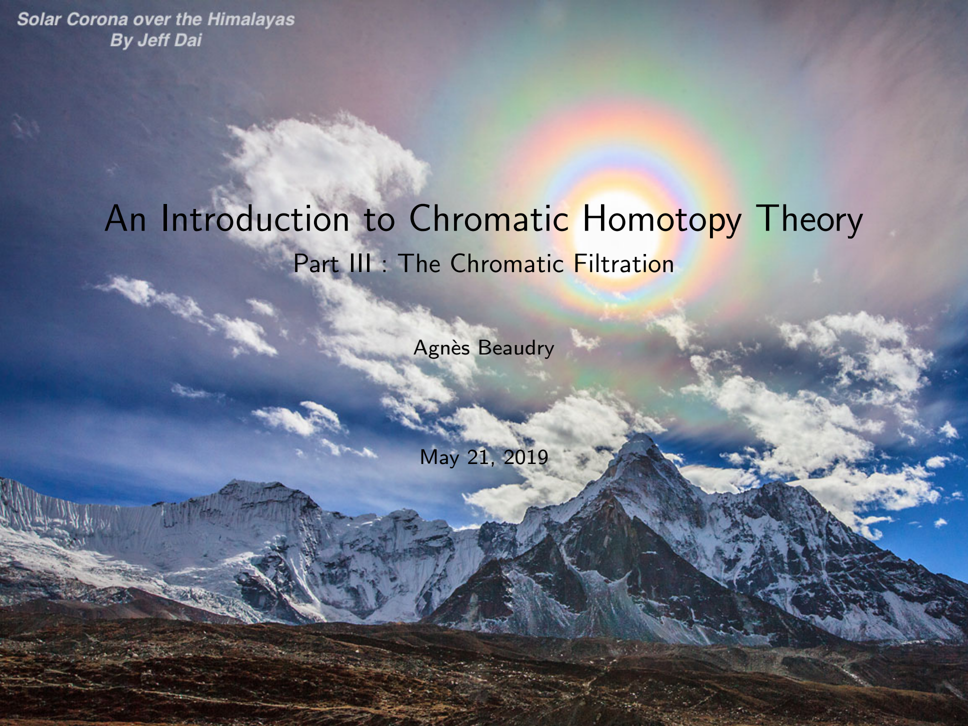

An Introduction to Chromatic Homotopy Theory

Part III : The Chromatic Filtration

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Last Time

(1) Complex Oriented Theories : Chern Classes and a formal group law

$$F_E(x, y) \in E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[\![x, y]\!]$$

$c_1(\ell_1 \otimes \ell_2) = F_E(c_1(\ell_1), c_1(\ell_2))$

(2) Universal Formal Group Law : carried by MU

(3) Elements v_n : For fixed p, $v_n \in MU_{2(p^n-1)}$ is the coefficient of x^{p^n} in

$$[p]_{F_{MU}}(x) = px + \ldots + v_1 x^p + \ldots + v_n x^{p^n} + \ldots .$$

$= x +_F \cdots +_F x$ p-times

(4) Height : A formal group law classified by

$$\phi \colon MU_* \to R_*$$

has height $\leqslant n$ if $\phi(v_n)$ is a unit and $= n$ if also $\phi(v_i) = 0$, $0 \leqslant i < n$.

(5) Landweber Exact Functor Theorem : Construct new theories

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, \overset{v_{n-1}}{\cancel{v_n}}, v_n^{\pm 1}] \text{ of height } \leqslant n.$$

(6) Morava K-theories : $K(n)$ for $0 \leqslant n \leqslant \infty$

- $K(n)_* \cong \mathbb{F}_p[v_n^{\pm 1}]$ of height $= n$.
- $K(n)_* K(m) = 0$ if $m \neq n$,
- X finite and $K(n)_* X = 0 \implies K(n-1)_* X = 0$.

Ravenel

Part III – The Chromatic Filtration

(1) Chromatic Fracture Square

(2) Algebraic Chromatic Filtration

(3) Thick Subcategories and Type n-Spectra

(4) Geometric Chromatic Filtration

(5) Telescope Conjecture

Ravenel 1984
Localization...

