Fall 2024: Quadratic curve counting

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Summary:

A classical problem in algebraic geometry is to ask how many rational curves of a given degree pass through some generally chosen points on the plane. In the case of a degree d rational curve passing through $3d-1$ points, the answer N_d is known to be finite, and classical counts up to $d=4$ date back to the 19th century and earlier. Within the last thirty years, the development of *Gromov–Witten theory* provided a new approach to these sorts of curve-counting problems, yielding a recursive formula for N_d.

Over the reals, the number of curves interpolating these points depends on the placement of the points. However Welschinger established that a certain *signed* count of the real curves was invariant of where the points were placed, using methods derived from symplectic geometry. These signs can be interpreted as the local Brouwer degree of an evaluation map out of the moduli of stable curves.

Recent work of Kass, Levine, Solomon, and Wickelgren produce a well-defined $\mathbb{A}^1$-degree of this evaluation map, providing a quadratically enriched count of rational curves interpolating points over an arbitrary field. This leverages the machinery of $\mathbb{A}^1$-*homotopy theory*, which provides an $\mathbb{A}^1$-Brouwer degree valued in the Grothendieck–Witt ring of quadratic forms over a field k.

Course goals:

- Learn about virtual fundamental classes and Gromov–Witten invariants, and their relationship to counting curves.
- Understand Welschinger invariants and counting real curves.
- Learn about $\mathbb{A}^1$-enumerative geometry, which provides enumerative answers valued in the Grothendieck-Witt ring $\mathrm{GW}(k)$ of a field which specializes to both the complex and real curve counts.
- Work through the details of a recent paper of Kass-Levine-Solomon-Wickelgren which defines a $\mathrm{GW}(k)$-valued count of rational curves on a del Pezzo surface interpolating points as the $\mathbb{A}^1$-degree of an evaluation map.

Week 1: (9/9/24) Overview (Thomas and Sabrina)

I. Gromov-Witten invariants

Week 2: (9/16/24) Quantum cohomology and Gromov-Witten classes

- Brief overview of stacks
- Define virtual fundamental classes

References: [Abr08]

Week 3: (9/23/24) Quantum cohomology and Gromov-Witten classes II

- Explain the recursive formula for N_d

References: [Abr08], Chapters 4 and 5 of Koch/Vainsencher

II. The real story

Week 4: (9/30/24) Interpolation over $\mathbb{R}$

- Discuss the Degtyarev–Kharlamov paper
- Define Welschinger invariants for planar real rational curves

References: [DK00], [Wel05]

Week 5: (10/7/24) Welschinger invariants

- Define symplectic manifolds and J-holomorphic curves
- Present Welschinger invariants via Welschinger's original work

References: [Wel05]

Week 6: (10/14/24) Welschinger invariants via open Gromov-Witten invariants

- Define open Gromov-Witten invariants.
- Sketch how to get a (relative) orientation from Spin/Pin-structures of the evaluation map [Sol06, Section 3].
- Relate open Gromov-Witten invariants to Welschinger invariants [Sol06, Theorem 1.8].

References: [Sol06], [Wel15]

Week 7: (10/21/24) $\mathbb{A}^1$-enumerative geometry

- Black box motivic spaces.
- Discuss the $\mathbb{A}^1$-degree and the Grothendieck-Witt ring of a field k.
- Recap traces on Grothendieck–Witt rings
- Discuss (quadratic) versions of relative orientations of vector bundles [KW21].
- Give some examples of $\mathbb{A}^1$-Euler numbers.

References: [KW21, Bra21, PW21, WW20].

Week 8: (10/28/24) Global and local $\mathbb{A}^1$-degrees

- Define a relative orientation of a map $f\colon X \to Y$
- Explain how to compute local and global $\mathbb{A}^1$-Brouwer degrees

References: [KLSW23a, Sections 2,3] [PW21, Section 8]

Week 9: (11/4/24) KLSW, part I ($S = \mathbb{P}^2$ and $D = \mathcal{O}(d)$)

- Carefully define the Konsevich moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, \mathcal{O}(d))$, and the evaluation map. Set $S = \mathbb{P}^2$ and $D = \mathcal{O}(d)$ for simplicity throughout.
- Define twists of the evaluation map [KLSW23a, Section 5].

References: [KLSW23a, KLSW23b, Section 4]

Week 10: (11/11/24) KLSW, part II ($S = \mathbb{P}^2$ and $D = \mathcal{O}(d)$)

- Prove the evaluation map is relatively oriented in characteristic zero and in positive characteristic.

References: [KLSW23a, KLSW23b]

Week 11: (11/18/24) KLSW, part III ($S = \mathbb{P}^2$ and $D = \mathcal{O}(d)$)

- Give Levine's quadratically enriched version of quadratic Welschinger weights (the thing inside the sum in [KLSW23a, Theorem 3]) and explain how it generalizes the Welschinger sign of a real curve.

- Compute the local degree of the evaluation map [KLSW23a, Theorem 1, Theorem 2] and show that it equals the quadratic Welschinger weight.
- State the main theorem [KLSW23a, Theorem 3] and give some example computations.

References: [KLSW23a]

Week 12: (11/25/24) Tropical correspondence theorems

- Define plane tropical curves and their dual subdivision.
- State Mikhalkin's tropical correspondence theorems over $\mathbb{C}$ and $\mathbb{R}$ for curves in $\mathbb{P}^2$.
- State the quadratically enriched tropical correspondence theorem and derive formulas for the quadratic curves counts in terms of the complex and real counts for curves in $\mathbb{P}^2$.
- If time permits, explain the idea of the tropical correspondence theorems.

References: [JPP24]

References

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