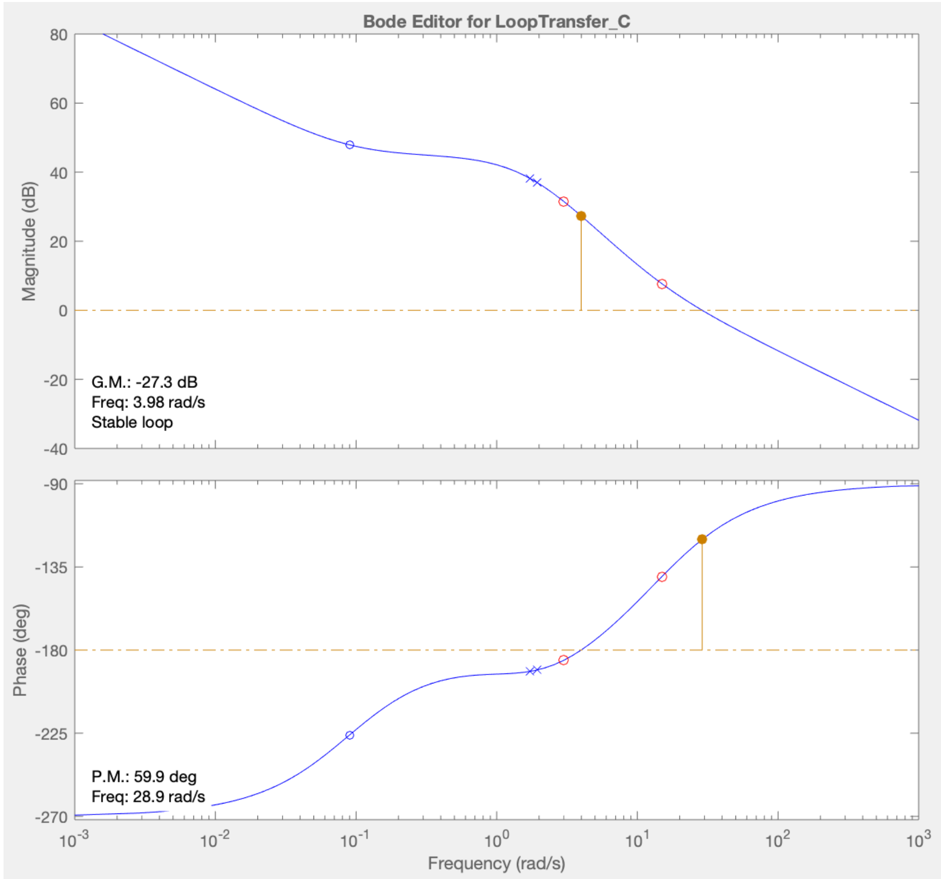

Figure 14: Magnitude and phase frequency plots. Phase Margin: 60 degrees Gain Margin:-30dB

Due to the nature of the unstable system, we weren't able to have a gain margin of at least 6dB, but we tried to limit it as much as we could. That being said, we easily met the +30 degree phase margin spec and padded it to have a total phase margin of +60 degrees.

6 Project Implementation

6.1 Compensator Implementation

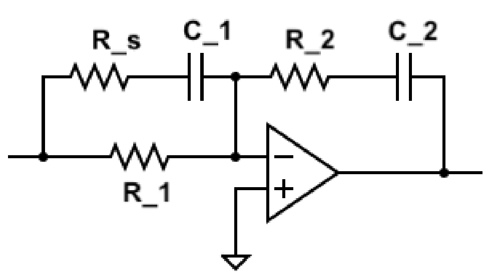

Figure 15: PID circuit diagram with labeled resistors and capacitors

With our PID controller defined we needed to devise a circuit that can accurately model it. For this, we used the PID circuit diagram provided to us in lab (Figure 4). Additionally, we used used the associated equations to solve for the needed resistor and capacitor values

Knowing that our low level gain was 1.45 (as determined in our PID controller design) we were able to solve for the ratio of our R_1 and R_2 resistors.