For

$$E = \{E_n,\ \omega_n^E : E_n \to \Omega E_{n+1}\} \qquad F = \{F_n, \omega_n^F : F_n \to \Omega F_{n+1}\}$$

the spectrum $E \vee F$ is the spectrification of $E_n \vee F_n$ with maps

$$E_n \vee F_n \xrightarrow{\omega_n^E \vee \omega_n^F} \Omega E_{n+1} \vee \Omega F_{n+1} \xrightarrow{\cong} \Omega(E_n \vee F_n)$$

The operation $\vee$ is both the product and the coproduct in $\mathcal{SH}$:

- $(E \vee F)_*(X) \cong E_*(X) \oplus F_*(X)$
- $E^*(X \vee Y) \cong E^*(X) \times E^*(Y)$
- Etc.

$$\underline{L_n X} := L_{\underbrace{K(0)\vee\dots\vee K(n)}} X$$

heuristic

Remark. Roughly, and this can be made precise for *MU*-module spectra:

$MU \wedge X$

- L_n is like inverting "v_n"
- $L_{K(n)}$ is like inverting "v_n" and completing at $(p, v_1, \dots, v_{n-1})$.

For every spectrum X,

- $L_n X \simeq L_{E(n)} X \simeq L_{v_n^{-1} MU_{(p)}} X$
- If $m < n$, then $L_{K(n)} L_m X = 0$

$$\begin{tikzcd} \Sigma^{2(p^n-1)} MU_{(p)} \arrow[r, "v_n \wedge 1"] \arrow[rr, bend right, "v_n"'] & MU_{(p)} \wedge MU_{(p)} \arrow[r, "m"] & MU_{(p)} \end{tikzcd}$$

There are natural transformations

$L_{E \vee F} \to L_E$

$$L_n X \to L_{n-1} X \qquad L_n X \to L_{K(n)} X$$

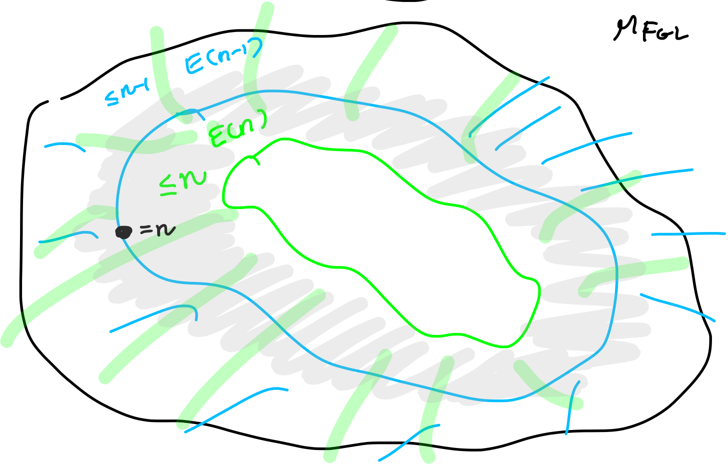
and a homotopy pull-back, the *Chromatic Fracture Square*:

$$\begin{tikzcd} L_n X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{K(n)} X \arrow[d] \\ L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X \end{tikzcd}$$

$v_n^{-1} MU_{(p)} = \operatorname{hocolim}\left(MU_{(p)} \xrightarrow{v_n} MU_{(p)} \xrightarrow{v_n} \cdots \right)$

$$\begin{tikzcd}
L_n X \arrow[r] \arrow[d] & L_{K(n)} X \arrow[d] \\
L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M}^{\leqslant n}_{FGL} & \widehat{\mathcal{M}}^{=n}_{FGL} \arrow[l] \\
\mathcal{M}^{\leqslant n-1}_{FGL} \arrow[u] & \mathcal{M}^{\leqslant n-1}_{FGL} \cap \widehat{\mathcal{M}}^{=n}_{FGL} \arrow[l] \arrow[u]
\end{tikzcd}$$

$\mathcal{M}_{FGL}$



Some Theorems of Hopkins and Ravenel

Localization Theorem. For every spectrum X,

$$L_nX \wedge MU_{(p)} \simeq X \wedge L_nMU_{(p)}$$

If X is $v_{n-1}^{-1}MU_{(p)}$-acyclic, then

$E(n-1)_* X = 0$
$(K(0) \vee \cdots \vee K(n-1))_* X = 0$

$$(MU_{(p)})_*(L_nX) = v_n^{-1}(MU_{(p)})_*(X).$$

Smash Product Theorem. For every spectrum X,

$$L_nX \simeq X \wedge L_nS^0$$

smashing localization

Chromatic Convergence Theorem. For every finite spectrum X,

$$X_{(p)} \simeq \operatorname{holim}_n L_nX$$

Orange book Ch8

Algebraic Chromatic Filtration

For a p-local spectrum X, let $\mathscr{C}_0^a(X) = \pi_*X$ and, for $n \geq 1$,

$$\mathscr{C}_n^a(X) = \ker(\pi_*X \to \pi_*L_{n-1}X).$$

Thick Subcategory

A full subcategory $\mathcal{C}$ of $\mathcal{SH}$ is *thick*:

- If $\mathcal{C}$ is closed under isomorphisms. weak equivalences
- If $X \xrightarrow{f} Y \to C_f \to \Sigma X$ is a distinguished triangle and two of $\{X, Y, C_f\}$ is in $\mathcal{C}$, then so is the third. cofibrations, extensions
- If $X \vee Y$ is in $\mathcal{C}$, then so are X and Y. retracts

Examples.

