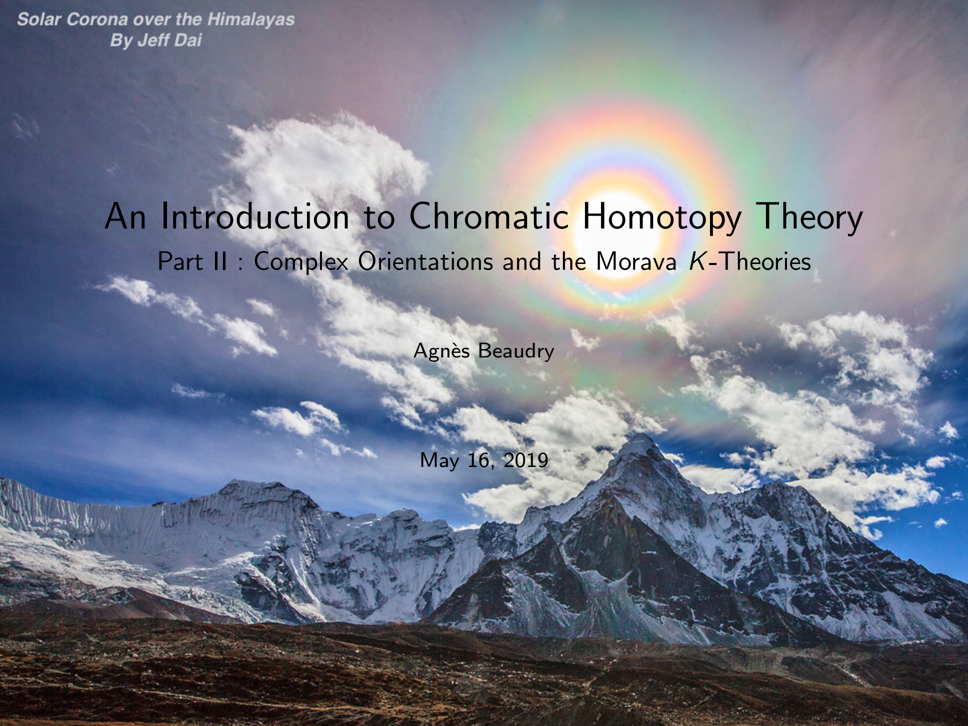

An Introduction to Chromatic Homotopy Theory

Part II : Complex Orientations and the Morava K-Theories

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Last Time

(1) Spectra Sp and the stable homotopy category $\mathcal{SH}$.

- $\mathcal{SH}$ is a closed symmetric monoidal triangulated category whose objects represent (co)homology theories.
- $\mathcal{SH}$ is obtained from Sp by inverting the $\pi_* = S^0_*$-isomorphisms.

(2) Bousfield Localization and E-local spectra $\mathcal{SH}_E$

- $\mathcal{SH}_E$ is obtained from Sp by inverting the E_*-isomorphisms.
- E-Local spectra Y: $E_*(Z) = 0 \implies [Z, Y] = 0$
- E-Localization $X \to L_E X$: E acyclic

$$\begin{tikzcd} X \arrow[r] \arrow[d, "E_*\text{-iso}"'] & Y \\ L_E X \arrow[ur, dotted, "\exists! \in \mathcal{SH}"'] & \end{tikzcd}$$

Y: E-local

$L_E X$: E-local

Part II – Complex Orientations and the Morava K-Theories

(1) Complex Orientations

(2) Formal Group Laws

(3) Height

(4) Landweber Exact Functor Theorem

(5) Morava K-Theories

(6) Chromatic Fracture Square

Ring Objects in $\mathcal{SH}$

A *homotopy ring spectrum* E is a ring object in $\mathcal{SH}$. In particular, E has a unit and a multiplication:

unit multiplication

$$S^0 \to E, \qquad E \wedge E \to E$$

which satisfy the diagrams of a ring object in $\mathcal{SH}$.

It is *homotopy commutative* if the following diagram commutes in $\mathcal{SH}$:

$$X \wedge Y \xrightarrow{\sigma} Y \wedge X$$

$$\begin{tikzcd}
E \wedge E \arrow[rr, "\sigma"] \arrow[dr] & & E \wedge E \arrow[dl] \\
& E &
\end{tikzcd}$$

For such E, $\pi_* E$ is a graded ring. It is graded commutative if E is commutative.

