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Topics in Stochastic Programming

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Topics in Stochastic Programming

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This monograph is a follow up to the recently published book [30], where additional technical details and proofs can be found. Hospitality of the Center for Operations Research and Econometrics (CORE) of Université Catholique de Louvain is greatly appreciated and acknowledged.

1 Introduction

In the last decade a considerable progress was made in the area of stochastic programming. At the same time an alternative and competing approach to optimization under uncertainty was developed in terms of Robust Optimization. One of the criticisms of stochastic programming is that in many applications the basic assumption of knowing, or even accurately estimating, the probability distribution of the uncertain data is unrealistic. Moreover, quite often the classical concept of probability distribution, based on the frequency approach, is not applicable – uncertainty does not necessarily mean randomness. On the other hand, the worst case approach of Robust Optimization could be too conservative. This motivated a renewed interest in a minimax approach to stochastic programming where one can identify a relevant family of probability distributions and consequently tries to solve the obtained worst probability stochastic programming problem. By duality techniques in some cases the minimax approach can be represented in terms of a risk averse stochastic programming. We will discuss this in various parts of this monograph, specifically for two stage problems in sections 2.4 and 2.5 and for multistage problems in section 3.4.

Another progress is related to development of Monte Carlo based randomization methods for solving stochastic programs and the associated complexity theory. For a long time approaches to modeling and solving stochastic programming problems were dominated by scenario generation methods. That is a finite, computationally manageable, number of scenarios, i.e., realizations of the data process with assigned probabilities, was generated and consequently the constructed optimization problem was solved by decomposition type methods. An argument is that considering many scenarios is certainly better than solving the problem for just one scenario which would be a deterministic optimization approach.

If one takes the position that generated scenarios represent reality in a reasonably accurate way, then there is no dramatic difference between two and multistage stochastic programming. Everybody would agree, however, that what will really happen in the future will be different with probability one from the set of generated scenarios. This raises the question of what does it mean to solve a stochastic programming problem? In that respect we may cite [3, p.413]: *"... it is absolutely unclear what the resulting solution [of a scenario based approximation of a multistage*

stochastic program] has to do with the problem we intend to solve. Strictly speaking, we even cannot treat this solution as a candidate solution, bad or good alike, to the original problem – the decision rules we end up with simply do not say what our decisions should be when the actual realizations of the uncertain data differ from the scenario realizations."

Of course, one can utilize only the computed first stage solution, which is deterministic, while recomputing (updating) it at every stage as a new information (realization of the uncertain data) becomes available. In such a rolling horizon approach one has to decide on how many stages to look ahead. From a computational point of view looking one stage ahead, i.e., solving two-stage problems, could be reasonably justified by employing Monte Carlo sampling randomization techniques. From the point of view of a number of (randomly generated) scenarios, computational complexity of two-stage stochastic programming problems is discussed in detail in [30, Chapter 5]. The conclusion is that certain classes of two-stage stochastic programming problems (in particular, linear two-stage stochastic programs with relatively complete recourse) can be solved with reasonable accuracy and reasonable computational effort.

From that point of view the number of scenarios, of the "true" problem, is irrelevant and can be astronomically large or even infinite. On the other hand, it turns out that computational complexity of multistage stochastic programming problems, measured in terms of required number of generated scenarios, is conceptually different. This gives a motivation for looking for other than scenario generation methods for solving in some reasonable sense multistage stochastic programming problems. One possible approach is to approximate dynamic programming equations. In section 5.2 we discuss one such method. Another approach is to construct a parameterized family of implementable policies. We will shortly discuss this in section 5.3.

2 Two Stage Problems

2.1 General Formulation

We consider the following robust formulation of stochastic problems

$$\operatorname*{Min}_{x\in\mathcal{X}} \left\{ f(x) := \sup_{P\in\mathfrak{M}} \mathbb{E}_P[F(x,\xi)] \right\}. \tag{2.1}$$

Here $\mathcal{X} \subset \mathbb{R}^n$ is a nonempty set, $F : \mathbb{R}^n \times \Xi \to \overline{\mathbb{R}}$ is an extended[1] real valued function and $\mathfrak{M}$ is a set of probability measures (distributions) on the set $\Xi \subset \mathbb{R}^d$. We assume

[1] By $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ we denote the extended real line.

that the set Ξ is closed and is equipped with its Borel sigma algebra. The expectation $\mathbb{E}_P[F(x,\xi)]$ is taken with respect to the probability distribution P of random vector ξ. We assume that for every $x \in \mathcal{X}$ and $P \in \mathfrak{M}$ this expectation is well defined. We use the same notation ξ for random vector and its particular realization, which of these two meaning is used will be clear from the context. If $\mathfrak{M} = \{P\}$ is a singleton, i.e., the set $\mathfrak{M}$ consists of single probability measure P, then (2.1) becomes

$$\operatorname*{Min}_{x \in \mathcal{X}} \big\{ f(x) := \mathbb{E}_P[F(x,\xi)] \big\}, \tag{2.2}$$

which is a standard formulation of a stochastic programming problem.

The set Ξ can be viewed as a set of possible realizations (called *scenarios*) of the random (uncertain) data vector ξ. If the set $\Xi = \{\xi_1, ..., \xi_K\}$ is finite, we say that the problem has a finite number of scenarios. Then every probability measure P on Ξ is defined by probabilities $p_i \geq 0$, $p_1 + ... + p_K = 1$, and $\mathbb{E}_P[F(x,\xi)] = \sum_{i=1}^K p_i F(x,\xi_i)$.

The above formulation is static in the sense that the objective function $F(x,\xi)$ is supposed to be given explicitly in a computable form. In two stage problems it is given as the optimal value of the corresponding second stage problem

$$\operatorname*{Min}_{y \in \mathcal{G}(x,\xi)} g(x,y,\xi), \tag{2.3}$$

where $g : \mathbb{R}^n \times \mathbb{R}^m \times \Xi \to \mathbb{R}$ and $\mathcal{G} : \mathbb{R}^n \times \Xi \rightrightarrows \mathbb{R}^m$ is a multifunction (point-to-set mapping). For example, linear two-stage problem can be formulated in the above form as

$$\operatorname*{Min}_{x \in \mathbb{R}^n} c^{\mathsf{T}} x + \mathbb{E}[Q(x,\xi)] \ \text{ subject to } Ax = b,\ x \geq 0, \tag{2.4}$$

where $Q(x,\xi)$ is the optimal value of the problem

$$\operatorname*{Min}_{y \geq 0} q^{\mathsf{T}} y \ \text{ subject to } Tx + Wy = h. \tag{2.5}$$

Here data vector ξ consists of elements of vectors q and h and matrices T and W, the functions $F(x,\xi) := c^{\mathsf{T}} x + Q(x,\xi)$ and $g(x,y,\xi) := q^{\mathsf{T}} y$ and the multifunction

$$\mathcal{G}(x,\xi) := \{y \in \mathbb{R}^m : Tx + Wy = h,\ y \geq 0\}. \tag{2.6}$$

The class of linear two-stage problems is especially important and we will discuss it in more details.

In principle it could happen that for some $x \in \mathcal{X}$ and $\xi \in \Xi$ the minimization problem (2.3) is unbounded from below and hence its optimal value $F(x,\xi) = -\infty$. This is a somewhat pathological situation meaning that for the first stage decision x the cost of the second stage problem could be arbitrary small. We assume that one

makes sure at the modeling stage that this does not happen. For example, the dual of the linear programming problem (2.5) is

$$\operatorname*{Max}_{\pi} \pi^{\mathsf{T}}(h - Tx) \text{ subject to } W^{\mathsf{T}}\pi \leq q, \tag{2.7}$$

and hence $Q(x, \xi) > -\infty$ if (and in fact only if provided the problem (2.5) is feasible) the system $W^{\mathsf{T}}\pi \leq q$ has a feasible solution.