- Finite spectra $\mathcal{SH}^c$. compact.
- E-local spectra and E-acyclic spectra
- A full subcategory of a thick subcategory

Thick Tensor Ideals

A thick subcategory $\mathcal{I} \subseteq \mathcal{C}$ is a *thick tensor ideal* if Spec($\mathcal{C}$)

- zero $* \in \mathcal{I}$
- If $X \in \mathcal{I}$ and $Y \in \mathcal{C}$, then $Y \wedge X \in \mathcal{I}$. ↳ Balmer

A proper thick tensor ideal $\mathcal{P} \subsetneq \mathcal{C}$ is *prime* if

- $X \wedge Y \in \mathcal{P}$ implies that $X \in \mathcal{P}$ or $Y \in \mathcal{P}$.

Thick Tensor Ideals of p-Local Spectra

p-local finite spectra.

Fix a prime p. For $n \geqslant 1$, let

$$\mathcal{P}_n = \{X \in \mathcal{SH}^c_{(p)} : K(n-1)_* X = 0\}.$$

Exercise. These are thick tensor prime ideals of $\mathcal{SH}^c_{(p)}$.

Classification of Thick Subcategories (Ravenel/Mitchell/Hopkins-Smith)

$K(n)_* X = 0 \Rightarrow K(n-1)_* X = 0$

Let $\mathcal{P}_0 = \mathcal{SH}^c_{(p)}$. Then $\supsetneq$ Ravenel $\supsetneq$ Mitchell

$$\mathcal{P}_0 \supsetneq \mathcal{P}_1 \supsetneq \ldots \supsetneq \mathcal{P}_n \supsetneq \mathcal{P}_{n+1} \supsetneq \ldots \supsetneq *$$

$K(n)$ are the primes of spectra

If $\mathcal{C} \subseteq \mathcal{SH}^c_{(p)}$ is a thick subcategory, then $\mathcal{C} = \mathcal{P}_n$ for some $n \geqslant 0$.

Thick subcategory Thm.

Type n Spectra

A finite p-local spectrum has *type n* if it is in $\mathcal{P}_n \backslash \mathcal{P}_{n+1}$. $\left\{ \begin{array}{l} K(n)_* X \neq 0 \\ K(n-1)_* X = 0 \end{array} \right.$

Example.

- $S^0_{(p)}$ has type 0 since $\overset{H\mathbb{Q}}{K(0)}_*(S^0_{(p)}) \neq 0$
- $S\mathbb{Z}/p$ has type 1 since $K(0)_* S\mathbb{Z}/p = 0$ and $K(1)_* S\mathbb{Z}/p \neq 0$.

$S^0 \xrightarrow{p} S^0 \to S\mathbb{Z}/p$

v_n^{-1} v_n^{-1} and complete $(p, \dots, v_{n-1})$ " = " 0

Exercise

Use the Chromatic Fracture Square to show $L_n X \simeq L_{K(n)} X$ for type n spectra.

Nilpotence I (Devinatz-Hopkins-Smith)

For a self map $f\colon \Sigma^d X \to X$, if f^k, given by

$$\Sigma^{kd} X \xrightarrow{\Sigma^{(k-1)d} f} \Sigma^{(k-1)d} X \longrightarrow \dots \xrightarrow{\Sigma^d f} \Sigma^d X \xrightarrow{f} X$$

is null for some $k > 0$, then f is *nilpotent*.

The *nilpotence theorem* states that for $X \in \mathcal{SH}^c_{(p)}$:

$$f \text{ is nilpotent if and only if } K(n)_* f = 0 \text{ for all } 0 \leqslant n < \infty$$

Nilpotence II Hopkins - Smith.

v_n-self maps

For $n \geqslant 0$, a *v_n-self map* for $X \in \mathcal{SH}^c_{(p)}$ is a map

$$f\colon \Sigma^d X \to X$$

such that

$$K(m)_* f = \begin{cases} 0 & n \neq m \\ v_n^{p^N} & n = m \end{cases}$$

Example.