Complex Orientations

A *complex orientation* for a homotopy ring spectrum E is a class

$$x = x^E \in \widetilde{E}^2(\mathbb{C}P^\infty)$$

$S^2 = \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$

whose restriction under the map

$$x \in E^2(\mathbb{C}P^\infty) \to E^2(\mathbb{C}P^1) \cong E^2(S^2) \cong \pi_0 E \ni 1$$

suspension iso.

is the unit. $x \longmapsto 1$

Eilenberg-MacLane Spectra

Let $\gamma_1 \to \mathbb{C}P^\infty$ be the tautological line bundle. Recall that

$a+b$

$a \in H^n, \ b \in H^m$

$$H\mathbb{Z}^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[x]$$

where

$$x = x^{H\mathbb{Z}} \in H\mathbb{Z}^2(\mathbb{C}P^\infty) \longrightarrow H\mathbb{Z}^2(\mathbb{C}P^1) = H\mathbb{Z}^2(S^2) = \mathbb{Z}$$

is the first Chern class of γ_1. This is a complex orientation. In fact,

$$H\mathbb{Z}^*(\mathbb{C}P^\infty) \cong H\mathbb{Z}^*[\![x]\!].$$

$\sim$

homo. homo.

$(\mathbb{Z}[x]) = (\mathbb{Z}[\![x]\!])$

$H\mathbb{Z}^* = H\mathbb{Z}_{-*} = \pi_{-*} H\mathbb{Z} = \mathbb{Z}$

Complex K-Theory

Let $\beta \in \pi_2 K$ be the Bott class and $\boxed{[\gamma_1 - 1]}^{m} \in \widetilde{K}^0(\mathbb{C}P^\infty)$. Then

$$K^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[\beta^{\pm 1}][\![[\gamma_1 - 1]]\!]$$

The class

$$x = x^K = \beta^{-1}[\gamma_1 - 1] \in \widetilde{K}^2(\mathbb{C}P^\infty) \longrightarrow \widetilde{K}^2(\mathbb{C}P^1)$$

is a complex orientation. In fact,

$$\boxed{K^*(\mathbb{C}P^\infty) \cong \underbrace{K^*}_{\mathbb{Z}[\beta^{\pm 1}]}[\![x]\!].}$$

Complex Cobordism MU(n)

Adams Blue Book Stable ... Parts I & II

Note that $\mathbb{C}P^\infty \simeq BU(1)$. The zero section

$$\mathbb{C}P^\infty \to \text{Thom}(\gamma_1) =: MU(1)$$

is a homotopy equivalence and gives a class

$$x = x^{MU} \in [\mathbb{C}P^\infty, MU_2] = \boxed{\widetilde{MU}^2(\mathbb{C}P^\infty)}$$

"2nd space of MU"

which is a complex orientation. In fact,

$$MU^*(\mathbb{C}P^\infty) \cong MU^*[\![x]\!]$$

$MU_* = \mathbb{Z}[x_1, x_2, \ldots]$, $x_i \in M_{2i}$

MU^{-*}

Exercise : Chern Classes

For a complex oriented theory E with orientation $x = x_E$,

$$E^*((\mathbb{C}P^\infty)^m) \cong E^*[\![x_1, \ldots, x_m]\!] \qquad x_i \in \widetilde{E}^2((\mathbb{C}P^\infty)^m)$$

coefficients

and

$$E^*(BU) \cong E^*[\![c_1, c_2, \ldots]\!] \qquad c_i \in \widetilde{E}^{2i}(BU)$$

Hint. Use the Atiyah–Hirzebruch spectral sequence.

The c_i are the *Chern classes* of E. The map

$$E^*(BU) \to E^*(BU(1)) \cong E^*(\mathbb{C}P^\infty)$$

maps c_1 to x.

1st chern class.