It also could happen that for some $x \in \mathcal{X}$ and $\xi \in \Xi$ the feasible set of problem (2.3) is empty, i.e., $\mathcal{G}(x, \xi) = \emptyset$. In that case, by the definition, the optimal value of problem (2.3) $F(x, \xi) = +\infty$. If for some $P \in \mathfrak{M}$ this happens with a positive probability, then of course $\mathbb{E}_P[F(x, \xi)] = +\infty$. We say that the two stage problem has *relatively complete recourse* [2] if $\mathcal{G}(x, \xi) \neq \emptyset$ for all $x \in \mathcal{X}$ and $\xi \in \Xi$. It is straightforward to see that the linear second stage problem (2.5) is feasible, i.e., $Q(x, \xi) < +\infty$, if and only if

$$h - Tx \in \operatorname{pos} W, \tag{2.8}$$

where

$$\operatorname{pos} W := \{u : u = Wy,\ y \geq 0\} \tag{2.9}$$

is the *positive hull* of matrix W. If the number of scenarios is finite and not too large, then for any given $x \in \mathcal{X}$ it could be possible to verify with a reasonable computational effort whether the feasible set $\mathcal{G}(x, \xi)$ of the second stage problem is nonempty for every $\xi \in \Xi$. The situation is different if ξ has a continuous distribution. If for a given $x \in \mathcal{X}$ and $P \in \mathfrak{M}$ the probability of the event $\mathcal{G}(x, \xi) = \emptyset$ is very small, say 10^{-6}, then we may never see that happens by generating a finite number of realizations (scenarios) of the random vector. On the other hand, if this probability is positive, does not matter how small, then we set $\mathbb{E}_P[F(x, \xi)] = +\infty$, i.e., such point x will be infeasible for the first stage problem. This motivates the discussion of the following section 2.2.

2.1.1 Interchangeability of Minimization and Expectation Operators

Consider an abstract set Ω and let $\mathcal{F}$ be a sigma algebra of subsets of Ω. We refer to $(\Omega, \mathcal{F})$ as a *measurable space*. If, moreover, a probability measure (distribution) P is defined on $(\Omega, \mathcal{F})$, then $(\Omega, \mathcal{F}, P)$ becomes a *probability space*. Of course, a closed set $\Xi \subset \mathbb{R}^d$ equipped with its Borel sigma algebra becomes a measurable space and for considered applications it could be sufficient to consider such spaces only. However, this is not essential at this point.

[2] If $\mathfrak{M} = \{P\}$ is a singleton, then the relatively complete recourse is often defined as that for all $x \in \mathcal{X}$ the corresponding second stage problem is feasible for P-almost every realization of the random data.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. It is said that a linear space $\mathfrak{Y}$ of $\mathcal{F}$-measurable functions (mappings) $\psi : \Omega \to \mathbb{R}^m$ is *decomposable* if for every $\psi \in \mathfrak{Y}$ and $A \in \mathcal{F}$, and every bounded and $\mathcal{F}$-measurable function $\gamma : \Omega \to \mathbb{R}^m$, the space $\mathfrak{Y}$ also contains the function $\eta(\cdot) := \mathbf{1}_{\Omega\setminus A}(\cdot)\psi(\cdot) + \mathbf{1}_A(\cdot)\gamma(\cdot)$. For example, for $p \in [1, \infty)$ the space of measurable mappings $\psi : \Omega \to \mathbb{R}^m$ such that $\int_\Omega \|\psi\|^p dP < +\infty$, denoted $\mathcal{L}_p(\Omega, \mathcal{F}, P; \mathbb{R}^m)$, is decomposable. Recall that an extended real valued function $g : \mathbb{R}^m \times \Omega \to \overline{\mathbb{R}}$ is said to be random lower semicontinuous if the associated epigraphical multifunction $\omega \mapsto \operatorname{epi} g(\cdot, \omega)$ is closed valued and measurable. Proof of the following theorem can be found in [21, Theorem 14.60].

Theorem 2.1 *Let $\mathfrak{Y}$ be a decomposable space and $g : \mathbb{R}^m \times \Omega \to \overline{\mathbb{R}}$ be a random lower semicontinuous function. Then*

$$\mathbb{E}\left[\inf_{y\in\mathbb{R}^m} g(y, \omega)\right] = \inf_{y(\cdot)\in\mathfrak{Y}} \mathbb{E}\big[g(y(\omega), \omega)\big], \tag{2.10}$$

provided that the right hand side of (2.10) *is less than $+\infty$. Moreover, if the common value of both sides in* (2.10) *is not $-\infty$, then $\bar{y}(\cdot) \in \arg\min_{y(\cdot)\in\mathfrak{Y}} \mathbb{E}[g(y(\omega), \omega)]$ if and only if*

$$\bar{y}(\omega) \in \arg\min_{y\in\mathbb{R}^m} g(y, \omega), \text{ for a.e. } \omega \in \Omega, \text{ and } \bar{y}(\cdot) \in \mathfrak{Y}. \tag{2.11}$$

The notation $y(\cdot)$ in the above theorem emphasizes that $y(\cdot)$ is considered as an element of the functional space $\mathfrak{Y}$, while $y \in \mathbb{R}^m$ is an m-dimensional vector.

2.2 Decision Rules

In this section we assume that the probability distribution of random vector ξ is (uniquely) specified, i.e., $\mathfrak{M} = \{P\}$ is a singleton, and all probability statements will be made with respect to the probability measure (distribution) P. By using interchangeability of minimization and expectation operators (see Theorem 2.1), the two-stage problem (2.1)–(2.3) can be written as one large problem

$$\begin{array}{ll} \underset{x, y(\cdot)}{\operatorname{Min}} & \mathbb{E}[g(x, y(\xi), \xi)] \\ \text{s.t.} & x \in \mathcal{X},\ y(\xi) \in \mathcal{G}(x, \xi) \text{ a.e. } \xi \in \Xi. \end{array} \tag{2.12}$$

Here the optimization is performed over $x \in \mathbb{R}^n$ and functions $y(\cdot) : \Xi \to \mathbb{R}^m$ belonging to a specified (decomposable) functional space $\mathfrak{Y}$ (cf., [30, Theorem 2.20]). If the number of scenarios $\Xi = \{\xi_1, ..., \xi_K\}$ is finite, we can identify functions $y(\cdot)$ with the set of vectors $y_i = y(\xi_i)$, $i = 1, ..., K$, and hence write problem (2.12) as

$$\begin{array}{ll} \underset{x, y_1, ..., y_K}{\operatorname{Min}} & \sum_{i=1}^K p_i g(x, y_i, \xi_i) \\ \text{s.t.} & x \in \mathcal{X},\ y_i \in \mathcal{G}(x, \xi_i),\ i = 1, ..., K. \end{array} \tag{2.13}$$