- $S^0_{(p)} \xrightarrow{p} S^0_{(p)}$ is a $\boxed{v_0 = p}$-self map.
- For p odd, the Adams map J(X) IV ~ image of J story

$$\alpha\colon \Sigma^{2(p-1)} S\mathbb{Z}/p \to S\mathbb{Z}/p$$

  is a v_1-self map with $K(1)_*(\alpha) = v_1$.
- For $p = 2$, the Adams map

$$\alpha\colon \Sigma^{8} S\mathbb{Z}/2 \to S\mathbb{Z}/2 \quad \neq 2(2-1)$$

  is a v_1-self map with $K(1)_*(\alpha) = v_1^4$.

$K(n)_*X \neq 0 \quad K(n-1)_*X = 0$

$f : \Sigma^d X \to X$

Periodicity Theorem (Hopkins-Smith)

- A spectrum X has type n if and only if it has a v_n-self map.
- If f and g are v_n-self maps, then

$$f^i = g^j$$

for some $i, j > 0$. That is, v_n-self maps are essentially unique.

Exercise

Prove that if X has type n and $f : \Sigma^d X \to X$ is a v_n-self map, the cofiber C_f

$$\Sigma^d X \xrightarrow{f} X \longrightarrow C_f \longrightarrow \Sigma^{d+1} X$$

has type $n+1$.

Hint. Examine the long exact sequence on $K(m)_*$ homology for the cofiber sequence, using the properties of $K(m)_*X$ and $K(m)_*f$.

Pr Orange

Constructing Type n-Spectra

If X has type n with v_n-self map $f\colon \Sigma^d X \to X$, then C_f has type $n+1$.

Examples.

- The cofiber $S\mathbb{Z}/p$ of $p\colon S^0_{(p)} \to S^0_{(p)}$ has type 1. It is also called $M(1)$.

  $= v_0$

- For p odd,

  $= S\mathbb{Z}/p$

  $$\alpha\colon \Sigma^{2(p-1)} M(1) \to M(1)$$

  satisfies $K(1)_*(\alpha) = v_1$. The cofiber has type 2 and is called $M(1,1)$.

- For $p = 2$,

  $\nexists M(1,1)$

  $$\alpha\colon \Sigma^8 M(1) \to M(1)$$

  satisfies $K(1)_*(\alpha) = v_1^4$. The cofiber has type 2 and is called $M(1,4)$.

Definition. $\boxed{M(i_0,\dots,i_n)}$ is the type $n+1$ spectrum defined inductively as the cofiber

$$\Sigma^{2(p^n-1)i_n} M(i_0,\dots,i_{n-1}) \xrightarrow{f_n} M(i_0,\dots,i_{n-1}) \longrightarrow M(i_0,\dots,i_n)$$

where f_n is a v_n-self map satisfying

$$\boxed{K(n)_*(f_n) = v_n^{i_n}.}$$

If it exists, $M(1,1,\dots,1)$ is called a *Smith-Toda complex*.

Nave

Torsion and Periodic Elements

Let $M_n := M(i_0, i_1, \dots, i_{n-1})$ with v_n-self map f_n so that $K(n)_* f_n = v_n^{i_n}$.

type n

For $X \in \mathcal{SH}_{(p)}$:

- $\omega \in \pi_r X$ is *v_{n-1}-torsion* if there is a diagram

$S^n \to X$

$S^0 \to \Sigma^{-r}X$

$$
\begin{tikzcd}
S_0 \arrow[r, "f_0"] & S_0 \arrow[r, "\omega"] \arrow[d] & \Sigma^{-r}X \\
\Sigma^{|f_1|}M_1 \arrow[r, "f_1"] & M_1 \arrow[d] \arrow[ur, dotted] & \\
 & \cdots \arrow[d] & \\
\Sigma^{|f_{n-1}|}M_{n-1} \arrow[r, "f_{n-1}"] & M_{n-1} \arrow[d] \arrow[uuur, dotted] & \\
\Sigma^{|f_n|}M_n \arrow[r, "f_n"] & M_n \arrow[uuuur, dotted, "\omega_n"'] &
\end{tikzcd}
$$

p^{i_0}

$=0$

$=0$

v_1-self

v_{n-1}-self

type $n-1$

v_n-periodic families

- If $\omega_n \circ f_n \neq 0$ for any v_n-self map f_n of M_n, then $\boxed{\omega \text{ is } v_n\text{-periodic}}$.

