MATH 2551 Studypalooza Key

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Problem 1. Find T, N and curvature for $r(t) = 3\sin(t)\hat{\imath} + 3\cos(t)\hat{\jmath} + 4t\hat{k}$.

Solution. The unit tangent vector is given by

$$T = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

Here, we can calculate $\vec{v}(t)$ by taking the derivative of $r(t)$ with respect to t. This gives us

$$\vec{v}(t) = \langle 3\cos t, -3\sin t, 4\rangle$$

The magnitude of $\vec{v}(t)$ is given by

$$|\vec{v}(t)| = \sqrt{(3\cos t)^2 + (-3\sin t)^2 + 4^2} = 5$$

Then using our formula for T, we obtain

$$T = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \langle \frac{3}{5}\cos t, -\frac{3}{5}\sin t, \frac{4}{5}\rangle$$

Next, we need to solve for the unit normal vector N. This is given by

$$N = \frac{\frac{dT}{dt}}{|\frac{dT}{dt}|}$$

First, we solve for the derivative of the unit tangent tangent vector. This gives us

$$\frac{dT}{dt} = \langle -\frac{3}{5}sint, -\frac{3}{5}cost, 0\rangle$$

Next, the magnitude of the above vector is given by

$$|\frac{dT}{dt}| = \sqrt{(-\frac{3}{5}sint)^2 + (-\frac{3}{5}cost)^2} = \frac{3}{5}$$

Then using our formula for N, we obtain

$$N = \frac{\frac{dT}{dt}}{|\frac{dT}{dt}|} = \langle -\sin t, -\cos t, 0\rangle$$

Finally, we solve for curvature, which is given by

$$k = \frac{|\frac{dT}{dt}|}{|\vec{v}(t)|}$$

Using values we obtained earlier, we obtain

$$k = \frac{3}{25}$$

□

Problem 2. Write the acceleration in terms of its tangential and normal components for $r(t) = (t+1)\hat{\imath} + 2t\hat{\jmath} + t^2\hat{k}$ at time $t = 1$.

Solution. We seek to find the tangential and normal components of acceleration, denoted a_T and a_N.

To begin, we have that a_T is given by the following, where $\frac{d|\vec{v}|}{dt}$ is the derivative of the magnitude of the velocity vector.

$$a_T = \frac{d|\vec{v}|}{dt}$$

First, we find $\vec{v}(t)$ by taking the derivative of the given position vector $r(t)$. This gives us

$$\vec{v}(t) = \langle 1, 2, 2t \rangle$$

Next, we calculate the magnitude of $\vec{v}(t)$. This gives us

$$|\vec{v}(t)| = \sqrt{1^2 + 2^2 + (2t)^2} = \sqrt{5 + 4t^2}$$

Finally, we can take the derivative of $|\vec{v}(t)|$ using chain rule, yielding

$$\frac{d|\vec{v}|}{dt} = \frac{1}{2}(5 + 4t^2)^{-\frac{1}{2}} \cdot (8t) = \frac{4t}{\sqrt{5 + 4t^2}}$$

Therefore, we have that

$$a_T = \frac{4t}{\sqrt{5 + 4t^2}}$$

Next, to calculate a_N, we can use the following formula relating the magnitude of acceleration to its tangential and normal components.

$$a_N = \sqrt{|\vec{a}(t)|^2 - a_T^2}$$

First, we need to calculate $|\vec{a}(t)|$. To obtain $\vec{a}(t)$, we can take the derivative of $\vec{v}(t)$, which we calculated earlier. This gives us

$$\vec{a}(t) = \langle 0, 0, 2 \rangle$$

The magnitude $|\vec{a}(t)|$ is then equal to 2.

Then using our formula for a_N, we obtain

$$a_N = \sqrt{2^2 - \frac{(4t)^2}{5+4t^2}} = 2\sqrt{\frac{5}{5+4t^2}}$$

Next, we can write the acceleration in terms of its tangential and normal components as

$$\vec{a}(t) = a_T\vec{T} + a_N\vec{N} = \frac{4t}{\sqrt{5+4t^2}}\vec{T} + 2\sqrt{\frac{5}{5+4t^2}}\vec{N}$$

Finally, we can evaluate this equation at time $t = 1$ to obtain

$$\vec{a}(t) = \frac{4}{3}\vec{T} + \frac{2\sqrt{5}}{3}\vec{N}$$

□

Problem 3. Find the following limit:

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$$

Solution. When evaluating a limit at a point, the first thing we can try is simply plugging in the point in question.

Here, we plug in the point $(0, 0)$ into the limit. This yields

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} = \frac{3(0)^2 - (0)^2 + 5}{(0)^2 + (0)^2 + 2} = \frac{5}{2}$$

Since this evaluates to a real, defined value, this value is the result of our limit, and the answer is $\frac{5}{2}$.

□

Problem 4. Find all second-order partial derivatives for $w = x\sin(x^2y)$.

Solution. First, we will calculate the first-order derivatives.