Tensor Product of Line Bundles

$(\gamma_1 \times 1) \otimes (1 \times \gamma_1)$

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\otimes} \mathbb{C}P^\infty$$

classifying the tensor product of line bundles gives map

$$E^*[\![x]\!] \cong E^*(\mathbb{C}P^\infty) \xrightarrow{E^*(\otimes)} E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[\![x,y]\!]$$

$$\boxed{E^*(\otimes)(x) = F(x,y) =: x +_F y}$$

For

$$\mathbb{C}P^\infty \xrightarrow{i} \mathbb{C}P^\infty$$

classifying the $\mathbb{C}$-linear dual of γ_1,

$$\boxed{E^*(i)(x) =: i(x)}$$

$c_1^{HZ}(\ell_1 \otimes \ell_2) = c_1^{HZ}(\ell_1) + c_1^{HZ}(\ell_2)$

In fact,

$$c_1(\ell_1 \otimes \ell_2) = F(c_1(\ell_1), c_1(\ell_2))$$

Properties of $\otimes$ imply:

(1) $(x +_F y) +_F z = x +_F (y +_F z)$ associative

(2) $x +_F y = y +_F x$ commutative $F(x,y) = F(y,x)$

(3) $x +_F 0 = x,\ 0 +_F y = y.$ $\ell \otimes \mathbb{C} = \ell$

(4) $x +_F i(x) = 0$ $\ell \otimes \ell^* = \mathbb{C}$

Formal Group Laws $x \in E^2(\mathbb{CP}^\infty) \rightsquigarrow -2$

For $R_* = R^{-*}$ a graded ring and $x, y \in (R_*[[x,y]])_{-2}$. Define a category

$$FGL(R_*)$$

Objects. A formal group law (fgl) over R_* is a power series

$$x +_F y = F(x,y) \in (R_*[[x,y]])_{-2}$$

satisfying the properties (1)-(4).

Morphisms. A morphism $f \colon F \to G$ of fgls is a power series

$$f(x) \in (xR_*[[x]])_{-2} \quad \text{such that} \quad f(x +_F y) = f(x) +_G f(y).$$

$f(F(x,y)) = G(f(x), f(y))$

The identity is x.

Base Change. If $\phi \colon R_* \to S_*$ is a ring homomorphism, applying ϕ to coefficients gives a functor

$$\phi \colon FGL(R_*) \to FGL(S_*)$$

$c_1(\ell_1 \otimes \ell_2) = c_1(\ell_1) + c_1(\ell_2)$

- $F_{H\mathbb{Q}}(x,y) = x + y$ is the *additive* fgl.
- $F_K(x,y) = \beta^{-1}((\beta x + 1)(\beta y + 1) - 1)$ is the *multiplicative* fgl.

$c_1(\ell) = (\ell - 1)\beta^{-1}$ $\quad = x + y + \beta x y$

Lazard and Quillen's Theorem – Algebra

MU carries the *universal* formal group law: (1) – (4)

$$\boxed{F_{MU}(x,y)} = \sum_{i,j} a_{i,j} x^i y^j \quad \leftarrow$$

over the Lazard ring

$a_{10} = 1 = a_{01}$

$$MU_* \cong L \cong \mathbb{Z}[a_{ij} : i,j \geqslant 0]/I$$

eg $a_{00} = 0$

where I is ideal of relations required for F_{MU} to be a fgl.

$$MU_* \cong \mathbb{Z}[x_1, x_2, x_3, \ldots], \qquad F_{MU}(x,y) = x + y + x_1 xy + \ldots$$

Quillen $\quad L^* \quad x_i \in \pi_{2i}$

Universality. For a $F \in FGL(R_*)$, there is a homomorphism $\phi \colon MU_* \to R_*$ such that

$$\phi F_{MU} = F.$$

$FGL(R_*) = \mathrm{Hom}(MU_*, R_*)$

Quillen's Theorem – Topology

For E complex oriented with fgl F_E, there is a map of homotopy ring spectra $f \colon MU \to E$ such that

$$(\pi_* f)F_{MU} = F_E(x,y)$$

Adams Part I A.H.S.S.

n-Series

For any $n \in \mathbb{N}$ and formal group law F,

$$[n]_F(x) = \underbrace{x +_F \ldots +_F x}_{n\text{-times}} = F(x, F(x, \cdots F(x,x))$$

is a morphism $[n]_F : F \to F$. This is called the n-series.