Geometric Chromatic Filtration

Let $\mathscr{C}_0^g(X) = \pi_* X$. For $n \geqslant 1$, let $\mathscr{C}_n^g(X)$ be the v_{n-1}-torsion elements.

$$\mathscr{C}_0^g(X) \supseteq \mathscr{C}_1^g(X) \subseteq \mathscr{C}_2^g(X) \supseteq \dots$$

Periodic Families

Let $\omega \in \pi_r X$ be v_n-periodic and

$$M = M_n = M(i_0, i_1, \ldots, i_{n-1})$$

with v_n-self map f_n so that

$$\Sigma^d M \xrightarrow{f_n} M \xrightarrow{\omega_n} \Sigma^{-r} X$$

is non-zero. Let M^r be the r-skeleton of M and

$$M^{r-1} \to M^r \to M^r_r \quad \text{and} \quad M^{r-1} \to M \to M_r$$

be cofiber sequences. There exists r and a diagram

Ravenel

“v_n-multiplication”

$$\begin{tikzcd}
 & \Sigma^d M \arrow[r, "f_n"] \arrow[d] & M \arrow[r, "\omega_n"] & \Sigma^{-r} X \\
S^k \cong \Sigma^d M^r_r \arrow[r, "i"] & \Sigma^d M_r \arrow[urr, dotted, "g"] & &
\end{tikzcd}$$

$\neq 0$

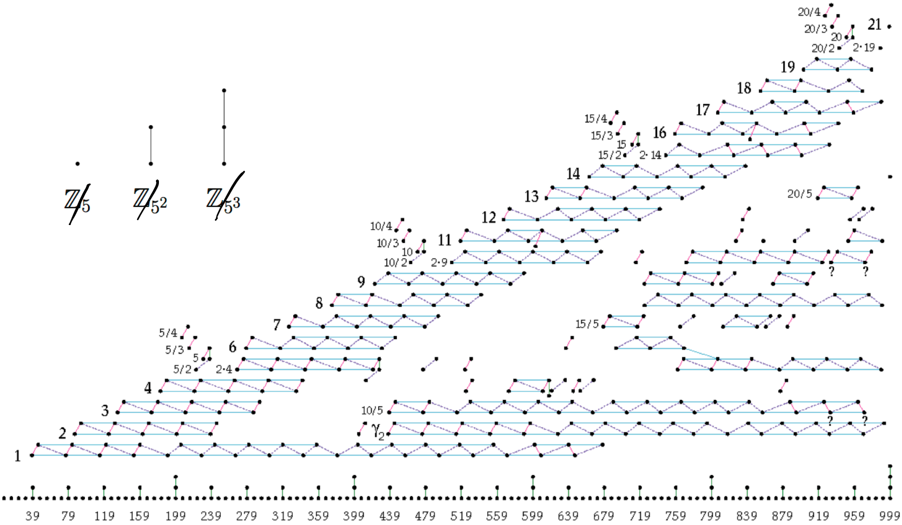
with $g \circ i$ non-trivial. That is, $\omega_n \circ f_n$ is non-trivial on some cell of M.

Such elements $g \circ i \in \pi_{k+r} X$ for the *v_n-periodic family of ω.*

Computation by Ravenel
Illustration by Hatcher
Edits by Behrens

$\pi_* S_{(5)}$

$p = 5$



$\longrightarrow \pi_* S^0_{(5)}$

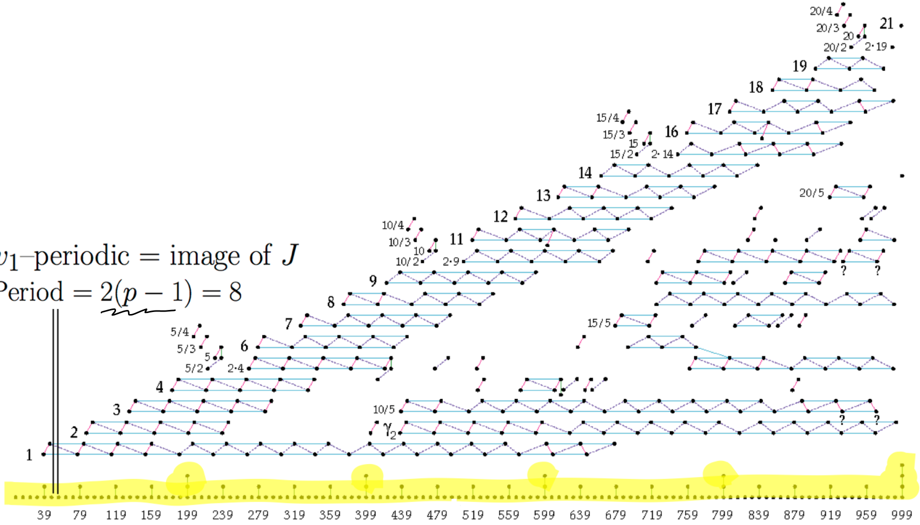
Computation by Ravenel
Illustration by Hatcher
Edits by Behrens