Using a combination of product rule and chain rule, we obtain

$$\frac{\partial w}{\partial x} = (1)\cdot(\sin(x^2y)) + (x)\cdot(2xy\cos(x^2y)) = \sin(x^2y) + 2x^2y\cos(x^2y)$$

Next, we use chain rule to obtain

$$\frac{\partial w}{\partial y} = (x)\cdot(x^2\cos(x^2y)) = x^3\cos(x^2y)$$

Next, we can take the derivatives of our first-order derivatives to find the second-order derivatives.

$$w_{xx} = (2xy)\sin(x^2y) + (4xy)\cos(x^2y) + (2x^2y)(-\sin(x^2y)) = (2xy - 2x^2y)\sin(x^2y) + 4xy\cos(x^2y)$$

$$w_{xy} = (x^2)\sin(x^2y) + (2x^2)\cos(x^2y) + (2x^2y)(-\sin(x^2y)) = (x^2 - 2x^2y)\sin(x^2y) + 2x^2\cos(x^2y)$$

$$w_{yx} = (3x^2)(\cos(x^2y)) + (x^3)(-2xy\sin(x^2y)) = 3x^2\cos(x^2y) - 2x^4y\sin(x^2y)$$

$$w_{yy} = (x^2)(-x^3\sin(x^2y)) = -x^5\sin(x^2y)$$

$\square$

Problem 5. Evaluate $\frac{dw}{dt}$ for $w = 2ye^x - \ln z$, $x = \ln(t^2+1)$, $y = \arctan t$, $z = e^t$ at time $t = 1$.

Solution. By the rule of chain rule for partial derivatives, we have that

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

As a result, we just need to calculate all of the above derivatives. First, we find each of the partial derivatives for w.

$$\frac{\partial w}{\partial x} = 2ye^x$$

$$\frac{\partial w}{\partial y} = 2e^x$$

$$\frac{\partial w}{\partial z} = -\frac{1}{z}$$

Next, we find the derivatives of each of x, y, z with respect to t.

$$\frac{dx}{dt} = (2t)\cdot\frac{1}{t^2+1} = \frac{2t}{t^2+1}$$

$$\frac{dy}{dt} = \frac{1}{t^2+1}$$

$$\frac{dz}{dt} = e^t$$

Next, we will plug these values into our formula to calculate $\frac{dw}{dt}$, yielding

$$\frac{dw}{dt} = (2ye^x)\left(\frac{2t}{t^2+1}\right) + (2e^x)\left(\frac{1}{t^2+1}\right) + \left(-\frac{1}{z}\right)(e^t)$$

Finally, we write each of x, y, z in terms of t and simplify to obtain

$$\frac{dw}{dt} = 2\arctan t e^{\ln(t^2+1)}\frac{2t}{t^2+1} + 2e^{\ln(t^2+1)}\frac{1}{t^2+1} + -\frac{e^t}{e^t}$$

$$\frac{dw}{dt} = \frac{4t(t^2+1)\arctan t + 2(t^2+1)}{t^2+1} - 1$$

$$\frac{dw}{dt} = 4t\arctan t + 1$$

The final step is to plug $t = 1$ into the expression for our derivative. This yields

$$\frac{dw}{dt} = 4(1)\arctan 1 + 3 = \pi + 1$$

$\square$

Problem 6. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ for $z^3 - xy + yz + y^3 - 2 = 0$ at the point $(1, 1, 1)$.

Solution. We have the following formula for $\frac{\partial z}{\partial x}$:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

First, we will calculate each of the partial derivatives of the surface.

$$F_x = -y$$

$$F_y = -x + z + 3y^2$$

$$F_z = 3z^2 + y$$

Using our formula, we obtain

$$\frac{\partial z}{\partial x} = \frac{y}{3z^2 + y}$$

Similarly, we have that

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{x - z - 3y^2}{3z^2 + y}$$

$\square$

Problem 7. Find the derivative of the function $f(x, y) = 2xy - 3y^2$ at the point $P = (5, 5)$ in the direction of $v = 4\hat{\imath} + 3\hat{\jmath}$.

Solution. This questions tasks us with finding the directional derivative of the function at a point. To do this, we use the following formula for the directional derivative in direction $\vec{u}$:

$$D_{\vec{u}} f(x, y) = \nabla f \cdot \vec{u}$$

The gradient ∇f is given by

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$$

To calculate the gradient ∇f, we need to calculate the partial derivatives of f. Taking the derivatives yields

$$\frac{\partial f}{\partial x} = 2y$$

$$\frac{\partial f}{\partial y} = 2x - 6y$$

This gives us the gradient as

$$\nabla f = \langle 2y, 2x - 6y \rangle$$

To get the direction of $\vec{v}$, we calculate the unit vector in the direction of $\vec{v}$, given by

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

$$\hat{v} = \langle \frac{4}{5}, \frac{3}{5} \rangle$$

Finally, we do our dot product to find the directional derivative.

$$D_{\hat{v}} f(x, y) = \langle 2y, 2x - 6y \rangle \cdot \langle \frac{4}{5}, \frac{3}{5} \rangle = \frac{8y}{5} + \frac{6x - 18y}{5}$$

$$D_{\hat{v}} f(x, y) = \frac{6x - 10y}{5}$$

The final step is to plug in our point P to obtain the final answer. Plugging in $(5, 5)$, we get that

$$D_{\hat{v}} f(5, 5) = -4$$

$\square$

Problem 8. Find the tangent plane and normal line for $2z - x^2 = 0$ given the point $P_0 = (2, 0, 2)$.

Solution. To begin, the equation of a tangent plane to a surface is

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

To use the above equation, we say that the function f is equal to the equation of the surface (in this case, $f(x, y, z) = 2z - x^2$. Then f_x, f_y, f_z are the partial derivatives of f with respect to each variable, and $P = (x_0, y_0, z_0)$.