For example:

- $F_{H\mathbb{Q}}(x, y) = x + y$ so
$$[n]_{F_{H\mathbb{Q}}}(x) = nx$$
- $F_K(x, y) = \beta^{-1}((\beta x + 1)(\beta y + 1) - 1)$ so
$$[n]_{F_K}(x) = \beta^{-1}((\beta x + 1)^n - 1)$$

Fix p a prime. Let $v_n \in MU_{2(p^n-1)}$ be the coefficient of x^{p^n} in

$$[p]_{F_{MU}}(x) = \underbrace{x +_F \dots +_F x}_{p\text{-times}} = px + \dots + v_1 x^p + \dots + v_n x^{p^n} + \dots .$$

$$v_n = \begin{cases} \overset{= v_0}{p} & n = 0 \\ x_{p^n-1} & \bmod (p, x_1, \dots, x_{p^n-2}) \end{cases}$$

Height

Let $F \in FGL(R_*)$ be classified by $\phi \colon \overset{v_n}{MU_*} \to R_*$. If $\phi(v_n)$ is a unit in R_*, then F has *height* $\leqslant n$ at p. If in addition $\phi(v_i) = 0$ for $0 \leqslant i < n$, then F has *height* $= n$.

- $F_{H\mathbb{Q}}$ has height 0:

$$[p]_{F_{H\mathbb{Q}}}(x) = px = v_0 x. \qquad p \in \mathbb{Q}$$

- F_K has height $\leqslant 1$:

$$\rightsquigarrow \quad [p]_{F_K}(x) = px + p(\dots) + \beta^{p-1}x^p.$$

$$K_* = \mathbb{Z}[\beta^{\pm 1}]$$

- $K\mathbb{Z}/p = K \wedge S\mathbb{Z}/p$ has height $= 1$:

$$[p]_{F_{K\mathbb{Z}/p}}(x) = \beta^{p-1}x^p.$$

For a ring homomorphism $\phi \colon MU_* \to R_*$ consider the functor $CW_+ \to \mathrm{Ab}$:

$$X \mapsto \boxed{R_* \otimes_{MU_*} \widetilde{MU}_*(X).}$$

This is a homology theory provided that Exactness holds.

Landweber Exact Functor Theorem

(1) If for every p and $n \geqslant 0$,

$$R_*/(p, v_1, \ldots, v_{n-1}) \xrightarrow{v_n} R_*/(p, v_1, \ldots, v_{n-1})$$

regular sequence.

is injective, then

$$X \mapsto R_* \otimes_{MU_*} MU_*(X)$$

is a homology theory.

(2) If E is complex oriented and $MU_* \to E_*$ satisfies (1), then

$$E_*(X) \cong E_* \otimes_{MU_*} MU_*(X).$$

Landweber exact Theory

Landweber Exact Functor Theorem

Localization w resp to Certain ... Ravenel

$L_{v_n^{-1}BP} = L_{E(n)}$

(1) If for every p and $n \geqslant 0$,

$$R_*/(p, v_1, \dots, v_{n-1}) \xrightarrow{v_n} R_*/(p, v_1, \dots, v_{n-1})$$

← check

is injective, then

$$X \mapsto R_* \otimes_{MU_*} MU_*(X)$$

is a homology theory.

(2) If E is complex oriented and $MU_* \to E_*$ is as in (1), then

$$E_*(X) \cong E_* \otimes_{MU_*} MU_*(X).$$

v_i unique mod $(p, v_1, \dots, v_{i-1})$

BP and Johnson-Wilson Spectra

$\pi_* MU_{(p)} = \mathbb{Z}_{(p)}[x_1, x_2, \dots]$

Fix p

- The Brown-Peterson BP is the spectrum coming from

$$MU_* \to BP_* = \pi_* MU_{(p)}/(x_i, i \neq p^k - 1) \cong \mathbb{Z}_{(p)}[v_1, v_2, v_3, \dots]$$

In fact, $MU_{(p)}$ splits as a wedge of suspensions of BP.

$MU_{(p)} \simeq \bigvee \Sigma^? BP$

- The Johnson-Wilson Spectrum $E(n)$ is the spectrum coming from

$$MU_* \to E(n)_* = v_n^{-1} BP_*/(v_{n+1}, \dots) \cong \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$$

$v_{n+2}, \dots$

x^{p^n}

The theory $E(n)$ has height $\leqslant n$.