In case of continuous distribution P, when the number of scenarios is infinite, problem (2.12) becomes an infinite dimensional problem. A particular function $y(\cdot) \in \mathcal{Y}$ is called a *policy* or a *decision rule*. For a given $x \in \mathcal{X}$ a decision rule $y(\cdot)$, satisfying the feasibility constraints $y(\xi) \in \mathcal{G}(x, \xi)$ for a.e. $\xi \in \Xi$, specifies the corresponding second stage decision for every possible realization of the random data. We can restrict the optimization in (2.12) to a chosen parametric family of decisions $y(\cdot) = \bar{y}(\cdot, \theta)$ parameterized by finite dimensional vector $\theta \in \mathbb{R}^k$. This leads to the following restriction of problem (2.12)

$$\operatorname*{Min}_{x,\theta} \mathbb{E}[g(x, \bar{y}(\xi, \theta), \xi)] \text{ subject to } x \in \mathcal{X},\ \theta \in \Theta(x), \tag{2.14}$$

where

$$\Theta(x) := \left\{\theta \in \mathbb{R}^k : \bar{y}(\xi, \theta) \in \mathcal{G}(x, \xi),\ \text{a.e. } \xi \in \Xi\right\}.$$

Of course the optimal value of problem (2.14) is greater than or equal to the optimal value of problem (2.12) and equality holds if the considered parameterization contains an optimal solution of the problem (2.12), i.e., for an optimal solution $\bar{x}$ of the first stage problem and some $\theta^* \in \mathbb{R}^k$ it holds that $\bar{y} = \bar{y}(\xi, \theta^*)$ is an optimal solution of the second stage problem (2.3) for $x = \bar{x}$ and a.e. $\xi \in \Xi$.

Example 1 Consider the linear two-stage stochastic problem (2.4)–(2.5). Assume that the $m \times 1$ vector q and $\ell \times m$ matrix W are deterministic (not random), i.e., the recourse is fixed, while elements of T and h form random vector ξ, and that the feasible set $\{\pi : W^{\mathsf{T}}\pi \leq q\}$ of the dual problem (2.7) is nonempty and hence $Q(x, \xi) > -\infty$ for all x and ξ. Let us discuss what will be optimal policies for such linear two-stage problems.

Let us fix a point $x \in \mathcal{X}$ and consider (random) vector $u := h - Tx$. We have that y and π are optimal solutions of problems (2.5) and (2.7), respectively, iff

$$Wy = u,\ y \geq 0,\ W^{\mathsf{T}}\pi - q \leq 0,\ y^{\mathsf{T}}(W^{\mathsf{T}}\pi - q) = 0. \tag{2.15}$$

It follows from (2.15) that the set of optimal solutions of (2.5) is unbounded iff the feasible sets of problems (2.5) and (2.7) are nonempty, and hence (2.5) has an optimal solution, and the system $Wy = 0$, $q^{\mathsf{T}}y = 0$, $y \geq 0$, has a nonzero solution. In any case, since the cone $\mathbb{R}^m_+$ does not contain linear spaces (except the trivial space consisting of the null vector), it follows that the set of optimal solutions of problem (2.5) does not contain linear spaces. Consequently if the optimal value of problem (2.5) is finite, then it attains its optimal value at an extreme point of its feasible set (of course, if (2.5) has more than one optimal solution, then some of them will be not extreme points of the feasible set).

Denote by $\mathfrak{O}(u)$ the set of optimal solutions of problem

$$\operatorname*{Min}_{y\geq 0} q^{\mathsf{T}}y \text{ subject to } Wy = u. \tag{2.16}$$

Then, of course, $\mathfrak{O}(h - Tx)$ is the set of optimal solutions of problem (2.5). The linear programming problem (2.16) has an optimal solution provided its optimal value is finite. Since we assume here that the dual of (2.16) is feasible, and hence the optimal value of (2.16) is greater than $-\infty$, it follows that the domain $\operatorname{dom}\mathfrak{O} := \{u : \mathfrak{O}(u) \neq \emptyset\}$ of the point-to-set mapping (multifunction) $\mathfrak{O}(\cdot)$ coincides with such u that problem (2.16) has a feasible solution, that is

$$\operatorname{dom}\mathfrak{O} = \operatorname{pos} W,$$

and hence $\operatorname{dom}\mathfrak{O}$ is a closed polyhedral cone.

By Hoffman's lemma we have that that the multifunction $\mathfrak{O}(\cdot)$ is Lipschitz continuous in the following sense: there is a constant $\kappa > 0$, depending only on matrix W, such that

$$\operatorname{dist}(y, \mathfrak{O}(u')) \leq \kappa\|u - u'\|, \ \forall\, u, u' \in \operatorname{dom}\mathfrak{O}, \ \forall y \in \mathfrak{O}(u).$$

In other words $\mathfrak{O}(\cdot)$ is Lipschitz continuous on its domain with respect to the Hausdorff metric.

Assume that the $\ell \times m$ matrix W has rank ℓ, i.e., rows of W are linearly independent. For an index set $\mathcal{I} \subset \{1, ..., m\}$ denote by $W_{\mathcal{I}}$ the submatrix of W formed by columns of W indexed by $\mathcal{I}$ and by $y_{\mathcal{I}}$ the subvector of y formed by components y_i, $i \in \mathcal{I}$. By the well known result of linear programming we have that a feasible point y is an extreme point (basic solution) of the feasible set of problem (2.16) iff there exists an index set $\mathcal{I} \subset \{1, ..., m\}$ of cardinality ℓ such that the matrix $W_{\mathcal{I}}$ is nonsingular and $y_i = 0$ for $i \in \{1, ..., m\} \setminus \mathcal{I}$. We also have that extreme points of the optimal set $\mathfrak{O}(u)$ are extreme points of the feasible set. Therefore if the optimal set $\mathfrak{O}(u)$ is nonempty, then it has an extreme point $\bar{y} = \bar{y}(u)$ and an index set $\mathcal{I} = \mathcal{I}(u)$ such that $W_{\mathcal{I}}\bar{y}_{\mathcal{I}} = u$ and $\bar{y}_i = 0$ for $i \in \{1, ..., m\} \setminus \mathcal{I}$. This can be written as $\bar{y}(u) = R_{\mathcal{I}}u$, where $R_{\mathcal{I}}$ is $m \times \ell$ matrix with the rows $[R_{\mathcal{I}}]_i = [W_{\mathcal{I}}^{-1}]_i$ for $i \in \mathcal{I}$ and $[R_{\mathcal{I}}]_i = 0$ for $i \in \{1, ..., m\} \setminus \mathcal{I}$.

If for some $\bar{u}$ the optimal set $\mathfrak{O}(\bar{u}) = \{\bar{y}\}$ is a singleton, then $\bar{y}(u)$ is continuous at $\bar{u}$ for any choice of $\bar{y}(u) \in \mathfrak{O}(u)$. In any case it is possible to choose an extreme point $\bar{y}(u) \in \mathfrak{O}(u)$ such that $\bar{y}(u)$ is continuous on $\operatorname{dom}\mathfrak{O}$. Indeed, it is possible to choose a vector $a \in \mathbb{R}^m$ such that $a^{\mathsf{T}}y$ has unique minimizer over $y \in \mathfrak{O}(u)$ for all $u \in \operatorname{dom}\mathfrak{O}$. This minimizer is an extreme point of $\mathfrak{O}(u)$ and is continuous in $u \in \operatorname{dom}\mathfrak{O}$. So we can choose a continuous policy $\bar{y}(u)$, $u \in \operatorname{dom}\mathfrak{O}$, and a finite collection $\mathfrak{I}$ of index sets $\mathcal{I} \subset \{1, ..., m\}$, of cardinality ℓ, such that $\bar{y}(u) = R_{\mathcal{I}}u$ for some $\mathcal{I} = \mathcal{I}(u) \in \mathfrak{I}$. In summary we can write (cf., [10]):

- It follows that under the specified assumptions and if the first stage problem has an optimal solution $\bar{x}$, then the linear two-stage stochastic problem (2.4)–(2.5) has an optimal policy given by a continuous piecewise linear function with pieces of the form $\bar{y}(\xi) = R_{\mathcal{I}}(h - T\bar{x})$, $h - T\bar{x} \in \operatorname{pos} W$.