First, we calculate the partial derivatives.

$$f_x = -2x$$

$$f_y = 0$$

$$f_z = 2$$

Next, we can evaluate these partial derivatives at our point $P = (2, 0, 2)$.

$$f_x(2, 0, 2) = -4$$

$$f_y(2, 0, 2) = 0$$

$$f_z(2, 0, 2) = 2$$

Finally, we can plug these values into our equation for the tangent plane, yielding

$$-4(x - 2) + 2(z - 2) = 0$$

Next, we need to find the normal line to this tangent plane. The equation of the normal line is given by

$$r(t) = \langle x_0, y_0, z_0 \rangle + t\nabla f(x_0, y_0, z_0)$$

Since the gradient is given by $\nabla f = \langle f_x, f_y, f_z \rangle$, we have everything we need to plug into this equation. Therefore, our equation for the normal line is

$$r(t) = \langle 2, 0, 2 \rangle + t\langle -4, 0, 2 \rangle$$

□

Problem 9. Find all local maxima, minima, and saddle points for the function $f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$.

Solution. To find the local maxima, minima, and saddle points for a function, we need to find the critical points of the function. These occur where all of the partial derivatives are equal to zero. Therefore, we first need to find the partial derivatives.

$$\frac{\partial f}{\partial x} = 2x + y + 3$$

$$\frac{\partial f}{\partial y} = x + 2y - 3$$

Next, we can set each of these expressions equal to zero and solve to find the critical points.

$$2x + y + 3 = 0$$

$$x + 2y - 3 = 0$$

Here, we solve the system with a matrix, but you can solve with any method.

$$\left(\begin{array}{cc|c} 2 & 1 & -3 \\ 1 & 2 & 3 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 3 \end{array}\right)$$

Thus $x = -3$ and $y = 3$, so we have one critical point, $(-3, 3)$.

Next, we decide the type of point by calculating the partial derivative f_{xx} and the "Hessian" $f_{xx}f_{yy} - f_{xy}$.

We obtain that

$$f_{xx} = 2$$

$$f_{xx}f_{yy} - f_{xy} = (2)(2) - (1)^2 = 3$$

We determine the type of critical point using the following chart, where the plus and minus signs indicate the sign of each term:

	f_{xx}	$f_{xx}f_{yy} - (f_{xy})^2$
min	$+$	$+$
max	$-$	$+$
saddle		$-$

Since both of our terms are positive, we see that the point $(-3, 3)$ is a local minimum.

$\square$

Problem 10. Find absolute maxima and minima for the function $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0$, $y = 2$, and $y = 2x$ in the first quadrant.

Solution. To find absolute maxima and minima for a function, we need to compare all of the critical points and find the least and greatest values of f at each point. Points where absolute maxima and minima can occur are local maxima and minima of the function in the given region, as well as the edges or corners of the region.

First, we will look for critical points. This requires us to find points at which all partial derivatives are equal to zero. We find our partial derivatives of f as

$$f_x = 4x - 4$$

$$f_y = 2y - 4$$

Setting these equal to zero, we find that this occurs only when $x = 1$ and $y = 2$, the point $(1, 2)$.

Looking at our triangle, we also see that the corners occur at points $(0, 0)$, $(0, 2)$, and $(1, 2)$.

Looking at all critical points and corners, we see that we need to compare $(0,0)$, $(0,2)$, and $(1,2)$. To do this, we just evaluate f at each point. This gives us

$$f(0,0) = 2(0)^2 - 4(0) + (0)^2 - 4(0) + 1 = 1$$

$$f(0,2) = 2(0)^2 - 4(0) + (2)^2 - 4(2) + 1 = -3$$

$$f(1,2) = 2(1)^2 - 4(1) + (2)^2 - 4(2) + 1 = -5$$

Thus $(1,2)$ is our absolute minimum with a value of -5, and $(0,0)$ is our absolute maximum with a value of 1. □

Problem 11. Find the maximum value of the function $f(x,y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$.

Solution. This problem can be solved using Lagrange multipliers. Here, we consider $f(x)$ to be our function which we want to maximize, and our our constraint to be the line given in the problem, which we will call $g(x,y) = x + 3y = 10$.

To use Lagrange multipliers, we solve the system of equations

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$
$$\nabla g(x,y) = k$$

Note that in the above, k is the constant attached to the constraint, in this case 10.

First, we find our gradients of f and g. We find that

$$\nabla f = \langle -2x, -2y \rangle$$

$$\nabla g = \langle `1, 3 \rangle$$

So considering the x and y components of the vectors as separate equations, we get 3 equations in our system:

$$-2x = \lambda$$
$$-2y = 3\lambda$$
$$x + 3y = 10$$

There are many ways to solve this system, but we will only show one way. One thing to be careful about when solving these systems is that you can sometimes miss solutions depending on what operations you use; especially zero solutions!

To begin, we can multiply the first equation on both sides by y, and the second equation on both sides by x. This leaves us with

