Fall 2023: Algebraic, motivic, and topological vector bundles

Thomas Brazelton and Morgan Opie

Course goals:

- Learn classical algebro-geometric methods for studying algebraic bundles
- Understand Quillen's proof of Serre's conjecture, and aspects of the Bass–Quillen conjecture generalizing Serre's conjecture
- Understand algebro-geometric versions of topological problems, for example questions about decomposing bundles as direct sums
- Learn about $\mathbb{A}^1$-invariance and the subtleties of using $\mathbb{A}^1$-invariant techniques to study vector bundles
- Learn about motivic techniques that can be used to address problems in bundle theory
- Understand algebraizability problems for topological vector bundles and some recent progress on this topic

---

Week 1: (9/7/23) Overview (Morgan Opie, UCLA and Thomas Brazelton, Harvard)

- Topological methods for studying and classifying vector bundles
- What are algebraic vector bundles? What are motivic vector bundles? (high-level)
- The difference between splitting, cancellation, and decomposability problems in topology vs in algebra
- Overview of themes and topics to be covered throughout the seminar

References: [AE17, Section 1, 2], [MS74]

I. Serre's Problem: statement, solutions, and generalizations

---

Week 2: (9/14/23) Serre's correspondence between algebraic vector bundles and projective modules (Ben Spitz, UCLA)

- Provide the basic definitions of algebraic varieties and vector bundles
- Give an overview of the correspondence between algebraic vector bundles on affine schemes and finitely generated projective modules over the ring of global sections
- Explain how faithfully flat descent is used in the Serre correspondence
- Motivate the transportation of topological questions about vector bundles into the algebraic setting and vice versa
- State *Serre splitting* as in [Ser58], and *Bass cancellation* [Bas68, V.3.5], compare and contrast with topological statements

References: [Ser55, Ser58, Bas68]

Week 3: (9/21/23) An intro to Serre's problem and Horrock's theorem (Anubhav Nanavaty, UC-Irvine)

- State Serre's problem about finitely generated vector bundles on polynomial rings over fields, and the heuristic that affine space should be "contractible" in some sense. Cover some low-dimensional examples:
    - Mention Seshadri's result that vector bundles over the affine plane are trivial

  - Mention Murthy-Towber's result that algebraic vector bundles over affine three-space over a field are trivial

- Spend the bulk of the talk stating and proving Horrock's theorem [Lam05, IV.2.1, IV.2.2] following the original cohomological proof [Hor64, Theorem 1] (see also [Aso, 6.1.4.1]).[1]
- If time allows, discuss other proofs of Horrock's theorem [Lam05, Chapter IV]

References: [Lam05, Chapters 1,4], [Ses58, Hor64, MT74]

Week 4: (9/28/23) Quillen's solution to Serre's problem (Zhong Zhang, UChicago)

- Introduce Zariski descent for modules
- Follow Quillen's proof of Serre's problem. This uses Horrock's theorem, and *Quillen patching* which should be explained

References: [Lam05, Chapters 4,5], [Qui76]

Week 5: (10/5/23) The Bass-Quillen conjecture: extending Serre's problem (Dan Marlowe, Warwick)

- State the Bass-Quillen conjecture
- Cover Lindel's theorem, explain what this means for vector bundles over smooth affine k-schemes [Aso, 6.3]. See also [Lam05, VIII.6]
- (Optional) explain how Lindel's étale neighborhood argument can be replaced with Gabber's geometric presentation lemma [AHW20, 2.2]

References: [Lam05, Chapter 8.6], [Lin82]

II. Building, classifying, and decomposing algebraic vector bundles

---

Week 6: (10/12/23) K-theory and Chow groups; Chow-valued Chern classes; cycle class maps (Krishna Kumar Madhavan Vijayalakhsmi, Milan)

- Cover the definition of algebraic K_0 (stress the difference between isomorphism and stable isomorphism). Define Chow groups, and give the axiomatic definitions of Chern classes valued in Chow groups, and some examples.
- Discuss cycle class maps, and algebraic Chern classes mapping to topological Chern classes.
- Define the Chern character map out of algebraic K-theory. Explain that, if Chern classes characterize the isomorphism type of bundles over some variety X, then cancellation holds over X. Thus understanding whether bundles can be constructed with unique Chern data directly leads to cancellation statements.