$\pi_* S_{(5)}$

$p = 5$

v_1–periodic = image of J

Period $= 2(p-1) = 8$



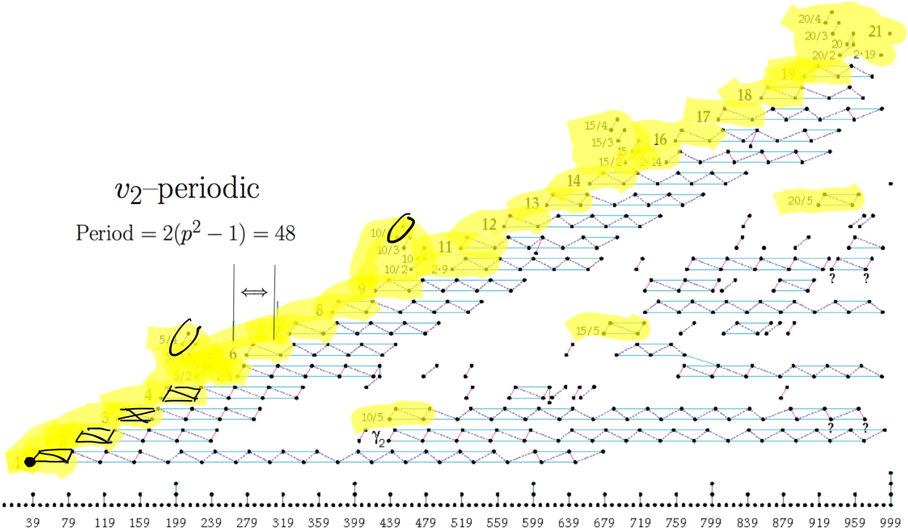
Computation by Ravenel
Illustration by Hatcher
Edits by Behrens

$\pi_* S_{(5)}$

$p = 5$

v_2–periodic

Period $= 2(p^2 - 1) = 48$



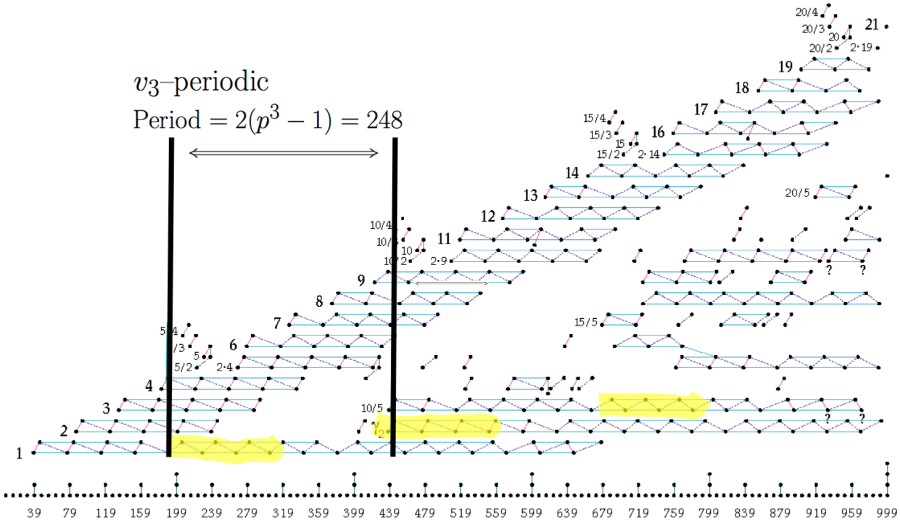
Computation by Ravenel
Illustration by Hatcher
Edits by Behrens