$$-2xy = \lambda y$$

$$-2xy = 3\lambda x$$

$$x + 3y = 10$$

Now, we can set the first two equations equal to each other, yielding

$$\lambda y = 3\lambda x$$

Thus y = 3x.
Now, we can plug this into the constraint equation, giving

$$x + 9x = 10$$

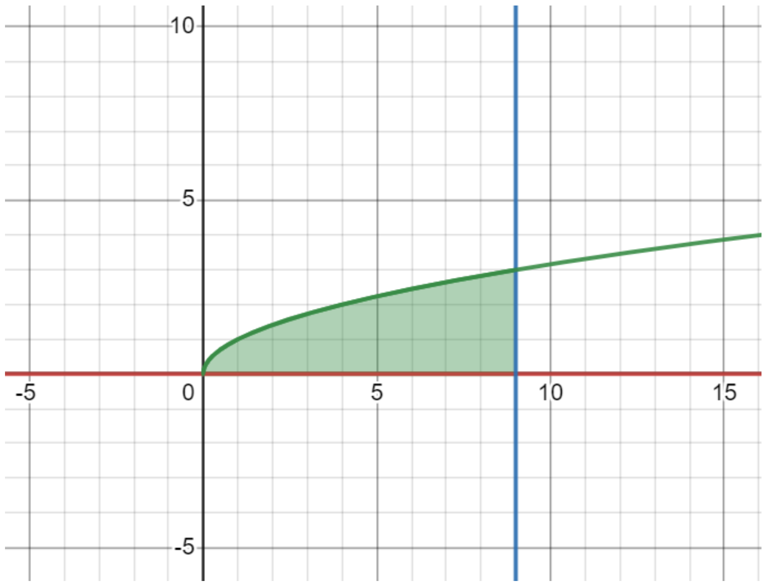
Thus $x = 1$, and $y = 3$.
In this case, we have just one solution point, $(1, 3)$.
Evaluating f at this point, we find that our maximum is

$$f(1, 3) = 39$$

$\square$

Problem 12. Write an iterated integral using (a) vertical cross-sections and (b) horizontal cross sections for the region bounded by $y = \sqrt{x}$, $y = 0$, $x = 9$.

Solution. To begin, we should sketch the region in question. This appears as follows:



To do vertical cross-sections, we first integrate with respect to y, and then to x.

Imagine that we enter the area through the bottom (crossing the line $y = 0$), and then leave through through the top (crossing the line $y = \sqrt{x}$). This defines our bounds for the vertical-cross sections of the integral when we integrate with respect to y.

Next, we need to figure out our bounds for x (the bounds over which we integrate these cross-sections). Referencing our drawing, we see that our area begins on the left at $x = 0$, and ends on the right at $x = 9$. Thus 0 and 9 are our bounds on x.

Finally, we write this double integral for the area, integrating first with respect to y.

$$Area = \int_0^9 \int_0^{\sqrt{x}} dydx$$

Next, we will use horizontal cross-sections to write the iterated integral.

Entering the area from the left, we first cross the line $y = \sqrt{x}$, and then we exit through the right, crossing the line $x = 9$. We need to restate the first bound so that it is in terms of x, which gives us the first bound as $x = y^2$. Thus our bounds for the horizontal cross-sections are y^2 and 9.

Next, we need to find the values for y over which we integrate these cross-sections. Notice that our region is bounded below by $y = 0$, and above at the intersection of the lines $x = y^2$ and $x = 9$, which occurs when $y = 3$. Thus our bounds on y are 0 and 3.

Finally, we can use these bounds to write our iterated integral in the following manner:

$$\int_0^3 \int_{y^2}^9 dxdy$$

$\square$

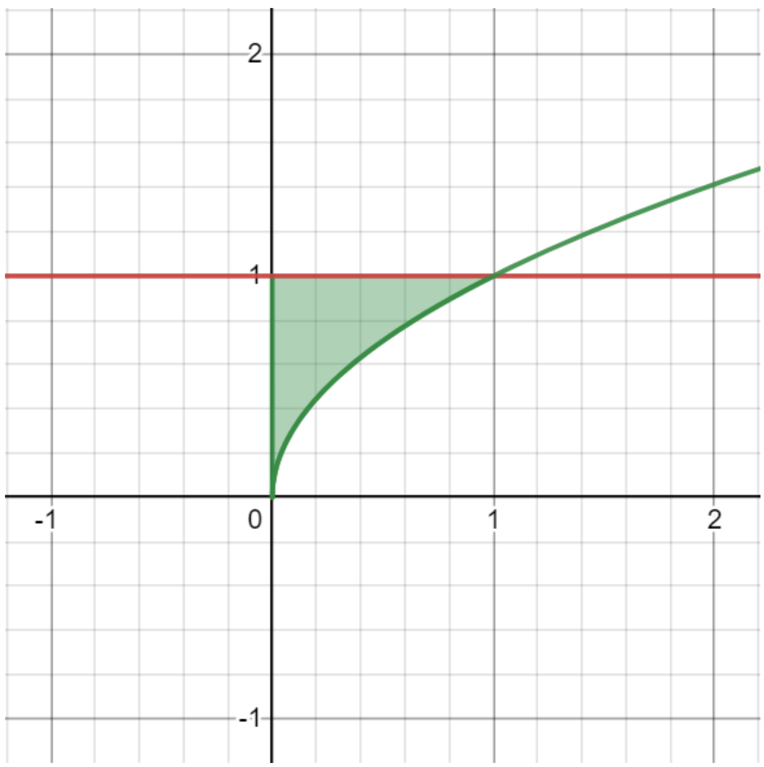
Problem 13. For the following integral, sketch the region implied by the bounds and evaluate the integral.

$$\int_0^1 \int_0^{y^2} 3y^3 e^{xy} dx dy$$

Solution. The bounds on x imply that horizontal cross-sections stretch from $x = 0$ to $x = y^2$; in our drawing, we observe that this is the area bounded on the left by $x = 0$, and on the right by $y = \sqrt{x}$ (this line is rearranged to be in terms of y).

Next, we see that y ranges from 0 to 1. This implies that the region is bounded below by $y = 0$, and above by $y = 1$.

This information allows us to sketch the region as below.



Next, we evaluate the integral.