Baas-Sullivan Construction

Morava K-Theory   Lecture 22   Lurie

Fix p and $n \geqslant 1$. The *n'th Morava K-theory* spectrum is constructed from $MU_{(p)}$ by killing

$$(p, x_1, x_2, \ldots, x_{p^n-2}, x_{p^n}, x_{p^n+1} \ldots)$$

and inverting $v_n = x_{p^n-1}$. It satisfies

$$\pi_* K(n) = K(n)_* \cong \mathbb{F}_p[v_n^{\pm 1}] \leftarrow \text{graded field}$$

and is a homotopy associative ring spectrum. It is complex oriented and

$$[p]_{F_{K(n)}}(x) = v_n x^{p^n} + \ldots$$

so $F_{K(n)}$ has height $= n$ at p.

- $K(0) \simeq H\mathbb{Q}$
- $K(1) \simeq K\mathbb{Z}/p^{C_{p-1}}$ is the Adams Summand
- $K(\infty) \simeq H\mathbb{Z}/p$
- $K(2)$ is $K(2)$

Note. These are not Landweber Exact Theories.

$\pi_* K\mathbb{Z}/p = \mathbb{F}_p[\beta^{\pm 1}]$

$\beta \in \pi_2$

$\pi_* K(1) = \mathbb{F}_p[v_1^{\pm 1}]$

$v_1 \in \pi_{2(p-1)}$

$v_1 = \beta^{p-1}$

- A spectrum is a *(p-local) finite spectrum* if it is isomorphic up to a shift in $\mathcal{SH}$ to (the p-localization) $\Sigma^\infty X$ for X a finite CW complex.
- E is a *field* if $E_* X$ is free over E_* for any X.

Some Properties of $K(n)$

- $K(n)$ is a field and any field E is a wedge of suspensions of $K(n)$'s.
- $K(n)_*$ has a Künneth Isomorphism:

$$K(n)_*(X \wedge Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y). \qquad \forall\, X, Y$$

Rav. Loc...

- $K(n) \wedge K(m) \simeq *$ if $m \neq n$
- Let $0 < n < \infty$. If X is finite, then

$$K(n)_* X = 0 \quad \Longrightarrow \quad K(n-1)_* X = 0.$$

Let $\mathcal{C}_0 = \mathcal{C}_{p,0} \subseteq Sp$ be the full subcategory of p-local finite spectra and

$$\mathcal{C}_n = \mathcal{C}_{p,n} = \{X \in \mathcal{C}_0 : K(n-1)_* X = 0\}$$

Then, $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$. A spectrum has *type n* if it is in $\mathcal{C}_n \backslash \mathcal{C}_{n+1}$.

An isomorphism $f\colon F \to G$, given by $f(x) \in xR_*[\![x]\!]$, is *strict* if $f(x) = x + \ldots$.
Let $\widetilde{FGL}(R_*)$ have objects formal group laws over R_* and morphisms strict isos:

$$\widetilde{FGL}\colon \text{Graded Rings} \longrightarrow \text{Groupoids}$$

Quillen's Theorem – Continued

$$MU_*MU \cong MU_*[b_1, b_2, b_3, \ldots] \qquad b_i \in MU_{2i}MU$$

where

$$f_{MU}(x) = x + \sum_{i \geqslant 1} b_i x^{i+1}$$

represents the universal strict isomorphism. That is,

$$\phi\colon MU_*MU \to R_*$$

is a the data:

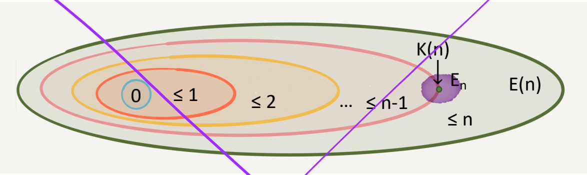
$$F, G \in FGL(R_*) \qquad f\colon F \xrightarrow[\simeq]{\text{s. iso.}} G$$

In particular, $\widetilde{FGL}$ is representable:

$\mathcal{M}_{FGL}$

$$\mathrm{obj}(\widetilde{FGL}(R^*)) \cong \mathrm{Hom}_{\mathrm{Rings}}(MU_*, R_*)$$
$$\mathrm{mor}(\widetilde{FGL}(R^*)) \cong \mathrm{Hom}_{\mathrm{Rings}}(MU_*MU, R_*)$$

The Moduli Stack Picture $\mathcal{M}_{FGL}$ at p



$$\begin{tikzcd}
\mathcal{M}_{FGL}^{\leqslant n} & \widehat{\mathcal{M}_{FGL}^{=n}} \arrow[l] \\
\mathcal{M}_{FGL}^{\leqslant n-1} \arrow[u] & \mathcal{M}_{FGL}^{\leqslant n-1} \cap \widehat{\mathcal{M}_{FGL}^{=n}} \arrow[l] \arrow[u]
\end{tikzcd}$$

Chromatic Fracture Square (Hopkins–Ravenel)

Let

$$L_n := L_{K(0)\vee\ldots\vee K(n)}$$

Then

$$L_n \simeq L_{E(n)} \simeq L_{v_n^{-1}MU}.$$

There is a natural transformations

$$L_nX \to L_{n-1}X \qquad L_nX \to L_{K(n)}X$$

In fact, for any X, there is a homotopy pull-back:

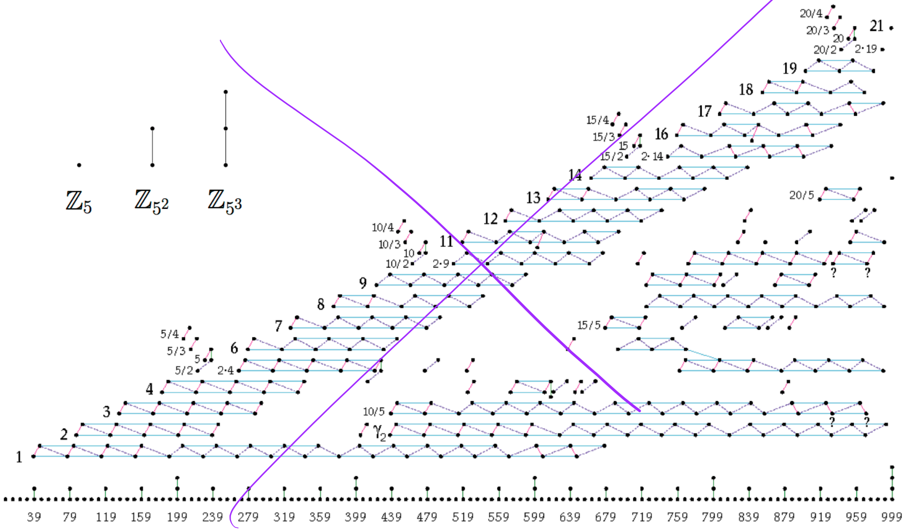
$$\begin{tikzcd}
L_nX \arrow[r] \arrow[d] & L_{K(n)}X \arrow[d] \\
L_{n-1}X \arrow[r] & L_{n-1}L_{K(n)}X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M}_{FGL}^{\leqslant n} & \widehat{\mathcal{M}_{FGL}^{=n}} \arrow[l] \\
\mathcal{M}_{FGL}^{\leqslant n-1} \arrow[u] & \mathcal{M}_{FGL}^{\leqslant n-1} \cap \widehat{\mathcal{M}_{FGL}^{=n}} \arrow[l] \arrow[u]
\end{tikzcd}$$


Jack Morava
Thank you!