$\Diamond$

2.2.1 Affine Decision Rules

Consider the linear two-stage stochastic problem with the second stage problem given in the form

$$\operatorname*{Min}_{y \in \mathbb{R}^m} q^{\mathsf{T}} y \ \text{ s.t. } \ Tx + Wy \leq h, \ y \geq 0. \tag{2.17}$$

Of course, it can be transformed into form (2.5) by adding slack variables:

$$\operatorname*{Min}_{y' \geq 0} q^{\mathsf{T}} y \ \text{ s.t. } \ Tx + W'y' = h. \tag{2.18}$$

where[3] $W' = [W, -I]$ and $y' = \begin{bmatrix} y \\ z \end{bmatrix}$.

We assume now that only the right hand side vector h is random, whose distribution is supported[4] on set $\mathcal{H} \subset \mathbb{R}^\ell$. By the discussion of Example 1 the considered two-stage problem has (under mild regularity conditions) a continuous piecewise affine optimal policy. It could be too difficult to handle piecewise affine policies (decision rules), so let us consider the following *affine decision rule* $y = Dh + d$, where vector d and matrix D are viewed as parameters defining a particular policy. Then the feasibility system of the second stage problem (2.5) takes the form

$$Tx + (WD - I)h + Wd \leq 0, \ Dh + d \geq 0, \ \ \forall h \in \mathcal{H}. \tag{2.19}$$

Note that constraints (2.19) are linear in (D, d). Therefore we can write constraints (2.19) in the following equivalent form[5] (cf., [3, pp. 11-12])

$$\begin{aligned} &T_i x + \max_{h \in \mathcal{H}} [WD - I]_i h + W_i d \leq 0, \ \ i = 1, ..., \ell, \\ &\max_{h \in \mathcal{H}} [-D]_i h - d_i \leq 0, \ \ i = 1, ..., \ell. \end{aligned} \tag{2.20}$$

For some "simple" sets $\mathcal{H}$ the maxima in (2.20) can be computed in a closed form. For example, if $\mathcal{H} := \{h : |h_i| \leq 1 : i = 1, ..., \ell\}$ is a box, then $\max_{h \in \mathcal{H}} a^{\mathsf{T}} h = \|a\|_1$

[3] By I we denote the identity matrix of an appropriate dimension.

[4] Support of probability distribution of random vector h is the smallest closed set $\mathcal{H} \subset \mathbb{R}^\ell$ such that $\mathsf{Pr}\{h \in \mathcal{H}\} = 1$.

[5] T_i denotes i-th row of matrix T.

for any $a \in \mathbb{R}^\ell$. If $\mathcal{H} := \{h : h^\mathsf{T} Q h \leq 1\}$ is ellipsoid (here Q is a symmetric positive definite matrix), then

$$\max_{h \in \mathcal{H}} a^\mathsf{T} h = \sqrt{a^\mathsf{T} Q^{-1} a}. \tag{2.21}$$

In that case constraints (2.20) become conic quadratic constraints (cf., [3, Example 1.3.3]).

Suppose now that the set $\mathcal{H}$ is defined by a finite number of linear constraints

$$\mathcal{H} := \{h \in \mathbb{R}^m : Vh + r \geq 0\}. \tag{2.22}$$

By duality we have that $\max_{h \in \mathcal{H}} [WD - I]_i h$ is equal to the optimal value of problem

$$\operatorname*{Min}_{\lambda \geq 0} r^\mathsf{T} \lambda \ \text{ subject to } V^\mathsf{T} \lambda + [WD - I]_i^\mathsf{T} = 0, \tag{2.23}$$

and $\max_{h \in \mathcal{H}} [-D]_i h$ to the optimal value of the problem

$$\operatorname*{Min}_{\mu \geq 0} r^\mathsf{T} \mu \ \text{ subject to } V^\mathsf{T} \mu - D_i^\mathsf{T} = 0. \tag{2.24}$$

Therefore system (2.19) is equivalent to

$$\begin{array}{rrr} T_i x + r^\mathsf{T} \lambda + W_i d \leq 0, & V^\mathsf{T} \lambda + [WD - I]_i^\mathsf{T} = 0, \ \ i = 1, ..., \ell, & \lambda \geq 0, \\ r^\mathsf{T} \mu \leq 0, & V^\mathsf{T} \mu - D_i^\mathsf{T} = 0, \ \ i = 1, ..., \ell, & \mu \geq 0. \end{array} \tag{2.25}$$

It follows that for the considered affine decision rule the corresponding (restricted) problem can be written as the following linear programming problem

$$\begin{array}{ll} \operatorname*{Min}\limits_{x, \lambda, \mu, D, d} & c^\mathsf{T} x + q^\mathsf{T} (D\eta + d) \\ \text{s.t.} & Ax = b, \ x \geq 0, \\ & T_i x + r^\mathsf{T} \lambda + W_i d \leq 0, \ \ i = 1, ..., \ell, \\ & V^\mathsf{T} \lambda + [WD - I]_i^\mathsf{T} = 0, \ \ i = 1, ..., \ell, \\ & V^\mathsf{T} \mu - D_i^\mathsf{T} = 0, \ \ i = 1, ..., \ell, \\ & r^\mathsf{T} \mu \leq 0, \ \lambda \geq 0, \ \mu \geq 0, \end{array} \tag{2.26}$$

where $\eta := \mathbb{E}[h]$.

2.2.2 Robust Formulation

Consider the following robust formulation of the first stage problem

$$\operatorname*{Min}_{x \in \mathcal{X}} \left\{ c^\mathsf{T} x + \max_{h \in \mathcal{H}} Q(x, h) \right\}, \tag{2.27}$$

with the second stage problem in the form (2.17). The above problem (2.27) can be formulated as

$$\underset{x\in\mathcal{X},\, v\in\mathbb{R}}{\text{Min}}\; c^{\mathsf{T}}x + v \;\text{ s.t. } Q(x,h) \le v,\; h \in \mathcal{H}. \tag{2.28}$$

The dual of problem (2.17) is the problem

$$\underset{\pi\ge 0}{\text{Max}}\; \pi^{\mathsf{T}}(Tx - h) \;\text{ s.t. } q + W^{\mathsf{T}}\pi \ge 0. \tag{2.29}$$

So $Q(x,h)$ is equal to the optimal value of problem (2.29) as well. Therefore we can write problem (2.28) as

$$\begin{array}{ll} \underset{x\in\mathcal{X},\, v\in\mathbb{R},\pi\ge 0}{\text{Min}} & c^{\mathsf{T}}x + v \\ \text{s.t.} & \pi^{\mathsf{T}}(Tx - h) \le v,\; h \in \mathcal{H}, \\ & q + W^{\mathsf{T}}\pi \ge 0. \end{array} \tag{2.30}$$