We begin from

$$\int_0^1 \int_0^{y^2} 3y^3 e^{xy} dx dy$$

Integrating with respect to x, we obtain $3y^2 e^{xy}$. Evaluating from 0 to y^2 gives

$$\int_0^1 (3y^2 e^{y^3} - 3y^2) dy$$

We split this into two integrals as follows.

$$\int_0^1 3y^2 e^{y^3} dy - \int_0^1 3y^2 dy$$

The integral on the right becomes y^3, which when evaluated is equal to 1.

For the integral on the left, we employ u-substitution. Let $u = y^3$, and $du = 3y^2dy$. We adjust our bounds by plugging them into the equation for u, and observe that the integral is bounded by $u = 0^3 = 0$ and $u = 1^3 = 1$. Then we have that the integral on the left is equal to

$$\int_0^1 e^u du$$

This integral is equal to e^u, which when evaluated is equal to $e - 1$. So we have that

$$\int_0^1 3y^2 e^{y^3} dy - \int_0^1 3y^2 dy = (e-1) - (1) = e - 2$$

Therefore, our final answer is $e - 2$.

□

Problem 14. For the following integral, change the region to polar coordinates and evaluate the integral.

$$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} dydx$$

Solution. We can approach this in a couple of different ways, either by sketching the region and describing it in polar coordinates, or by just plugging in our substitutions $x = r\cos\theta$ and $y = \sin\theta$. Regardless of the approach you use, remember to add the Jacobian r into the integral!

Here, we will approach the problem by sketching the region and describing it in polar coordinates.

We see that our vertical cross-sections are bounded below by the line $y = 0$, and above by the semi-circle of radius 1, $y = \sqrt{1-x^2}$. Additionally, x is bounded by $x = -1$ and $x = 1$, which are the same as the points where the semi-circle intersects with the x-axis. As such, we can sketch our region as below.

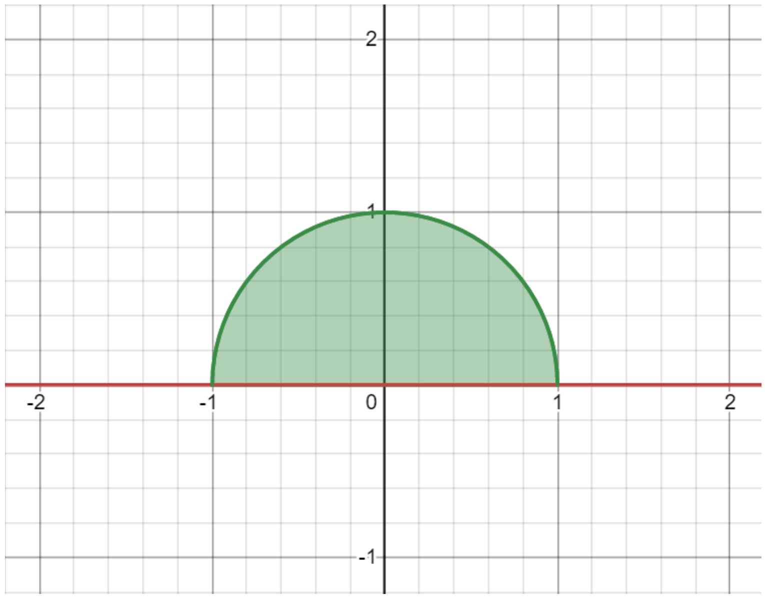
In polar coordinates, we see that our region begins at $r = 0$ since the region is shaded in the center, and ends at $r = 1$ since the circle has radius 1. For the bounds on θ, we see that we begin from $\theta = 0$ on the unit circle, and rotate around until $\theta = \pi$.

Also remember that when we perform a change of coordinates, we need to multiply the integrand by the Jacobian; for polar, this is equal to r.

Therefore, our final integral is

$$\int_0^\pi \int_0^1 rdrd\theta$$

First, we integrate with respect to r, yielding $\frac{r^2}{2}$, which evaluates to $\frac{1}{2}$.



We then have

$$\int_0^1 \frac{1}{2} d\theta$$

This integrates to $\frac{\theta}{2}$, which evaluates to $\frac{\pi}{2}$.

Therefore, our area is $\frac{\pi}{2}$. Notice: this agrees with the formula for the area of a half-circle with radius 1!

$\square$

Problem 15. Integrate the function $f(x, y, z) = 3 - 4x$ over the region bounded above by $z = 4 - xy$, below by $z = 0$, and inside $0 \leq x \leq 2$ and $0 \leq y \leq 1$.

Solution. From the question, we see that our bounds on z are 0 and $4 - xy$, that our bounds for x are 0 and 2, and that our bounds for y are 0 and 1. As a result, we write our triple integral as follows.

$$\int_0^1 \int_0^2 \int_0^{4-xy} (3 - 4x) dz dx dy$$

Integrating first with respect to z, this becomes

$$\int_0^1 \int_0^2 (12 - 16x - 3xy + 4x^2 y) dx dy$$

Next, integrating with respect to x gives $12x - 8x^2 - \frac{3}{2}x^2 y + \frac{4}{3}x^3 y$, which evaluates to $24 - 32 - 6y + \frac{32}{3}y = \frac{14}{3}y - 8$.

We now have

$$\int_0^1 (\frac{14}{3}y - 8) dy$$

This integrates to $\frac{7}{3}y^2 - 8y$, which evaluates to $-\frac{17}{3}$.

Therefore, our result is $-\frac{17}{3}$.