Computation by Ravenel
Illustration by Hatcher
Edits by Behrens

$$\pi_* S_{(5)}$$

$p = 5$



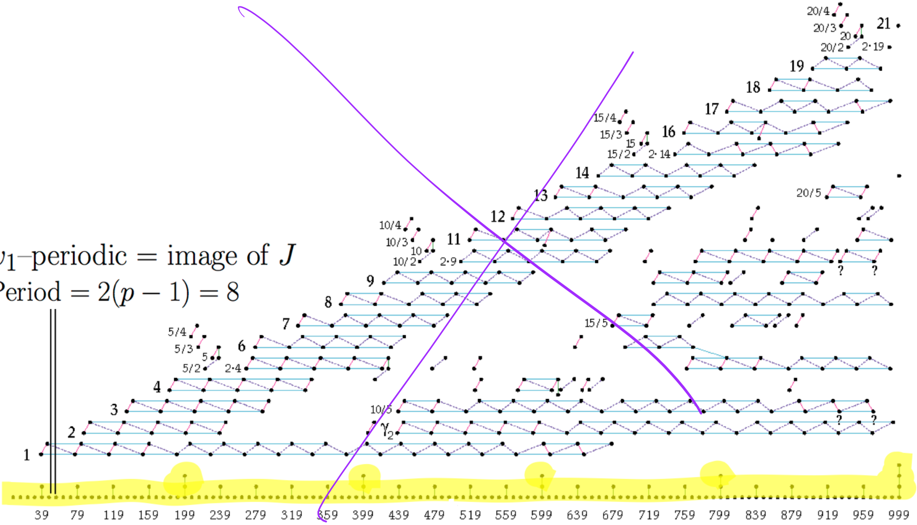
Computation by Ravenel
Illustration by Hatcher
Edits by Behrens

$\pi_* S_{(5)}$

$p = 5$

v_1–periodic = image of J

Period $= 2(p-1) = 8$



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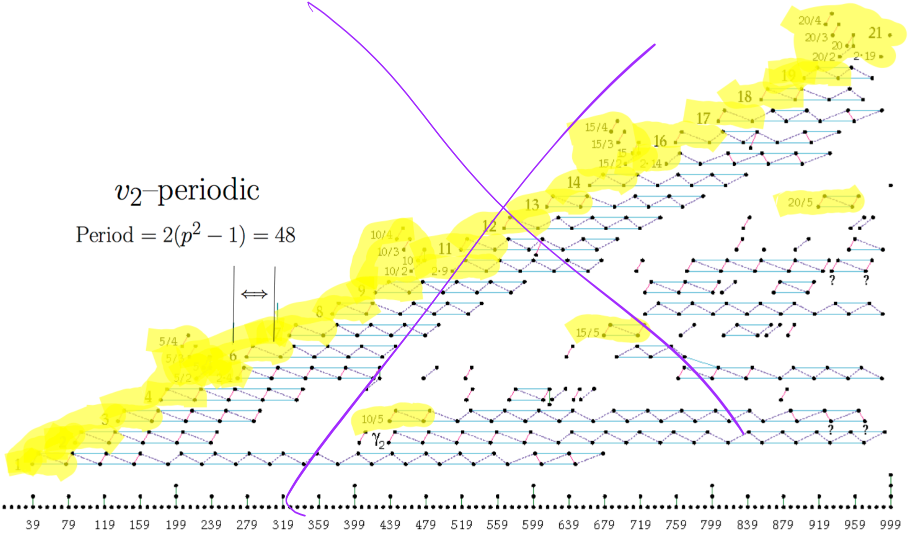
$\pi_* S_{(5)}$