Note that since $\mathcal{H}$ is a bounded polyhedral set, it is sufficient to verify constraints $\pi^{\mathsf{T}}(Tx-h) \le v,\; h \in \mathcal{H}$, at vertexes of $\mathcal{H}$. Therefore problem (2.28) can be formulated as a linear programming problem as follows. Let $h_k,\; k = 1, ..., K$, be vertexes of $\mathcal{H}$. Then (2.28) is equivalent to

$$\underset{x\in\mathcal{X},\, v\in\mathbb{R}}{\text{Min}}\; c^{\mathsf{T}}x + v \;\text{ s.t. } Q(x,h_k) \le v, \;\; k = 1, ..., K. \tag{2.31}$$

The constraint $Q(x,h_k) \le v$ means that there exists $y_k \ge 0$ such that $Tx + Wy_k \le h_k$ and $q^{\mathsf{T}}y_k \le v$. Therefore problem (2.31) can be formulated as the following large linear program

$$\begin{array}{ll} \underset{x,v,y_1,...,y_K}{\text{Min}} & c^{\mathsf{T}}x + v \\ \text{s.t.} & Ax = b,\; x \ge 0, \\ & q^{\mathsf{T}}y_k \le v,\; y_k \ge 0, \;\; k = 1, ..., K, \\ & Tx + Wy_k \le h_k,\; k = 1, ..., K. \end{array} \tag{2.32}$$

Problem (2.27), although convex, could be difficult to solve. In formulation (2.30) the constraint $\pi^{\mathsf{T}}(Tx-h) \le v$ is not convex, and in formulation (2.32) the number of vertexes can increase exponentially with increase of the dimension of h. Using affine decision rule $y = Dh + d$ we can write the following approximation of problem (2.28)

$$\begin{array}{ll} \underset{x,v,D,d}{\text{Min}} & c^{\mathsf{T}}x + v \\ \text{s.t.} & Ax = b,\; x \ge 0, \\ & q^{\mathsf{T}}(Dh + d) \le v, \;\; h \in \mathcal{H}, \\ & Dh + d \ge 0, \;\; h \in \mathcal{H}, \\ & Tx + W(Dh + d) \le h,\; h \in \mathcal{H}. \end{array} \tag{2.33}$$

Since the set $\mathcal{H}$ is polyhedral, the constraints of the above problem involving vector h can be treated in the same way as in section 2.2.1 (compare with (2.19)–(2.26)).

2.3 The SAA Method

Consider the linear two-stage stochastic problem (2.4)–(2.5). As in section 2.2.1 assume that only the right hand side vector h is random, with distribution P supported on set $\mathcal{H} \subset \mathbb{R}^\ell$. We have that for a given first stage decision x the second stage problem (2.5) has a feasible solution iff $h - Tx \in C$, where $C := \operatorname{pos} W$. Denote

$$\mathfrak{H}(x) := \{h \in \mathcal{H} : h - Tx \notin C\} \text{ and } p(x) := \mathsf{Pr}\{\mathfrak{H}(x)\}, \tag{2.34}$$

i.e., $p(x)$ is the probability of the event that the feasible set of the second stage problem is empty. Recall that it is said that the recourse is relatively complete, if $p(x) = 0$ for every $x \in \mathcal{X}$.

Note that C is a closed convex polyhedral cone. Therefore if $h - Tx \notin C$ for some $h \in \mathcal{H}$ and $x \in \mathcal{X}$, then $h' - Tx \notin C$ for all h' in a neighborhood of h, and hence $p(x) > 0$. Thus we can write problem (2.4) in the form

$$\operatorname*{Min}_{x \in \mathcal{X}} c^\mathsf{T} x + \mathcal{Q}(x) \text{ s.t. } h - Tx \in C,\ h \in \mathcal{H}, \tag{2.35}$$

where $\mathcal{Q}(x) := \mathbb{E}[Q(x,h)]$ and $Q(x,h)$ is the optimal value of the second stage problem (2.5). Note that since it was assumed that the feasible set of the dual problem (2.7) is nonempty, and hence $Q(x,h) > -\infty$, we have that for every feasible x of the above problem (2.35) the optimal value $Q(x,h)$ is finite for all $h \in \mathcal{H}$. We assume that the expectation $\mathcal{Q}(x) = \mathbb{E}[Q(x,h)]$ is finite valued for all these feasible x.

We can apply the *Sample Average Approximation* (SAA) method to the problem (2.35). That is an iid sample $h^1, ..., h^N$ of random vector h is generated and problem (2.35) is approximated by

$$\operatorname*{Min}_{x \in \mathcal{X}} c^\mathsf{T} x + \hat{\mathcal{Q}}_N(x), \tag{2.36}$$

where $\hat{\mathcal{Q}}_N(x) := \frac{1}{N} \sum_{j=1}^N Q(x, h^j)$. We can write the above SAA problem (2.36) as the following linear programming problem

$$\begin{array}{ll} \operatorname*{Min}\limits_{x, y_1, ..., y_N} & c^\mathsf{T} x + \frac{1}{N} \sum_{j=1}^N q^\mathsf{T} y_j \\ \text{s.t.} & Ax = b,\ x \geq 0, \\ & Tx + W y_j = h^j,\ y_j \geq 0,\ j = 1, ..., N. \end{array} \tag{2.37}$$

Let us make the following observations. Problem (2.36) is equivalent to the following problem

$$\operatorname*{Min}_{x \in \mathcal{X}} c^\mathsf{T} x + \hat{\mathcal{Q}}_N(x) \text{ s.t. } h^j - Tx \in C,\ j = 1, ..., N. \tag{2.38}$$

Unless relatively complete recourse is ensured, it still may happen that there exists $h \in \mathcal{H}$ such that $T\hat{x}_N - h \notin C$, i.e., the probability $p(\hat{x}_N)$, that the corresponding second stage problem is infeasible, is positive. In that case $\mathcal{Q}(\hat{x}_N) = +\infty$ and $\hat{x}_N$ is not a feasible point of the true problem (2.35), does not matter how small the probability $p(\hat{x}_N)$ is. If the probability $p(\hat{x}_N)$ is small, then one may still be satisfied with the computed solution. This motivates to introduce chance constraint $p(x) \leq \alpha$, at a chosen significance level $\alpha \in (0, 1)$, and to restrict the optimization to such x that the second stage problem is feasible. This leads to the following "chance constraint" variant of the two stage problem (2.35):

$$\operatorname*{Min}_{x \in \mathcal{X}} c^{\mathsf{T}} x + \mathfrak{Q}(x) \;\text{ subject to } p(x) \leq \alpha, \tag{2.39}$$

where $\mathfrak{Q}(x)$ is the expectation of the second stage optimal value *conditional* on the event $h - Tx \in C$, i.e.,

$$\mathfrak{Q}(x) := \mathbb{E}\left[Q(x,h) \middle| h - Tx \in C\right] = \frac{1}{1 - p(x)} \int_{h \in \mathcal{H} \cap (Tx + C)} Q(x,h) dP(h).$$

2.4 Minimax Analysis

In this section we discuss a general theory of minimax problems of the form (2.1). We assume in this section that the expectation

$$\phi(x, P) := \mathbb{E}_P[F(x, \xi)]$$

is finite for all $x \in \mathcal{S}$ and $P \in \mathfrak{M}$, where $\mathcal{S}$ is an open set containing the set $\mathcal{X}$. In particular, this implies that $F(x, \xi)$ is finite valued for a.e. $\xi \in \Xi$ with respect to every $P \in \mathfrak{M}$. The problem (2.1), to which we refer as the *primal* problem, is

$$\operatorname*{Min}_{x \in \mathcal{X}} \left\{ f(x) := \sup_{P \in \mathfrak{M}} \phi(x, P) \right\}. \tag{2.40}$$

Its *dual* problem is obtained by interchanging the *min* and *max* operators:

$$\operatorname*{Max}_{P \in \mathfrak{M}} \left\{ g(P) := \inf_{x \in \mathcal{X}} \phi(x, P) \right\}. \tag{2.41}$$

Clearly for any $(x, P) \in \mathcal{X} \times \mathfrak{M}$ we have that $f(x) \geq g(P)$, and hence it follows that the optimal value of the primal problem (2.40) is greater than or equal to the optimal value of the dual problem (2.41). The difference between the optimal values of problems (2.40) and (2.41) is called the *duality gap* between these two problems. It is said that there is no duality gap if the optimal values of problems (2.40) and (2.41) are equal to each other.

