Final arguments for the existence of θ_6

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joint work with Weinan Lin and Guozhen Wang

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Existence of θ_6

Theorem (Lin–Wang–Xu)

h_6^2 survives to the E_∞-page in the Adams spectral sequence.

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  - Xu, Isaksen–Wang–Xu: $2 \cdot \theta_5 = 0$ in $\pi_{62}S^0$ for every θ_5.

Inductive Approach for θ_6

Theorem (Barratt–Jones–Mahowald, Burklund–Xu)

1. *The element* h_6^2 *survives to the* E_{r+3}*-page of the classical Adams spectral sequence if and only if for some* θ_5*,*

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$$\lambda\eta\theta_5^2 = 0 \text{ in } \pi_{125,125+4}S^{0,0}.$$

- In fact, the expression $\lambda\eta\theta_5^2$ is consistent for every choice of θ_5.

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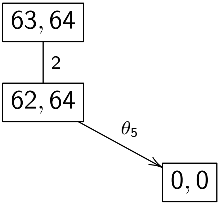

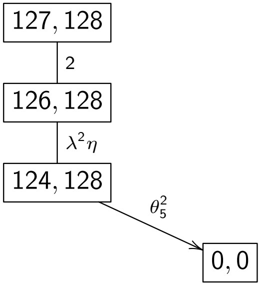

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$$h_6^2 \longmapsto \lambda\eta\theta_5^2$$

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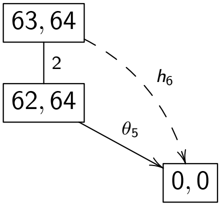

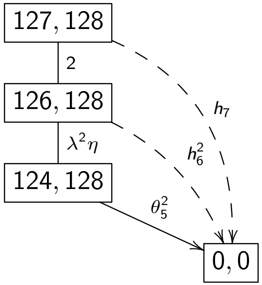

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  - If $\mathrm{AF}(\lambda^3\eta[h_0^2 x_{124,8}]) > 14$, then it is zero.

An η-extension

Proposition A

Exactly one of (1) and (2) is true:

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(2) $d_{12}(h_6^2) = h_1 h_4 x_{109,12} \neq 0.$

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(4) $\theta_5^2 = \lambda^6[h_0^2 x_{124,8}] \neq 0 \in \pi_{124,124+4} S^{0,0}$.

(5) $\lambda^3 \eta [h_0^2 x_{124,8}] = \lambda^6 [h_1 h_4 x_{109,12}] \in \pi_{125,125+8} S^{0,0}$.

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1. $$\begin{aligned} \lambda^3\eta \cdot \alpha_1 &= \lambda^3[h_0^2 x_{124,8}] + \lambda^6 \alpha_2 && \in \pi_{124,124+7} S^{0,0}/\lambda^9, \\ \eta \cdot \alpha_2 &= \lambda \cdot \alpha_3 && \in \pi_{125,125+14} S^{0,0}/\lambda^9, \end{aligned}$$

Proof of Proposition B

For the sake of contradiction, we assume (3) and (5) are both true.

Lemma 1

There exists $\alpha_1 = [x_{123,9} + h_0 x_{123,8}] \in \pi_{123,123+9} S^{0,0}/\lambda^9$,

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2. $\lambda^3 \cdot \alpha_1 \cdot [h_0] = 0 \quad \in \pi_{123,123+7} S^{0,0}/\lambda^9.$

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Lemma 2

The synthetic Toda bracket

$$\langle \lambda^3 \alpha_1, [h_0], \eta \rangle \subset \pi_{125,125+7} S^{0,0}/\lambda^9$$

does not contain zero, and is detected by $\lambda^4 h_0^2 x_{125,9,2}$.

Proof of Proposition B

Lemma 3 (Corollary of Lemma 2)

$[\lambda^4 h_0^2 x_{125,9,2}] \cdot [h_0] = \lambda^6 [h_1 h_4 x_{109,12}] \neq 0 \in \pi_{125,125+8} S^{0,0}/\lambda^9.$

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Lemma 4

$[\lambda^4 h_1 x_{121,7}] \cdot [h_2] = \lambda[\lambda^5 h_0^2 x_{125,9,2}] \in \pi_{125,125+5} S^{0,0}/\lambda^9.$

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$$\lambda^6 [h_6 M d_0] \text{ in AF} = 11, \quad \lambda^7 [h_5 x_{91,11}] \text{ in AF} = 12.$$

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- $\Rightarrow$ in $\mathrm{ASS}(S^0/\nu)$, $h_1 h_4 x_{109,12}[0]$ must be killed by d_r for $r \leqslant 5$.