$\square$

Problem 16. Use the change of coordinates $x = \frac{u}{v}$, $y = uv$ to evaluate the integral $\int\int \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy$ over the region in the first quadrant bounded by $xy = 1$, $xy = 9$, $y = x$, $y = 4x$.

Solution. Our first step will be to restate our bounds and integrand in the new coordinates.

Our integrand is $\sqrt{\frac{y}{x}}+\sqrt{xy}$, so this becomes $v+u$. We also need to multiply the integrand by the Jacobian, so we will calculate that now.

The formula for the Jacobian given a change of coordinates $u = f(x, y)$, $v = g(x, y)$ is

$$\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Here, this is equal to

$$\det \begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{pmatrix} = \frac{u}{v} + \frac{u}{v} = \frac{2u}{v}$$

So our final integrand is equal to $\frac{2u}{v}(v + u)$.

Next, we consider our bounds after the transformation. The bound $xy = 1$ becomes $u^2 = 1$, and $xy = 9$ becomes $u^2 = 9$; since x and y are both positive in the first quadrant, we can rewrite these as $u = 1$ and $u = 3$.

The bound $y = x$ becomes $v^2 = 1$, and the bound $y = 4x$ becomes $v^2 = 4$. Again, since x and y are both positive, we have that $v = 1$ and $v = 2$.

As a result, our new region is a rectangle in the uv plane.

Finally, we evaluate the new integral. We have

$$\int_1^2 \int_1^3 \frac{2u}{v}(u + v)dudv$$

First integrating with respect to u, we get $\frac{2u^3}{3v} + \frac{u^2}{v^2}$, which evaluates to $\frac{2(27)}{3v} + \frac{(9)}{v^2} - \frac{2(1)}{3v} - \frac{(1)}{v^2} = \frac{52}{3v} + \frac{8}{v^2}$.

We now have

$$\int_1^2 (\frac{52}{3v} + \frac{8}{v^2})dv$$

This integrates to $\frac{52 \ln v}{3} - \frac{8}{v}$, which evaluates to $\frac{52 \ln 2}{3} + 8$.

Therefore, the final answer is $\frac{52 \ln 2}{3} + 8$.

$\square$

Problem 17. Evaluate the line integral $\int_C (xy + y + z)ds$ along the curve $r(t) = 2t\hat{i} + t\hat{j} + (2 - 2t)\hat{k}$, for time $0 \leq t \leq 1$.

Solution. The formula for evaluating a line integral is as follows. Given functions such that $x = g(t)$, $y = h(t)$, and $z = p(t)$ on $a \leq t \leq b$, we have that

$$\int_C f(x,y,z)ds = \int_a^b f(g(t),h(t),p(t))|r'(t)|dt$$

First, we will find $|r'(t)|$. By taking the derivatives of each component, we see that

$$r'(t) = \langle 2, 1, -2 \rangle$$

The magnitude is then

$$|r'(t)| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$$

So, plugging in our expressions for x, y, z in the curve, we rewrite our integral as

$$\int_0^1 ((2t)(t) + (t) + (2 - 2t))(3)dt = 3\int_0^1 (2t^2 - t + 2)dt$$

Next, we solve the integral.
Integrating, we see that we obtain

$$3(\frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t)\Big|_0^1$$

Which evaluates to $\frac{13}{2}$.

□

Problem 18. Find the flow of the field $F = -4xy\hat{\imath} + 8y\hat{\jmath} + 2\hat{k}$ along the curve $r(t) = t\hat{\imath} + t^2\hat{\jmath} + \hat{k}$, for time $0 \leq t \leq 2$.

Solution. To calculate the flow of a vector field along a path, we use the following formula:

$$Flow = \int_a^b F \cdot r'(t)dt$$

First, we will calculate $r'(t)$. This gives us

$$r'(t) = \langle 1, 2t, 0 \rangle$$

Next, we can rewrite F as a function of t using the x, y, and z components of $r(t)$. This gives us

$$F = \langle -4(t)(t^2), 8(t^2), 2 \rangle = \langle -4t^3, 8t^2, 2 \rangle$$

Finally, we can take the dot product and take our integral. This gives us

$$Flow = \int_0^2 \langle -4t^3, 8t^2, 2 \rangle \cdot \langle 1, 2t, 0 \rangle dt$$

$$Flow = \int_0^2 12t^3 dt$$

$$Flow = 48$$

□

Problem 19. Find the circulation and flux of the field $F = x\hat{\imath} + y\hat{\jmath}$ around $r(t) = \cos t\hat{\imath} + \sin t\hat{\jmath}$ for tie $0 \leq t \leq 2\pi$.

Solution. Given a vector field $F = M\hat{\imath} + N\hat{\jmath}$ and a path $r(t) = x\hat{\imath} + y\hat{\jmath}$, the circulation of the field around the path is given for time $a \leq t \leq b$ by

$$Circulation = \int_a^b (M\frac{dx}{dt} + N\frac{dy}{dt})dt$$

Note that in the above formula, $\frac{dx}{dt}$ is the derivative of the x component of $r(t)$, and $\frac{dy}{dt}$ is the derivative of the y component of $r(t)$.

First, we calculate the derivative of $r(t)$ with respect to t. We obtain that

$$r'(t) = \langle -\sin t, \cos t\rangle$$

Currently, our vector field is stated in terms of x and y. At steps along the path $r(t)$, our x and y are given by the x and y components of the path vector. Thus we can write x and y according to the x and y components of $r(t)$. This gives that $x = \cos t$ and $y = \sin t$, so we have that the vector field in terms of t is given by

$$F = \langle \cos t, \sin t\rangle$$

Finally, we can plug our numbers into the formula for circulation to obtain the following integral:

$$\int_0^{2\pi} (\cos t(-\sin t) + \sin t \cos t)dt$$

This is equivalent to

$$\int_0^{2\pi} 0dt = 0$$

Thus the circulation is equal to zero.