Definition 2.1 It is said that $(\bar{x}, \bar{P}) \in \mathcal{X} \times \mathfrak{M}$ is a *saddle point* of problem (2.40) if

$$\phi(\bar{x}, P) \leq \phi(\bar{x}, \bar{P}) \leq \phi(x, \bar{P}), \tag{2.42}$$

for all $(x, P) \in \mathcal{X} \times \mathfrak{M}$.

Equivalently $(\bar{x}, \bar{P}) \in \mathcal{X} \times \mathfrak{M}$ is a saddle point iff

$$\bar{x} \in \arg\min_{x \in \mathcal{X}} \phi(x, \bar{P}) \ \text{ and } \ \bar{P} \in \arg\max_{P \in \mathfrak{M}} \phi(\bar{x}, P). \tag{2.43}$$

Proof of the following classical result is rather straightforward.

Theorem 2.2 *The minimax problem* (2.40) *has a saddle point iff there is no duality gap between problems* (2.40) *and* (2.41) *and both problems have optimal solutions. In the last case the set of saddle points coincides with the direct (Cartesian) product of the sets of optimal solutions of problems* (2.40) *and* (2.41).

Let us assume now that the set $\mathcal{X}$ is *convex* and closed and $F(x, \xi)$ is *convex* in x for every $\xi \in \Xi$. Then the expectation function $\phi(x, P)$ is convex in x, and hence the max-function $f(x)$ is convex, and thus problem (2.40) is a convex problem. Note also that the maximum $\sup_{P \in \mathfrak{M}} \phi(x, P)$ is not changed if the set $\mathfrak{M}$ is replaced by its convex hull. Therefore we can assume without loss of generality that the set $\mathfrak{M}$ is *convex*, i.e., if $P, P' \in \mathfrak{M}$, then $tP + (1-t)P' \in \mathfrak{M}$ for any $t \in [0, 1]$. Since $\phi(x, P)$ is linear in P, it follows that the set

$$\mathfrak{M}^*(x) := \arg\max_{P \in \mathfrak{M}} \phi(x, P) \tag{2.44}$$

is also convex.

Let $\bar{x}$ be an optimal solution of the primal problem (2.40). Assume that the function $f(x)$ is finite valued for all x in a neighborhood of $\bar{x}$. Then we can write the following first order optimality conditions

$$0 \in \partial f(\bar{x}) + \mathcal{N}_{\mathcal{X}}(\bar{x}), \tag{2.45}$$

where $\mathcal{N}_{\mathcal{X}}(\bar{x})$ denotes the normal cone to the set $\mathcal{X}$ at the point $\bar{x} \in \mathcal{X}$ and $\partial f(\bar{x})$ is the subdifferential of $f(x)$ at $x = \bar{x}$. Recall that we assumed that the function $f(x)$ is finite valued for all x near $\bar{x}$, and thus by convexity arguments the optimality conditions (2.44) are necessary and sufficient. Of course, if the function $f(x)$ is finite valued for all x in a neighborhood of $\bar{x}$, then the functions $\phi(x, P)$, $P \in \mathfrak{M}$, are

also finite valued for all x in that neighborhood. Suppose, further, that the following formula for the subdifferential of the max-function $f(\cdot)$ holds

$$\partial f(\bar{x}) = \text{conv}\left\{\bigcup_{P\in\mathfrak{M}^*(\bar{x})} \partial_x \phi(\bar{x}, P)\right\}, \tag{2.46}$$

where $\text{conv}\{A\}$ denotes the convex hull of set A. In particular, (2.46) requires the set $\mathfrak{M}^*(\bar{x})$, of maximizers of $\phi(\bar{x}, P)$ over $P \in \mathfrak{M}$, to be nonempty. We will discuss conditions ensuring validity of formula (2.46) later.

Formula (2.46) implies that the optimality conditions (2.45) can be formulated as follows: there exist $P_1, ..., P_k \in \mathfrak{M}^*(\bar{x})$ and nonnegative weights $w_1, ..., w_k$ summing up to one such that

$$0 \in \sum_{i=1}^{k} w_i \partial_x \phi(\bar{x}, P_i) + \mathcal{N}_{\mathcal{X}}(\bar{x}).$$

Moreover, by the Moreau-Rockafellar Theorem we have that

$$\textstyle\sum_{i=1}^k w_i \partial_x \phi(\bar{x}, P_i) = \partial_x \left(\sum_{i=1}^k w_i \phi(\bar{x}, P_i)\right).$$

Also $\sum_{i=1}^k w_i \phi(x, P_i) = \phi(x, \bar{P})$, where $\bar{P} := \sum_{i=1}^k w_i P_i$. Since the set $\mathfrak{M}^*(\bar{x})$ is convex, and hence $\bar{P} \in \mathfrak{M}^*(\bar{x})$, it follows that optimality conditions (2.45) can be written as follows: there exists $\bar{P} \in \mathfrak{M}^*(\bar{x})$ such that

$$0 \in \partial_x \phi(\bar{x}, \bar{P}) + \mathcal{N}_{\mathcal{X}}(\bar{x}). \tag{2.47}$$

By convexity arguments condition (2.47) is necessary and sufficient for the first of the conditions (2.43). The second of the conditions (2.43) also holds by the definition of the set $\mathfrak{M}^*(\bar{x})$. Therefore it follows, under the specified assumptions, that $(\bar{x}, \bar{P})$ is a saddle point of the problem (2.40).

In order to verify validity of formula (2.46) we need several nontrivial results. In particular, we need to verify compactness of the set $\mathfrak{M}$ in an appropriate topology. We use here the weak topology of probability measures defined on the set $\Xi \subset \mathbb{R}^d$ equipped with its Borel sigma algebra, so all topological properties, such as compactness, convergence, continuity, of probability measures on Ξ will be considered with respect to the weak topology.

The weak topology can be defined by a metric and hence can be described in terms of convergent sequences. A sequence $\{P_n\}$ of probability measures on Ξ converges, in the weak topology, to a probability measure P if

$$\lim_{n\to\infty} \int_{\Xi} h(z) dP_n(z) = \int_{\Xi} h(z) dP(z) \tag{2.48}$$

for any bounded continuous function $h : \Xi \to \mathbb{R}$. We refer to Billingsley [4] for a discussion of the involved concepts and results. By Prohorov's theorem the set $\mathfrak{M}$ is compact iff $\mathfrak{M}$ is tight and closed, [4, Section 5]. It is said that $\mathfrak{M}$ is *tight* if for any $\varepsilon > 0$ there exists a compact set $\Xi' \subset \Xi$ such that $P(\Xi') > 1 - \varepsilon$ for every $P \in \mathfrak{M}$. Recall that a subset of a finite dimensional vector space is compact iff it is bounded and closed. Therefore, in particular, if the set Ξ is bounded and closed, then $\mathfrak{M}$ is compact iff it is closed in the weak topology. Closedness of $\mathfrak{M}$ means that if $P_n \in \mathfrak{M}$ is a sequence of probability measures convergent weakly to a probability measure P, then $P \in \mathfrak{M}$.