References: [Bas64], [Ful98], [Sri96]

Week 7: (10/19/23) Cancellation and Suslin's conjecture (Yang Hu, New Mexico State)

- State what "cancellation" means for vector bundles
- Bass ([Bas64, Theorem 9.3], also in [Lam05, V.4.8]) proves that if $X = \mathrm{Spec}(R)$ with R Notherian, of Krull dimension $d < \infty$, then cancellation holds for bundles of rank $> d$.
- Suslin proves that, if X is a finite type affine k-scheme of dimension d over a C_1-field, then rank d bundles on X are stably isomorphic if and only if they are isomorphic. [Sus82, 2.4]

[1] We will need to state the classification of vector bundles on $\mathbb{P}^1$, but we can defer the proof to Week 8.

- Suslin conjectures in [Sus80] that the correct bound for cancellative bundles of rank r is $r \geq \frac{d+1}{2}$. This turns out not to be correct (Kumar [MK85] gave examples of bundles of rank $r = d-2$ on dimension $d \geq 2$ smooth rational affine k-varieties over $k = \bar{k}$, which answered Suslin's conjecture in the negative and collapsed the open range of his cancellation conjecture to just the case $r = d-1$.)
- A preprint of Fasel proves cancellation for bundles of rank $r = d-1$ on a smooth variety over a field where $d!$ is a unit [Fas21].

References: [Bas64], [Fas21]. Look at [Lam05, VIII.2] for historical context. If you speak Russian, [Sus77a, Sus77b, Sus80]

Week 8: (10/26/23) Algebraic vector bundles on the projective line and projective plane (Yuyuan Luo, MIT)

- Present Grothendieck's classification of vector bundles on $\mathbb{P}^1$ [Aso, §2.4.3, §2.4.4]
- Explain the constraints on Chern classes of bundles arising from Riemann-Roch (a good reference for this is Schwarzenberger's appendix to Hirzebruch's book [Hir95, §23])
- As an example, discuss vector bundles on $\mathbb{P}^2$ following Schwarzenberger

References: [Aso], [Sch61b, Sch61a]

Week 9: (11/2/23) Plane bundles on $\mathbb{P}^3$ and higher-dimensional questions (Natalia Pacheco-Tallaj, MIT)

- Present Atiyah and Rees' classification of topological plane bundles on $\mathbb{P}^3$.
- Define the α-invariant as in Atiyah-Rees, explain how it classifies topological bundles [AR76, 2.8, 3.3]. Discuss how to compute the α-invariant for algebraic vector bundles [AR76, 5.4].
- Discuss how the α-invariant, together with Horrocks' construction for locally free sheaves, can be combined to show that all topological vector bundles of rank 2 on $\mathbb{P}^3$ admit algebraic structures [Hor68].
- Explain the connection between studying vector bundles on projective space and complete intersections in projective space. State Hartshorne's conjecture about plane bundles on projective space [Har74, 6.3].

References: [AR76, Har74]

III. Simplicial and motivic methods: applications to classification and algebraicity

---

Week 10: (11/9/23) Representability of torsors in simplicial varieties (Kyle Ormsby, University of Washington)

- Discuss simplicial presheaves on varieties, and the Zariski and Nisnevich topologies. Define τ-local weak equivalences following Jardine.
- Argue that the bar construction for a group scheme G classifies G-torsors in the τ-local homotopy category.
- Argue that GL_n-torsors in the Nisnevich topology classify algebraic vector bundles of rank n.[2]
- (Optional) Show that étale torsors for a group scheme are not $\mathbb{A}^1$-invariant. [Par78]

References: [Jar15]

Week 11: (11/16/23) $\mathbb{A}^1$-localization, motivic spaces, and intro to affine representability (Alexander Ziegler, Wuppertal)

[2] This uses a string of identifications $\mathrm{Vect}_n^{\mathrm{alg}}(X) = H^1_{\mathrm{Zar}}(X, \mathrm{GL}_n) = H^1_{\mathrm{Nis}}(X, \mathrm{GL}_n)$.

- Building off of work from last week, localize at $\mathbb{A}^1$ to define the category of motivic spaces
- Explore when the functor $X \mapsto \mathrm{Vect}(X)$ is and is not $\mathbb{A}^1$-invariant. Give a counterexample (e.g. $\mathbb{P}^1$) and state Morel's result about representability over smooth affine schemes
- State the affine representability theorem

References: [AE17], [Mor06, Mor04]

Week 12: (11/30/23) Affine representability I and II (Federico Ernesto Moccheti, Milan/Osnabrück)

- Go through the proof of affine representability for vector bundles [AHW17] and, as time allows, affine representability for principle bundles [AHW18]
- Discuss the local-to-global principle for torsors for group schemes, compare this to results about Quillen patching we have seen earlier
- As a special case of affine representability, show that K_0 is represented by $\mathbb{Z} \times \mathrm{BGL}$ for smooth affine schemes. The theorem is true for all smooth schemes (not necessarily affine)!

