1. First lecture

(1) Let $A$ be an abelian group with a filtration whose associated graded object is a (graded) $\mathbb{F}_2$-vector space of dimension $n$. Prove that $A$ has order $2^n$.

(2) Let $A$ be an abelian group with a filtration whose associated graded object is a (graded) $\mathbb{F}_2$-vector space. Suppose that the associated graded object is zero in filtrations below 1 and above $n$. Prove that $2^n \cdot A$ is zero.

(3) Let $A$ be an abelian group with a filtration whose associated graded object is a (graded) $\mathbb{F}_2$-vector space. Suppose that the dimensions of $\text{Gr}_0 A$, $\text{Gr}_1 A$, and $\text{Gr}_2 A$ are 1, 1, and 2 respectively ($\text{Gr} A$ is zero in all other degrees). Find all possible values of $A$. (See the diagram below.)

(4) The diagram below represents the map $\text{Gr} f : \text{Gr} A \to \text{Gr} B[1]$ induced by a map $f : A \to B$ of abelian groups. The solid line indicates a non-zero value of $\text{Gr} f$, while the dashed line indicates a hidden value.

Analyze the map $f$, as in the example in the middle of page 3 of the lecture notes. Give names to elements of $A$ and $B$, and determine the values of $f$ in terms of these names.

[Note: This is an example of a “crossing” value. It complicates analyses of associated graded objects.]
5) The figure below represents the associated graded object of an abelian group $A$. Each dot represents a copy of $F_2$ in $\text{Gr}A$, and vertical lines indicate the effect of multiplication by 2 as a map $\text{Gr}A \to \text{Gr}A[1]$. What are the possible values of $A$?

[Note: This problem is not just interesting in principle. It actually arises in the 45-stem!]

6) What changes if the two components in the previous diagram are separated by more than one filtration degree?

2. Second Lecture

7) Check that the expression $a_{01}a_{14} + a_{02}a_{24} + a_{03}a_{34}$ in the definition of a fourfold Massey product is a cycle.

8) Prove that

$$a_{01}(a_{12}, a_{23}, a_{34}) = \langle a_{01}, a_{12}, a_{23} \rangle a_{34}$$

when both brackets are defined.

[Hint: First show that the indeterminacies are the same. Then show that they contain a common element.]
(9) The $E_1$-page of the May spectral sequence that converges to $\text{Ext}_{A(2)}(F_2, F_2)$ is
\[ F_2[h_{01}, h_{12}, h_{23}, h_{02}, h_{13}, h_{03}], \]
with differentials
\begin{align*}
dh_{02} &= h_{01}h_{12} \\
dh_{13} &= h_{12}h_{23} \\
dh_{03} &= h_{01}h_{13} + h_{02}h_{23}.
\end{align*}
The degrees of the generators are (0, 1), (1, 1), (3, 1), (2, 1), (5, 1), and (6, 1) respectively. Compute the Massey product $\langle h_{01}, h_{12}, h_{23}, h_{12} \rangle$.

[Note: The resulting non-zero element lies in the May $E_2$-page, and it is sometimes called $h_0(1).$]

(10) Make precise the idea that the differential graded algebra of problem (9) is the “universal threefold Massey product that contains zero”.

[Hint 1: Characterize maps $E_1 \to B$ in terms of the Massey product structure on $B$.]

[Hint 2: Not all differential graded algebras are commutative, but the May $E_1$-page is commutative. This limitation has to be accounted for.]

(11) Show that the homology of the differential graded algebra from problem (9) is
\[ F_2[h_{01}, h_{12}, h_{23}, b_{02}, b_{13}, b_{03}, h_0(1)] \]
\[ h_{01}h_{12}, h_{12}h_{23}, h_{23}b_{02} + h_{01}h_0(1), h_{23}h_0(1) + h_{01}b_{13}, h_0(1)^2 = b_{02}b_{13} + h_{12}^2b_{03}, \]
where $b_{ij} = h_{ij}^2$ and $h_0(1) = h_{03}h_{13} + h_{12}h_{03}$.

[Note: This is a hard, or at least lengthy, problem. But if you carry it out, then you are well on your way to computing the homotopy of tmf.]

(12) Using the Moss Convergence Theorem and Adams differentials, find Toda bracket decompositions for the following elements. [Hint: You don’t need to worry about the technical conditions of the Moss Convergence Theorem. They don’t pertain in these situations.]

(a) $\eta \bar{\kappa}$ detected by $h_2f_0 = h_{14}$ in the 21-stem.

(b) $\nu \kappa$ detected by $h_0e_0 = h_{20}d_0$ in the 17-stem.

(c) A homotopy element detected by $h_2^2h_5$.

[Note: There are two entirely different solutions to (c).]

3. Third lecture

(13) Let $v(n)$ be the 2-adic valuation of $n$, i.e., $2^{v(n)}$ is the highest power of 2 that divides $n$. Show that
\[ v(3^k - 1) = \begin{cases} 
1 & \text{if } v(k) = 0 \\
2 + v(k) & \text{if } v(k) > 0.
\end{cases} \]

The goal of this series of problems is to compute the cohomology of $A(1)$, which is the subalgebra of the Steenrod algebra that is generated by $Sq^1$ and $Sq^2$. Equivalently, this is a computation of the May spectral sequence (and Adams spectral sequence) for the homotopy of the real connective $K$-theory spectrum $ko$. 
The May $E_1$-page consists of $\mathbb{F}_2[h_{01}, h_{12}, h_{02}]$, where $h_{01}$, $h_{12}$, and $h_{02}$ have degrees $(0,1)$, $(1,1)$, and $(2,1)$ respectively. Moreover, $h_{01}$ and $h_{12}$ have May filtration 0, and $h_{02}$ has May filtration 1. The May $d_r$ differential decreases the May filtration by $r$, and it changes bidegrees by $(-1,1)$. (In other words, May differentials point one unit left and one unit up.)

(14) To begin, we need that $d_1(h_{02}) = h_{01} h_{12}$. Using this formula, compute that the $E_2$-page is $\mathbb{F}_2[h_{01}, h_{12}, b_{02}] / h_{01} h_{12}$, where $b_{02} = h_{02}^2$.

(15) Use the Massey product shuffle $h_{12} \langle h_{01}, h_{12}, h_{01} \rangle = \langle h_{12}, h_{01}, h_{12} \rangle h_{01}$ to deduce that $h_{12}^3$ must be hit by a differential. Conclude that $d_2(b_{02}) = h_{12}^3$.

(16) Compute that the $E_3$-page is

$$\frac{\mathbb{F}_2[h_{01}, h_{12}, a, b]}{h_{01} h_{12}, h_{12}^3, h_{12} a, a^2 + h_{01} b^2},$$

where $a = h_{01} b_{02}$ and $b = b_{02}^2$.

(17) Verify that there are no possible higher May differentials.

(18) You have now obtained the $E_2$-page of the Adams spectral sequence for $ko$. Verify that there are no possible Adams differentials, so you have obtained the homotopy of $ko$.

(19) Find a threefold Massey product decomposition for $a$. Find a fourfold Massey product decomposition for $b$.

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