$p = 5$



Computation by Ravenel
Illustration by Hatcher
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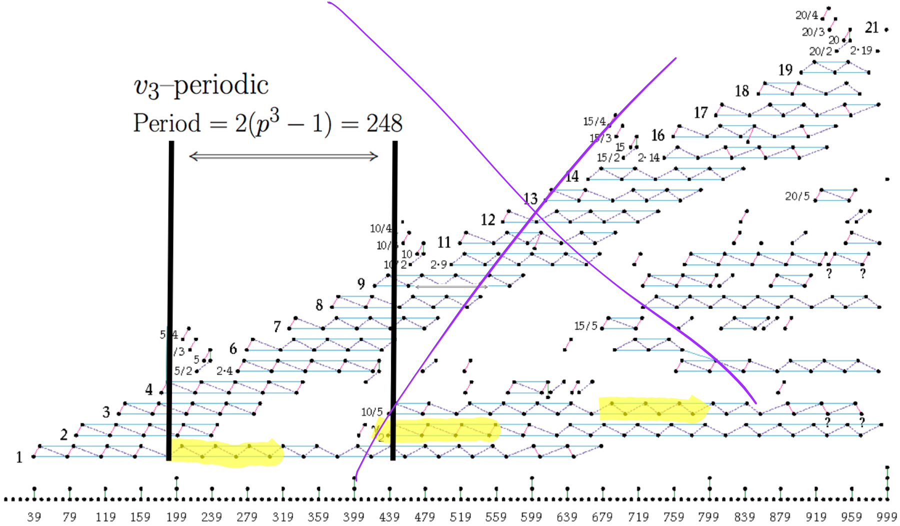
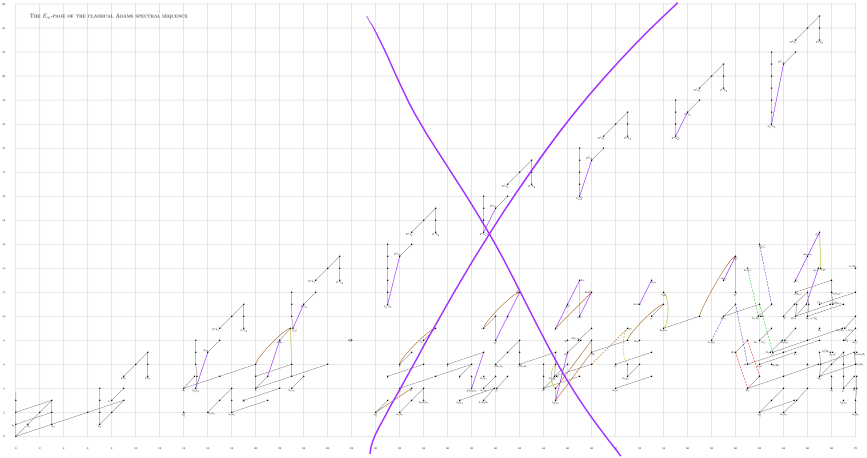
$\pi_* S_{(5)}$

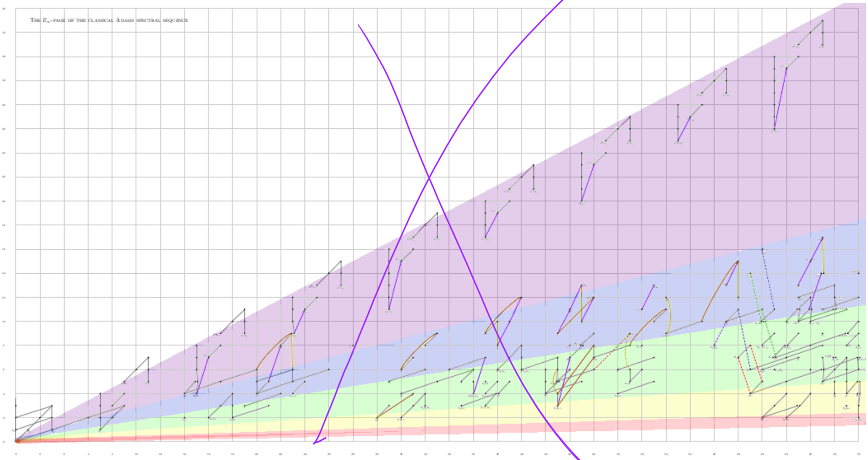
$p = 5$



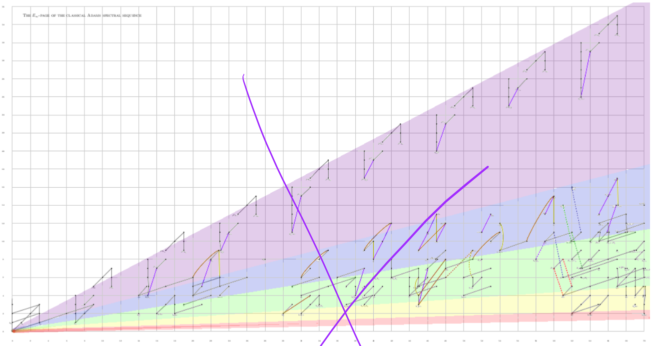
$\pi_* S_{(2)}$ (Illustration by Isaksen)



$\pi_* S_{(2)}$ (Illustration by Isaksen)



$\pi_* S_{(2)}$ (Illustration by Isaksen)



Telescope Conjecture (Ravenel)

The first n-rays are detected by $L_n S^0$.

Chromatic Splitting Conjecture (Hopkins)

The gluing data for the chromatic layers is simple.