References: [AHW17, AHW18]

Week 13: (12/7/23) Obstructions to algebraizicity (Gabriela Guzmán, CIMAT)

- Establish that motivic vector bundles are an intermediate setting for studying lifts of topological bundles to the algebraic setting
- Explore motivic obstruction theory for the first two Chern classes
- Sketch the construction of the main result of [AFH19]: that algebraizability of Chern classes is not enough to guarantee algebraizability of the bundle

References: [AFH19].

References

[AE17] Benjamin Antieau and Elden Elmanto, *A primer for unstable motivic homotopy theory*, Surveys on recent developments in algebraic geometry, Proc. Sympos. Pure Math., vol. 95, Amer. Math. Soc., Providence, RI, 2017, pp. 305–370. MR 3727503

[AFH19] A. Asok, J. Fasel, and M. J. Hopkins, *Obstructions to algebraizing topological vector bundles*, Forum Math. Sigma **7** (2019), Paper No. e6, 18. MR 3925497

[AHW17] Aravind Asok, Marc Hoyois, and Matthias Wendt, *Affine representability results in $\mathbb{A}^1$-homotopy theory, I: vector bundles*, Duke Math. J. **166** (2017), no. 10, 1923–1953. MR 3679884

[AHW18] ———, *Affine representability results in $\mathbb{A}^1$-homotopy theory, II: Principal bundles and homogeneous spaces*, Geom. Topol. **22** (2018), no. 2, 1181–1225. MR 3748687

[AHW20] ———, *Affine representability results in $\mathbb{A}^1$-homotopy theory III: finite fields and complements*, Algebr. Geom. **7** (2020), no. 5, 634–644. MR 4156421

[AR76] M. F. Atiyah and E. Rees, *Vector bundles on projective 3-space*, Invent. Math. **35** (1976), 131–153. MR 419852

[Aso] Aravind Asok, *Algebraic geometry from an a1-homotopic viewpoint.*

[Aut] Stacks Project Authors, *Stacks project.*

[Bas64] H. Bass, *K-theory and stable algebra*, Inst. Hautes Études Sci. Publ. Math. (1964), no. 22, 5–60. MR 174604

[Bas68] Hyman Bass, *Algebraic K-theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1968. MR 249491

[BP87] Constantin Bănică and Mihai Putinar, *On complex vector bundles on projective threefolds*, Invent. Math. **88** (1987), no. 2, 427–438. MR 880960

[Bro73] Kenneth S. Brown, *Abstract homotopy theory and generalized sheaf cohomology*, Trans. Amer. Math. Soc. **186** (1973), 419–458. MR 341469

[Fas21] Jean Fasel, *Suslin's cancellation conjecture in the smooth case*, 2021.

[Ful98] William Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR 1644323

[Har74] Robin Hartshorne, *Varieties of small codimension in projective space*, Bull. Amer. Math. Soc. **80** (1974), 1017–1032. MR 384816

[Hir95] Friedrich Hirzebruch, *Topological methods in algebraic geometry*, english ed., Classics in Mathematics, Springer-Verlag, Berlin, 1995, Translated from the German and Appendix One by R. L. E. Schwarzenberger, Appendix Two by A. Borel, Reprint of the 1978 edition. MR 1335917

[Hor64] G. Horrocks, *Projective modules over an extension of a local ring*, Proc. London Math. Soc. (3) **14** (1964), 714–718. MR 169878

[Hor68] ———, *A construction for locally free sheaves*, Topology **7** (1968), 117–120. MR 227180

[Hor78] ———, *Examples of rank three vector bundles on five-dimensional projective space*, J. London Math. Soc. (2) **18** (1978), no. 1, 15–27. MR 502651

[Jar15] John F. Jardine, *Local homotopy theory*, Springer Monographs in Mathematics, Springer, New York, 2015. MR 3309296

[KM82] N. Mohan Kumar and M. Pavaman Murthy, *Algebraic cycles and vector bundles over affine three-folds*, Ann. of Math. (2) **116** (1982), no. 3, 579–591. MR 678482

[Lam05] T. Y. Lam, *Serre's problem on projective modules*, Springer Monogr. Math., Berlin: Springer, 2005 (English).