(Thinking conceptually, notice that the vector field always points straight out from the origin, perpendicular to the path of the circle whose center is the origin. Since the vector field is always perpendicular to the path, there is no circulation along the path.)

Next, the flux is given by

$$Flux = \int_a^b M\frac{dy}{dt} - N\frac{dx}{dt}$$

Again plugging into our formula, we obtain the following integral for flux:

$$\int_0^{2\pi} (\cos t \cos t - \sin t(-\sin t))dt$$

This is equivalent to

$$\int_0^{2\pi} (\cos^2 t + \sin^2 t)dt$$

This simplifies to

$$\int_0^{2\pi} dt = 2\pi$$

Thus the flux is equal to 2π.

(Thinking conceptually, notice that our vector field is always perpendicular to our path. As a result, we expect to see some flux across the path, so our answer makes sense!)

□

Problem 20. Find the potential function for the field $F = e^{y+2z}(\hat{\imath} + x\hat{\jmath} + 2x\hat{k})$.

Solution. The potential function of a vector field is a function such that the gradient of the potential function is the vector field. In a sense, if taking the gradient of a function is like taking the derivative of a function, then finding the potential function of a vector field is like finding the integral of the vector field.

To begin, we have from the definition of the potential function that $f_x = e^{y+2z}$, $f_y = xe^{y+2z}$, and $f_z = 2xe^{y+2z}$. These are the corresponding partial derivatives of each variable given as a vector field in the problem.

Next, we can integrate f_x with respect to x to get our function $f(x, y, z)$ as a sum of some function and a function $g(y, z)$. Let's try this.

$$f(x, y, z) = \int f_x dx$$

$$f(x, y, z) = \int e^{y+2z} dx$$

Evaluating, we get

$$f(x, y, z) = xe^{y+2z} + g(y, z)$$

Just to check our work, notice that the partial derivative of f with respect to x is e^{y+2z}, which is what we should expect if F is the gradient of f. We have to add $g(y, z)$ because we treat y and z as constants when we differentiate with respect to x, so $g(y, z)$ is like our constant of integration when we integrate with respect to x.

From here, let's differentiate the above equation with respect to y. This yields

$$f_y = xe^{y+2z} + g_y$$

Note that here, g_y is the partial derivative of g with respect to y. Next, we can plug in our expression for f_y given in the problem to solve for g_y. This yields

$$xe^{y+2z} = xe^{y+2z} + g_y$$

Thus $g_y = 0$. Since the partial derivative of g with respect to y is zero, we know that g does not contain y, and is therefore a function only of z. So $g(y, z) = h(z)$ for some function h, and we can rewrite an earlier equation as

$$f(x, y, z) = xe^{y+2z} + h(z)$$

Here, let's take the derivative of the whole equation with respect to z. This yields

$$f_z = 2xe^{y+2z} + h_z$$

Next, we plug in the expression given in the problem for f_z, giving

$$2xe^{y+2z} = 2xe^{y+2z} + h_z$$

Thus $h_z = 0$. This means that the function $h(z)$ must be some constant C. Thus we have that

$$f(x, y, z) = xe^{y+2z} + C$$

This is our final answer, but we could check our answer by calculating the gradient of f and confirming that $\nabla f = F$.

□

Problem 21. Use Green's Theorem to find counterclockwise circulation and outward flux for the field $F = (y^2 - x^2)\hat{\imath} + (x^2 + y^2)\hat{\jmath}$ and the curve C defined by $y = 0$, $x = 3$, $y = x$.

Solution. The circulation form of Green's Theorem states that given a closed curve C bounding a region D, we have that

$$Circulation = \oint_C \vec{F} \cdot \hat{T} ds = \int\int_D (N_x - M_y) dA$$

Note that here, we have that $\vec{F} = \langle M, N \rangle$.

We will use the rightmost side of this identity to solve the problem.

Our first step is to calculate N_x and M_y, the partial derivatives of N and M with respect to x and y respectively. Here, we obtain the following:

$$N_x = 2x$$

$$M_y = 2y$$

Next, we define the bounds of our region using the given curve in the problem. Here, the curve C bounds the triangle bounded by the lines $y = 0$, $x = 3$, $y = x$. This allows us to define our bounds with vertical cross sections, observing that y ranges from 0 to x. Then we can integrate over these cross sections on $0 \leq x \leq 3$ to obtain our answer.

We write our integral as follows:

$$\int_0^3 \int_0^x (2x - 2y) dy dx$$

Solving the inner integral, we find that this is equivalent to the following:

$$\int_0^3 (2x^2 - x^2)dx$$

$$\int_0^3 x^2 dx$$

Solving, we obtain

$$\int_0^3 x^2 dx = 9$$

Thus the circulation is equal to 9.
Next, we calculate the flux. The flux form of Green's Theorem states that

$$Flux = \oint_C \vec{F} \cdot \hat{N} ds = \int \int_D (M_x + N_y) dA$$

First, we calculate M_x and N_y.

$$M_x = -2x$$

$$N_y = 2y$$

Using the same bounds as before, we write our flux integral as

$$\int_0^3 \int_0^x (2y - 2x) dy dx$$

Solving the inner integral, we obtain

$$\int_0^3 -x^2 dx$$

Solving, we see that

$$\int_0^3 -x^2 dx = -9$$

Thus the flux is equal to -9.