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- $\Rightarrow$ in $\mathrm{ASS}(S^0/\nu)$, $h_1 h_4 x_{109,12}[0]$ must be killed by d_r for $r \leqslant 5$. Contradiction!

Sketch Proof of Lemma 1

Lemma 1

There exists $\alpha_1 = [x_{123,9} + h_0 x_{123,8}], \alpha_2, \alpha_3 \in \pi_{*,*} S^{0,0}/\lambda^9$*, such that*

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Facts: (1) $x_{123,9} + h_0 x_{123,8}$ survives to E_{12}, not killed by any differential.

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- Let $\alpha_1 = [x_{123,9} + h_0 x_{123,8}] \in \pi_{123,123+9} S^{0,0}/\lambda^{11}$ and its images in $\pi_{123,123+9} S^{0,0}/\lambda^r$ for $1 \leqslant r \leqslant 10$.

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- $\lambda\eta\alpha_1 + \lambda[h_0^2 x_{124,8}] \in \pi_{124,124+9}S^{0,0}/\lambda^9$ lies in AF $\geqslant 11$.

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- $\lambda\eta\alpha_1 + \lambda[h_0^2 x_{124,8}] \in \pi_{124,124+9}S^{0,0}/\lambda^9$ lies in AF $\geqslant 11$.
- $\lambda^3\eta\alpha_1 + \lambda^3[h_0^2 x_{124,8}] \in \pi_{124,124+7}S^{0,0}/\lambda^9$ has AF $\geqslant 13$ by inspection,

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- $\lambda \eta \alpha_1 + \lambda [h_0^2 x_{124,8}] \in \pi_{124,124+9} S^{0,0}/\lambda^9$ lies in AF $\geqslant 11$.
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There exists $\alpha_1 = [x_{123,9} + h_0 x_{123,8}], \alpha_2, \alpha_3 \in \pi_{*,*}S^{0,0}/\lambda^9$, *such that*

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- $\lambda^3 \eta \alpha_1 + \lambda^3 [h_0^2 x_{124,8}] \in \pi_{124,124+7} S^{0,0}/\lambda^9$ has AF $\geqslant 13$ by inspection, and is detected by $\lambda^6 e_0 \Delta h_6 g$ if AF $= 13$.
- in Ext, $h_1 \cdot e_0 \Delta h_6 g = 0$.
- $\lambda^3 \cdot \alpha_1 \cdot [h_0] \in \pi_{123,123+7} S^{0,0}/\lambda^{11}$ has AF $\geqslant 17$ by inspection.

Sketch Proof of Lemmas 2 and 3

Lemma 2

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- $\Rightarrow$ Lemma 3:
$[\lambda^4 h_0^2 x_{125,9,2}]\cdot[h_0] = \lambda^6[h_1 h_4 x_{109,12}] \neq 0 \in \pi_{125,125+8}S^{0,0}/\lambda^9$.

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Sketch Proof of Lemma 4

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$[\lambda^4 h_1 x_{121,7}] \cdot [h_2] = \lambda[\lambda^5 h_0^2 x_{125,9,2}] \in \pi_{125,125+5} S^{0,0}/\lambda^9.$

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$$\begin{array}{ccccc} h_0^2 x_{125,9,2} & \longmapsto & h_0^2 x_{125,9,2}[0] & & \\ & & \uparrow{\scriptstyle d_3} & & \\ & & h_1 x_{121,7}[4] & & \\ & & +x_{126,8}[0] & \longmapsto & h_1 x_{121,7} \\ & & +x_{126,8,2}[0] & & \end{array}$$

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- Generalized Mahowald Trick:
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  $[h_1 x_{121,7}] \cdot [h_2] = \lambda^2 [h_0^2 x_{125,9,2}] \in \pi_{125,125+9} S^{0,0}/\lambda^3.$
- Lift via $S^{0,0}/\lambda^5 \to S^{0,0}/\lambda^3$, push via $\lambda^4 : \Sigma^{0,-4} S^{0,0}/\lambda^5 \to S^{0,0}/\lambda^9$

Thank you!