$\pi_* S_{(5)}$

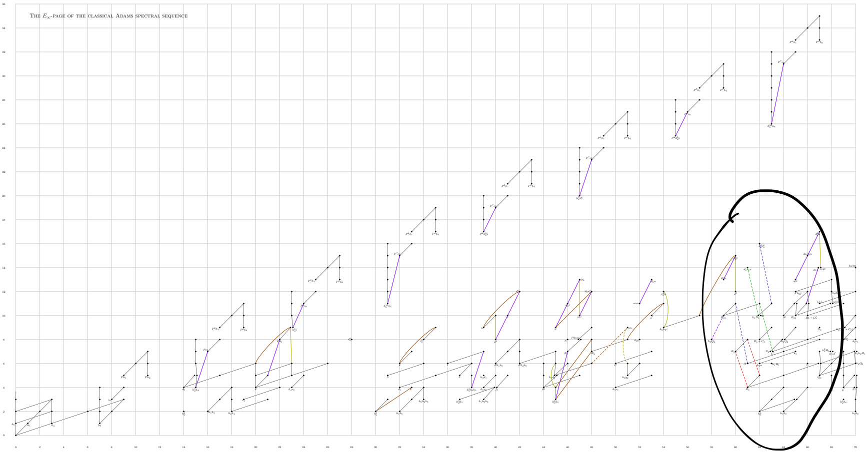
$p = 5$

v_3–periodic

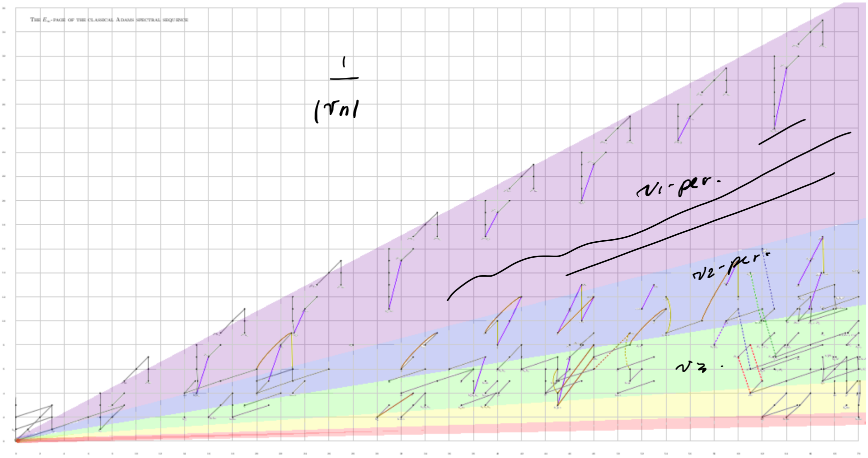
Period $= 2(p^3 - 1) = 248$



$\pi_* S^{\mathcal{U}}_{(2)}$ (Illustration by Isaksen)



$\pi_* S_{(2)}$ (Illustration by Isaksen)



Telescope of a Map

If $f\colon \Sigma^d X \to X$ is a self map, the *telescope of f* is

$$f^{-1}X = \operatorname{hocolim}\left(X \xrightarrow{\Sigma^{-d}f} \Sigma^{-d}X \xrightarrow{\Sigma^{-2d}f} \Sigma^{-2d}X \xrightarrow{\Sigma^{-3d}f} \cdots\right)$$

Telescopic Localization

For $M_n = M(i_0, i_1, \ldots, i_{n-1})$ with self map f_n, let $\mathrm{Tel}(n) = f_n^{-1}M_n$ and

BT

$$L_n^f X = L_{\mathrm{Tel}(0)\vee\cdots\vee\mathrm{Tel}(n)}X$$

Example. If X is a type n complex with v_n-self map f, then

$$L_n^f X = f^{-1}X.$$ ← Telescoping Localization.

Geometric Chromatic Filtration

Let X be a p-local spectrum. Then $\mathscr{C}_0^g(X) = \pi_* X$ and, for $n \geqslant 1$,

$$\mathscr{C}_n^g(X) = \ker(\pi_* X \to \pi_* L_{n-1}^f X).$$

Algebraic Chromatic Filtration

For a p-local spectrum X, let $\mathscr{C}_0^a(X) = \pi_* X$ and, for $n \geqslant 1$,

$$\mathscr{C}_n^a(X) = \ker(\pi_* X \to \pi_* L_{n-1} X)$$

for $L_n X = L_{K(0)\vee\dots\vee K(n\cancel{-1})} X$.

Geometric Chromatic Filtration

Let X be a p-local spectrum. Then $\mathscr{C}_0^g(X) = \pi_* X$ and, for $n \geqslant 1$,

$$\mathscr{C}_n^g(X) = \ker(\pi_* X \to \pi_* L_{n-1}^f X)$$

for $L_n^f X = L_{\mathrm{Tel}(0)\vee\dots\vee \mathrm{Tel}(n\cancel{-1})} X$.