Now by the Levin-Valadier Theorem we have that, under the specified assumptions of convexity, formula (2.46) holds if the set $\mathfrak{M}$ is compact and for every x in a neighborhood of $\bar{x}$ the function $\phi(x, P)$ is continuous in $P \in \mathfrak{M}$. Note that by the definition of weak convergence we have that if, for a given $x \in \mathcal{X}$, the function $F(x, \cdot)$ is bounded and continuous on Ξ, then $\phi(x, P)$ is continuous in P. By summarizing the above discussion we have the following result (cf., [27]).

Theorem 2.3 *Let $\bar{x}$ be an optimal solution of the primal problem* (2.40)*. Suppose that: (i) the set $\mathcal{X}$ is convex and closed, the set $\mathfrak{M}$ is convex and $F(x, \xi)$ is convex in x for every $\xi \in \Xi$, (ii) the set $\mathfrak{M}$ is tight and closed (in the weak topology), (iii) for every x in a neighborhood of $\bar{x}$ the function $F(x, \cdot)$ is bounded and continuous on Ξ. Then there is no duality gap between problems* (2.40) *and* (2.41)*, both problems have optimal solutions and the set of saddle points coincides with the direct product of the sets of optimal solutions of problems* (2.40) *and* (2.41)*.*

Remark 1 Suppose that the set Ξ is compact, i.e., is bounded and closed in $\mathbb{R}^d$. Then the set $\mathfrak{M}$ is tight and hence condition (ii) holds, provided $\mathfrak{M}$ is closed. Suppose, further, that $F(x, \xi)$ is continuous in $\xi \in \Xi$ for every $x \in \mathcal{X}$. Then $F(x, \cdot)$ is bounded on Ξ, and hence condition (iii) holds. $\diamondsuit$

Also by using conjugate duality it is possible to proof the following sufficient conditions for the no duality gap property (e.g., [7, Theorem 7.10]).

Theorem 2.4 *Suppose that the set $\mathcal{X}$ is convex and closed, the set $\mathfrak{M}$ is convex, $F(x, \xi)$ is convex in x for every $\xi \in \Xi$, and the problem* (2.40) *has a nonempty and bounded set of optimal solutions. Then there is no duality gap between problems* (2.40) *and* (2.41)

2.5 Coherent Risk Measures

There is another way to look at robust formulations of stochastic problems. Consider a probability space $(\Omega, \mathcal{F}, P)$. Recall that a measurable function $Z : \Omega \to \mathbb{R}$ can be

viewed as a *random variable*. With every random variable $Z = Z(\omega)$ we associate a number, denoted $\rho(Z)$, indicating our preference between possible realizations of random variables. That is, $\rho(\cdot)$ is a real valued function defined on a space of measurable functions $Z : \Omega \to \mathbb{R}$. We refer to $\rho(\cdot)$ as a *risk measure*. For example, for a chosen probability measure P we can employ the expected value $\rho(Z) := \mathbb{E}_P[Z]$ as a risk measure. We assume that "smaller is better", so eventually we would like to perform a minimization with respect to a chosen risk measure. The term "risk measure" is somewhat unfortunate since it could be confused with the concept of probability measures. However, it became quite standard, so we will use it here.

In the subsequent analysis we equip $(\Omega, \mathcal{F})$ with a probability measure P, to which we refer as the *reference probability measure* or *reference distribution*, and unless stated otherwise all probabilistic statements will be made with respect to the reference measure P. Moreover, we have to specify a space of random variables $Z : \Omega \to \mathbb{R}$ on which a considered risk measure will be defined. In that respect it is natural to consider spaces $\mathcal{L}_p(\Omega, \mathcal{F}, P)$ of random variables $Z(\omega)$ having finite p-th order moment, $p \in [1, \infty)$. Note that two random variables $Z(\omega)$ and $Z'(\omega)$ are undistinguishable if $Z(\omega) = Z'(\omega)$ for a.e. $\omega \in \Omega$ (i.e., for all $\omega \in \Omega$ except on a set of P-measure zero). Therefore $\mathcal{L}_p(\Omega, \mathcal{F}, P)$ consists of classes of random variables $Z(\omega)$ such that $Z(\omega)$ and $Z'(\omega)$ belong to the same class if $Z(\omega) = Z'(\omega)$ for a.e. $\omega \in \Omega$, and $\mathbb{E}|Z|^p = \int_\Omega |Z(\omega)|^p dP(\omega)$ is finite. The space $\mathcal{L}_p(\Omega, \mathcal{F}, P)$ equipped with the norm $\|Z\|_p := \left(\int_\Omega |Z(\omega)|^p dP(\omega)\right)^{1/p}$ becomes a Banach space.

We also consider space $\mathcal{L}_\infty(\Omega, \mathcal{F}, P)$ of essentially bounded functions. That is, $\mathcal{L}_\infty(\Omega, \mathcal{F}, P)$ consists of random variables with finite sup-norm $\|Z\|_\infty := \operatorname{ess\,sup} |Z|$, where the essential supremum of a random variable $Z(\omega)$ is defined as

$$\operatorname{ess\,sup}(Z) := \inf \left\{ \sup_{\omega \in \Omega} Z'(\omega) : Z'(\omega) = Z(\omega) \text{ a.e. } \omega \in \Omega \right\}. \tag{2.49}$$

A set $\mathfrak{A} \subset \mathcal{L}_p(\Omega, \mathcal{F}, P)$ is said to be *bounded* if there exists constant $c > 0$ such that $\|Z\|_p \leq c$ for all $Z \in \mathfrak{A}$. Unless stated otherwise we work with the space $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$ of random variables for some $p \in [1, \infty]$, and all topological statements related to the space $\mathcal{L}_p(\Omega, \mathcal{F}, P)$ will be made with respect to its strong (norm) topology.

With each space $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in [1, +\infty)$, is associated its dual space $\mathcal{Z}^* := \mathcal{L}_q(\Omega, \mathcal{F}, P)$, where $q \in (1, \infty]$ is such that $1/p + 1/q = 1$. For $Z \in \mathcal{Z}$ and $\zeta \in \mathcal{Z}^*$ their scalar product is defined as

$$\langle Z, \zeta \rangle := \int_\Omega Z(\omega)\zeta(\omega) dP(\omega) = \mathbb{E}_P[Z\zeta]. \tag{2.50}$$

Note that for $p \in (1, \infty)$ the dual of $\mathcal{Z}^* = \mathcal{L}_q(\Omega, \mathcal{F}, P)$ coincides with $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$, i.e., the space $\mathcal{L}_p(\Omega, \mathcal{F}, P)$ is *reflexive*. On the other hand the dual of the space

$\mathcal{L}_\infty(\Omega, \mathcal{F}, P)$ is significantly bigger than the space $\mathcal{L}_1(\Omega, \mathcal{F}, P)$, and the spaces $\mathcal{L}_1(\Omega, \mathcal{F}, P)$ and $\mathcal{L}_\infty(\Omega, \mathcal{F}, P)$ are not reflexive. For the space $\mathcal{L}_\infty(\Omega, \mathcal{F}, P)$ it also makes sense to consider its weak* topology (for reflexive Banach spaces the weak and weak* topologies coincide). Recall that by Banach-Alaoglu Theorem any bounded and closed in the weak* topology subset of $\mathcal{Z}^*$ is weakly* compact.