[Lin82] Hartmut Lindel, *On the Bass-Quillen conjecture concerning projective modules over polynomial rings*, Invent. Math. **65** (1981/82), no. 2, 319–323. MR 641133

[MK85] N. Mohan Kumar, *Stably free modules*, Amer. J. Math. **107** (1985), no. 6, 1439–1444. MR 815767

[Mor04] Fabien Morel, *An introduction to $\mathbb{A}^1$-homotopy theory*, Contemporary developments in algebraic K-theory, ICTP Lect. Notes, vol. XV, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004, pp. 357–441. MR 2175638

[Mor06] ———, *$\mathbb{A}^1$-algebraic topology*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 1035–1059. MR 2275634

[Mor12] ———, *$\mathbb{A}^1$-algebraic topology over a field*, Lecture Notes in Mathematics, vol. 2052, Springer, Heidelberg, 2012. MR 2934577

[MS74] JOHN W. MILNOR and JAMES D. STASHEFF, *Characteristic classes. (am-76)*, Princeton University Press, 1974.

[MS76] M. Pavaman Murthy and Richard G. Swan, *Vector bundles over affine surfaces*, Invent. Math. **36** (1976), 125–165. MR 439842

[MT74] M. Pavaman Murthy and Jacob Towber, *Algebraic vector bundles over A^3 are trivial*, Invent. Math. **24** (1974), 173–189. MR 422276

[Mur66] M. Pavaman Murthy, *Projective a[x]-modules*, Journal of the London Mathematical Society **s1-41** (1966), no. 1, 453–456.

[Mur94] ———, *Zero cycles and projective modules*, Ann. of Math. (2) **140** (1994), no. 2, 405–434. MR 1298718

[Par78] S. Parimala, *Failure of a quadratic analogue of Serre's conjecture*, Amer. J. Math. **100** (1978), no. 5, 913–924. MR 517136

[Pop89] Dorin Popescu, *Polynomial rings and their projective modules*, Nagoya Math. J. **113** (1989), 121–128. MR 986438

[Qui76] Daniel Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167–171. MR 427303

[Sch61a] R. L. E. Schwarzenberger, *Vector bundles on algebraic surfaces*, Proc. London Math. Soc. (3) **11** (1961), 601–622. MR 137711

[Sch61b] ———, *Vector bundles on the projective plane*, Proc. London Math. Soc. (3) **11** (1961), 623–640. MR 137712

[Ser55] Jean-Pierre Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2) **61** (1955), 197–278. MR 68874

[Ser58] J.-P. Serre, *Modules projectifs et espaces fibrés à fibre vectorielle.*, Séminaire P. Dubreil, M.-L. Dubreil-Jacotin et C. Pisot, 1957/58, Fasc. 2,, ,, 1958, pp. Exposé 23, 18. MR 177011

[Ses58] C. S. Seshadri, *Triviality of vector bundles over the affine space K^2*, Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 456–458. MR 102527

[Sri96] V. Srinivas, *Algebraic K-theory*, second ed., Progress in Mathematics, vol. 90, Birkhäuser Boston, Inc., Boston, MA, 1996. MR 1382659

[Sus76] A. A. Suslin, *Projective modules over polynomial rings are free*, Dokl. Akad. Nauk SSSR **229** (1976), no. 5, 1063–1066. MR 0469905

[Sus77a] ———, *A cancellation theorem for projective modules over algebras.*, Dokl. Akad. Nauk SSSR (1977), no. no. 4,, 808–811. MR 466104

[Sus77b] ———, *Stably free modules.*, Mat. Sb. (N.S.) (1977), no. no. 4,, 537–550, 632. MR 441949

[Sus80] ———, *The cancellation problem for projective modules, and related questions.*, Proceedings of the International Congress of Mathematicians (Helsinki, 1978),, ,, 1980, pp. 323–330,. MR 562623

[Sus82] ———, *Cancellation for affine varieties*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) (1982), 187–195, 222, Modules and algebraic groups. MR 669571

[VS76] L. N. Vaseršteĭn and A. A. Suslin, *Serre's problem on projective modules over polynomial rings, and algebraic K-theory*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), no. 5, 993–1054, 1199. MR 0447245