There is a natural transformation

$$L_n^f X \longrightarrow L_n X$$

which is known to be an equivalence if

- X is $E(m)$-local for some $m \geqslant 0$ $X = L_m X$ m
- X is an MU-module spectrum (see the Localization Theorem)

Telescope Conjecture (Ravenel)

For every spectrum X,

$$L_n^f X \xrightarrow{\simeq} L_n X.$$

- $n = 0,\ p \geqslant 2$ Bousfield since $\mathrm{Tel}(0) = S\mathbb{Q} = H\mathbb{Q} = K(0)$.
- $n = 1,\ p > 2$ Miller
- $n = 1,\ p = 2$ Mahowald
- $n > 1,\ p \geqslant 2$ Open

Attempts at disproofs from Rav.

Let $n \geqslant 1$. The following are equivalent:

- If $L_{n-1}^f \simeq L_{n-1}$, then $L_n^f \simeq L_n$
- There exists a type n spectrum X with $f^{-1}X \simeq L_n X$.

Thick subcategory argument.

The Comparison Map

Let X have type n with v_n-self map $f\colon \Sigma^d X \to X$.

Exercise. $L_n X \simeq L_{K(n)} X$

By definition, the localization $X \to L_{K(n)} X$ is a $K(n)_*$-isomorphism.

Localization gives a map

$$f^{-1}X \longrightarrow L_{K(n)} f^{-1}X \simeq (L_{K(n)} f)^{-1} L_{K(n)} X \simeq L_{K(n)} X$$

where the last equivalence comes from the fact that $K(n)_* f$ is an isomorphism.

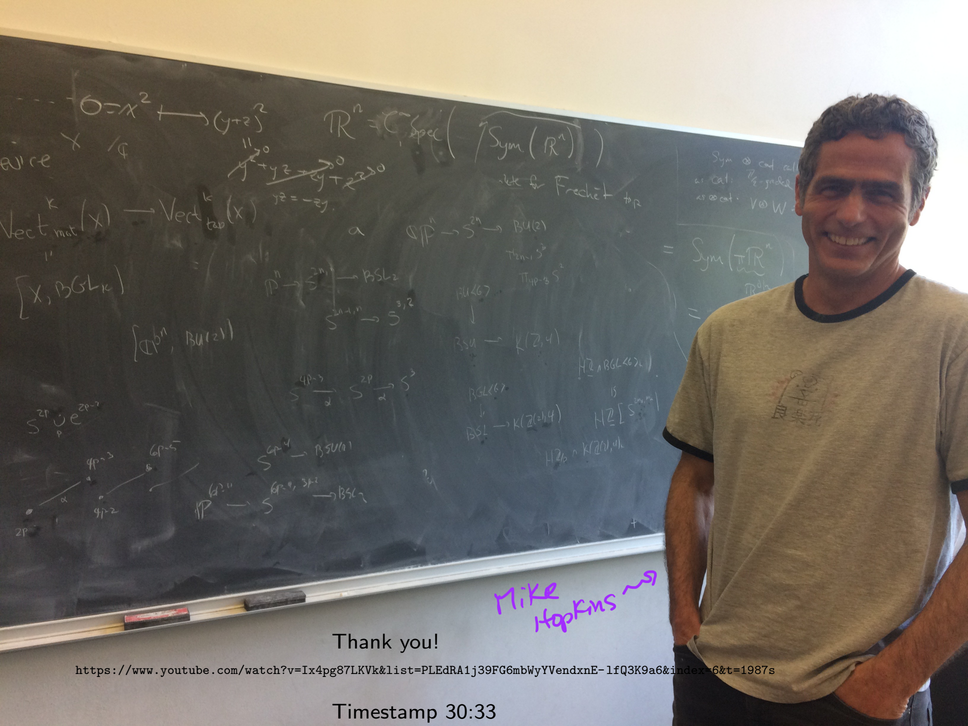
This gives the map

$$L_n^f X = f^{-1}X \longrightarrow L_{K(n)} X = L_n X.$$

Next Time

$$\begin{tikzcd} L_n X \arrow[r] \arrow[d] & L_{K(n)} X \arrow[d] \\ L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X \end{tikzcd}$$

- Discuss the gluing data $L_{n-1}X \to L_{n-1}L_{K(n)}X$ of the fracture square.
- Discuss the building blocks $L_{K(n)}X$
- Higher p-complete K-theories E_n and higher Adams operations $\mathbb{G}_n$.



Thank you!

https://www.youtube.com/watch?v=Ix4pg87LKVk&list=PLEdRA1j39FG6mbWyYVendxnE-lfQ3K9a6&index=6&t=1987s

Timestamp 30:33