$$Low\,Level\,Gain = 1.45 = \frac{R_2}{R_1} \tag{16}$$

Looking at the zeros of our compensator, we knew that their location in the frequency domain would be at $3\frac{rad}{sec}$ and $15\frac{rad}{sec}$ for the $s = -3$ and $s = -15$ zeros respectively. Given the nature of RC circuits, we then related these frequency values to their decay constants, τ_1 and τ_2, by the following equations:

$$15\frac{rad}{sec} \rightarrow \tau_1 = 70ms = R_1 * C_1 \tag{17}$$

$$3\frac{rad}{sec} \rightarrow \tau_2 = 330ms = R_2 * C_2 \tag{18}$$

With equations 16, 17, 18 we were finally able to assign values for our circuit. Starting with equation 16 we easily assigned $R_2 = 14k\Omega$ and $R_1 = 10k\Omega$. We then plugged those values into equations 17 and 18 to get our corresponding capacitor values; $C_1 = 33\mu f$ and $C_2 = 4.7\mu f$.

Lastly, when implementing our PID controller we needed introduce an extra pole to keep the number of poles and zeros in our compensator equal. Given that our farthest main system pole is at $s = -15$ we chose to add this extra pole at $s = -100$ as to not have an effect on the main system. sing the same criteria as before, we relate this pole to a value for tau.

$$10ms = R_s * C_1 \tag{19}$$

Knowing our pre-calculated value for C_1 we were able to find a good value for R_s. This gives us to following final resistor and capacitor values for our PID controller:

$$R_1 = 10k\Omega$$
$$R_2 = 14.7k\Omega$$
$$R_s = 300\Omega$$
$$C_1 = 33\mu f$$
$$C_2 = 4.6\mu f$$

6.2 Inverter Implementation

Aside from the compensator we also needed to create an inverter to switch the polarity of the signal such that the motor will move in the correct direction to catch the pendulum. This can be done easily by feeding two resistors across an operational amplifier as follows:

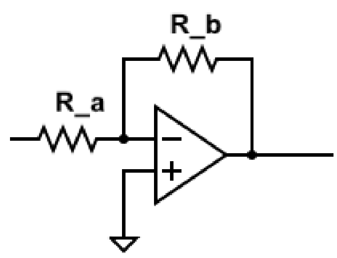

Figure 16: Inverter circuit diagram with labeled resistor values

In the diagram shown in figure 4 the output of the circuit is described by the equation

$$V_{out} = -\frac{R_b}{R_a} V_{in} \tag{20}$$

Therefore, in order to simply invert the signal provided by our compensator we just need to make sure that $R_a = R_b$. This will provide a -1 gain to the system and thereby invert the signal. It is also important to note that at the the inverter stage of the circuit, we can tweak the values of R_a and R_b to apply a non-unity gain to the system. This became useful to us for any quick adjustments we needed to make in the system.

6.3 Final Circuit

With all of our individual components and parts solved for we implemented the complete circuit on a bread board.

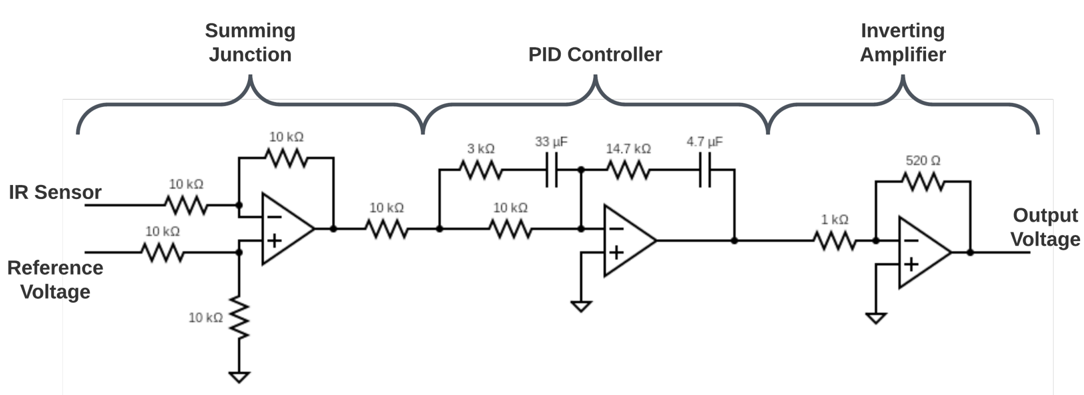

Figure 17: Full circuit diagram with calculated resistor and capacitor values. Starting on the left, the summing junction is created sing a differential amplifier, the output of the summing junction is then fed to the PID controller which conditions the signal, lastly the signal is sent through the inverter before being sent to the motor

Presented above is the complete schematic of the electronic system. Our reference voltage is combined with feedback voltage (IR Sensor Signal) in the summing junction (provided to us in the project). The summing junction in question is a differential amplifier circuit that will output the difference in the IR and reference signals.

from there, the signal is sent through the PID controller which will react to any changes in the signal by sending a response signal to the motor (ideally to catch wherever the pendulum is tipping).