□

Problem 22. Use a parameterization to express the area of the surface S using a double integral, where S is the portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 6$.

Solution. The goal of parameterization of a surface is to express a surface using two parameters as a vector valued function $\vec{r}(t) = \langle x(u,v), y(u,v), z(u,v)\rangle$ where x, y, and z are three functions of two bounded variables u and v.

Here, notice that the surface S is a cone, and that we are given z as a function of x and y. We will attempt this parameterization by converting our coordinates to cylindrical coordinates, expressing each of x, y, and z as a function of r and θ. Since we are given z as a function of x and y, we should be able to express every variable this way.

First, let $x = r\cos\theta$ and $y = r\sin\theta$. Next, we can substitute for x and y in our equation for z to solve for the function of z. This gives us the following:

$$z = 2\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}$$

$$z = 2r\sqrt{\cos^2\theta\sin^2\theta}$$

$$z = 2r$$

Finally, we need to define the bounds on r and θ. Since the cone forms a full circle around the origin, we know $0 \leq \theta \leq 2\pi$.

Next, since we are given in the problem that $2 \leq z \leq 6$ and we have that $r = \frac{z}{2}$, we see that $1 \leq r \leq 3$.

Therefore, our parameterization of the surface S is given by

$$\vec{r}(t) = \langle r\cos theta, r\sin theta, 2r\rangle; 1 \leq r \leq 3, 0 \leq \theta \leq 2\pi$$

The final step is to use this parameterization to express the area of the surface as a double integral. Given a parameterization $\vec{r}(u,v)$ of a surface S, we can express the area of S as follows:

$$\int\int_D ||\vec{r}_u \times \vec{r}_v|| dA$$

Note that this is a special case of the following formula (when calculating area, we let $f(x,y,z) = 1$):

$$\int\int_S f(x,y,z)dS = \int\int_D f(\vec{r}(u,v))||\vec{r}_u \times \vec{r}_v||dA$$

First, observe that

$$\vec{r}_r = \langle\cos\theta, \sin\theta, 2\rangle$$

$$\vec{r}_\theta = \langle -r\sin\theta, r\cos theta, 0\rangle$$

Taking the cross product, we get that

$$\vec{r}_r \times \vec{r}_\theta = \langle -2r\cos\theta, -2r\sin\theta, r$$

Finally, taking the magnitude, we obtain

$$||\vec{r}_r \times \vec{r}_\theta|| = \sqrt{5}r$$

Then plugging into our formula, we obtain the surface area as the following double integral:

$$\int_1^3 \int_0^{2\pi} \sqrt{5}r d\theta dr$$

$\square$

Problem 23. Evaluate the double integral $\int\int 2ydS$ over S, the surface given by the portion of the surface $y^2 + z^2 = 4$ between the planes $x = 0$ and $x = 3 - z$.

Solution. Our first step will be to find a parameterization of the surface S.

Notice that our surface is bounded by the cylinder centered on the x axis of radius 2, as well as by bounds on x. We will express our parameterization using a version of cylindrical coordinates where $y = r\sin theta$, $z = r\cos theta$, and $x = x$. In this case, we know that $r = 2$ since our circle has radius 2, so our parameterization is in terms of θ and x.

We obtain our parameterization as the following:

$$\vec{r}(t) = \langle x, 2\sin\theta, 2\cos\theta\rangle; 0 \leq x \leq 3 - 2\cos\theta, 0 \leq \theta \leq 2\pi$$

Next, we calculate $\vec{r}_x$ and $\vec{r}_\theta$.

$$\vec{r}_x = \langle 1, 0, 0\rangle$$

$$\vec{r}_\theta = \langle 0, 2\cos\theta, -2\sin\theta\rangle$$

Next, we find $\vec{r}_x \times \vec{r}_\theta$.

$$\vec{r}_x \times \vec{r}_\theta = \langle 0, 2\sin\theta, 2\cos\theta$$

Finally, we find $||\vec{r}_x \times \vec{r}_\theta||$.

$$||\vec{r}_x \times \vec{r}_\theta|| = 2(\sin^2\theta + \cos^2\theta) = 2$$

Next, we use the following formula to convert our double integral into one which uses our parameterization:

$$\int\int_S f(x, y, z)dS = \int\int_D f(\vec{r}(u, v))||\vec{r}_u \times \vec{r}_v||dA$$

Using the right side of the equation, we write our integral as the following:

$$\int_0^{2\pi} \int_0^{3-2\cos\theta} 2(2\sin\theta)(2)dxd\theta$$

$$\int_0^{2\pi} \int_0^{3-2\cos\theta} 8\sin\theta dxd\theta$$

Finally, we can evaluate the integral. Performing the inner integral, we obtain

$$\int_0^{2\pi} 8\sin\theta(3-2\cos\theta)d\theta$$

Next, we can use u substitution to evaluate the integral. Letting $u = 3 - 2\cos\theta$, we obtain that $du = 2\sin thetady$. This allows us to rewrite the integral as

$$\int_1^1 4udu = 0$$

Then since the integral bound is from 1 to 1, we see that the final result is 0.

Therefore, the result of the double integral is 0.

$\square$

Hi to all students, TAS employees, and volunteers! We hope that this solution sheet is a valuable resource to you all! Thank you for support, and good luck!

Best, Alex and John. ♡