But before reaching the motor, we invert the signal in the inverting amplifier in order to correct the polarity of the signal; making sure the motor runs in the right direction.

7 Acknowledgements

We would like offer a special thanks to ENGS26 Professor Laura Ray for the lessons and guidance provided in this class, as well as ENGS26 Lab Instructor Bob Barry for offering ever so helpful insight into all of our labs and projects.

May your lives be more **stable** and **robust** than any system we've compensated in your class :)

8 Appendix

Figure 18: Final working cart modeled in the final project

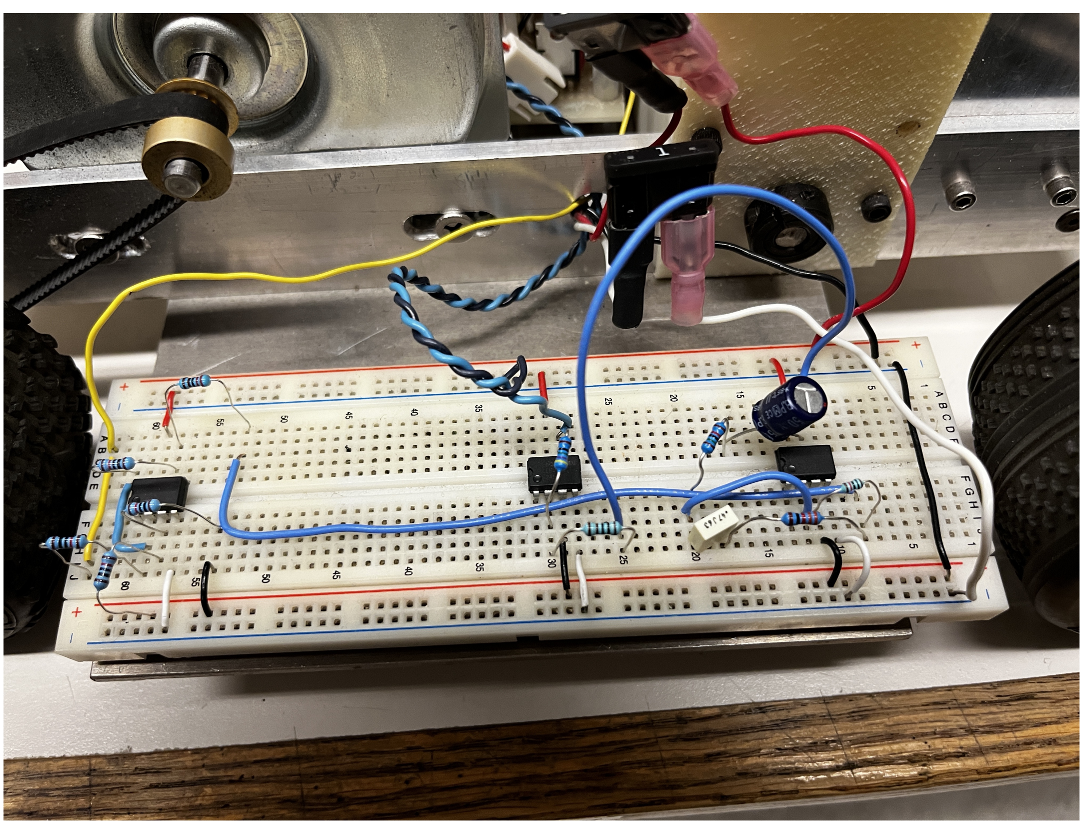

Figure 19: Final bread boarded circuit. From left to right: The summing junction, inverting op-amp, and PID contrtoller

References

[1] K. Ogata, *Modern control engineering*, 5th ed., ser. Prentice-Hall electrical engineering series. Instrumentation and controls series. Boston: Prentice-Hall, 2010.