Formally, risk measure is a real valued function $\rho : \mathcal{Z} \to \mathbb{R}$, where $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$ for some $p \in [1, \infty]$. It is also possible to consider risk measures taking values $\rho(Z) = +\infty$ for some $Z \in \mathcal{Z}$. However, with virtually every interesting risk measure is associated in a natural way an $\mathcal{L}_p(\Omega, \mathcal{F}, P)$ space on which it is finite valued. It was suggested in Artzner et al [1] that a "good" risk measure should satisfy the following axioms, and such risk measures were called *coherent.*

(A1) *Monotonicity:* If $Z, Z' \in \mathcal{Z}$ and $Z \succeq Z'$, then $\rho(Z) \geq \rho(Z')$.

(A2) *Convexity:*

$$\rho(tZ + (1-t)Z') \leq t\rho(Z) + (1-t)\rho(Z')$$

for all $Z, Z' \in \mathcal{Z}$ and all $t \in [0, 1]$.

(A3) *Translation Equivariance:* If $a \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then $\rho(Z + a) = \rho(Z) + a$.

(A4) *Positive Homogeneity:* If $t \geq 0$ and $Z \in \mathcal{Z}$, then $\rho(tZ) = t\rho(Z)$.

Here the notation $Z \succeq Z'$ means that $Z(\omega) \geq Z'(\omega)$ for a.e. $\omega \in \Omega$. Monotonicity property (axiom (A1)) is a natural condition that a risk measure should satisfy (recall that we deal here with minimization rather than maximization formulations of optimization problems). Convexity property is also a natural one. Because of (A4) the convexity axiom (A2) holds iff the following subadditivity property holds

$$\rho(Z + Z') \leq \rho(Z) + \rho(Z'), \quad \forall\, Z, Z' \in \mathcal{Z}. \tag{2.51}$$

That is, risk of the sum of two random variables is not bigger than the sum of risks. Axioms (A3) and (A4) postulate position and scale properties, respectively, of risk measures.

We have the following basic duality result associated with coherent risk measures. Denote by

$$\mathfrak{P} := \left\{ \zeta \in \mathcal{Z}^* : \int_\Omega \zeta(\omega) dP(\omega) = 1,\ \zeta \succeq 0 \right\} \tag{2.52}$$

the set of probability density functions in the dual space $\mathcal{Z}^*$.

Theorem 2.5 *Let $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$, and $\rho : \mathcal{Z} \to \mathbb{R}$ be a coherent risk measure. Then ρ is continuous and there exists a bounded set $\mathfrak{A} \subset \mathfrak{P}$ such that*

$$\rho(Z) = \sup_{\zeta \in \mathfrak{A}} \langle Z, \zeta \rangle, \quad \forall Z \in \mathcal{Z}. \tag{2.53}$$

Conversely if the representation (2.53) *holds for some nonempty bounded set $\mathfrak{A} \subset \mathfrak{P}$, then ρ is a (real valued) coherent risk measure.*

Remark 2 The dual representation (2.53) follows from the classical Fenchel-Moreau theorem. Note that monotonicity (axiom (A1)) and convexity (axiom (A2)) imply continuity (in the strong topology) of the risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ for any $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty]$ (cf., [23]). If the representation (2.53) holds for some bounded set $\mathfrak{A}$, then it also holds if the set $\mathfrak{A}$ is replaced by the weak* topological closure of its convex hull. We refer to the (weak* closed convex) set $\mathfrak{A}$ in the dual representation (2.53) as the *dual set* of the coherent risk measure ρ. The dual set $\mathfrak{A}$ can be written in the form

$$\mathfrak{A} = \big\{ \zeta \in \mathcal{Z}^* : \langle Z, \zeta \rangle \leq \rho(Z), \ \ \forall Z \in \mathcal{Z} \big\}. \tag{2.54}$$

Recall that a convex subset of a Banach space is closed in the strong topology iff it is closed in the weak topology. Therefore for reflexive spaces $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in (1, \infty)$, it suffices to assume that the dual set $\mathfrak{A}$ is closed in the strong topology. Note also that by the Banach-Alaoglu Theorem the dual set $\mathfrak{A}$ is weakly* compact, and hence the maximum in the right hand side of (2.53) is attained.

Remark 3 From the point of view of convex analysis the representation (2.53) means that $\rho(\cdot)$ is the so-called support function of the set $\mathfrak{A} \subset \mathcal{Z}^*$, and the following properties hold (e.g., [7, Proposition 2.116]).

(i) If $\rho_1 : \mathcal{Z} \to \mathbb{R}$ and $\rho_2 : \mathcal{Z} \to \mathbb{R}$ are coherent risk measures and $\mathfrak{A}_1 \subset \mathcal{Z}^*$ and $\mathfrak{A}_2 \subset \mathcal{Z}^*$ are the respective dual sets, then $\rho_1(\cdot) \leq \rho_2(\cdot)$ iff $\mathfrak{A}_1 \subset \mathfrak{A}_2$, and $\rho_1(\cdot) = \rho_2(\cdot)$ iff $\mathfrak{A}_1 = \mathfrak{A}_2$.

(ii) Let $\rho_i : \mathcal{Z} \to \mathbb{R}$, $i = 1, ..., m$, be coherent risk measures, $\mathfrak{A}_i \subset \mathcal{Z}^*$ be the respective dual sets and λ_i, $i = 1, ..., m$, be nonnegative numbers such that $\sum_{i=1}^m \lambda_i = 1$. Then $\rho(\cdot) := \sum_{i=1}^m \lambda_i \rho_i(\cdot)$ is also a coherent risk measure and its dual set $\mathfrak{A} \subset \mathcal{Z}^*$ is given[6] by $\mathfrak{A} := \sum_{i=1}^m \lambda_i \mathfrak{A}_i$.

[6]For sets $\mathfrak{A}_1 \subset \mathcal{Z}^*$ and $\mathfrak{A}_2 \subset \mathcal{Z}^*$ and numbers λ_1 and λ_2 the sum $\lambda_1 \mathfrak{A}_1 + \lambda_2 \mathfrak{A}_2$ is defined as the set $\{\lambda_1 \zeta_1 + \lambda_2 \zeta_2 : \zeta_1 \in \mathfrak{A}_1, \ \zeta_2 \in \mathfrak{A}_2\}$.

(iii) Let $\rho_i : \mathcal{Z} \to \mathbb{R}$, $i \in \mathcal{I}$, be coherent risk measures and $\mathfrak{A}_i \subset \mathcal{Z}^*$ be the respective dual sets, and let $\rho(\cdot) := \sup_{i\in\mathcal{I}} \rho_i(\cdot)$ be real valued, i.e., $\rho(Z) < \infty$ for all $Z \in \mathcal{Z}$. Then $\rho(\cdot)$ is also a coherent risk measure and its dual set $\mathfrak{A}$ is given by the topological closure of the convex hull of the set $\cup_{i\in\mathcal{I}}\mathfrak{A}_i$.

For $\zeta \in \mathfrak{P}$ the scalar product $\langle Z, \zeta\rangle$ can be understood as the expectation $\mathbb{E}_Q[Z]$ taken with respect to the probability measure $dQ = \zeta dP$. Therefore the representation (2.53) can be written as

$$\rho(Z) = \sup_{Q\in\mathfrak{Q}} \mathbb{E}_Q[Z], \quad \forall Z \in \mathcal{Z}, \tag{2.55}$$

where

$$\mathfrak{Q} := \{Q : dQ = \zeta dP,\ \zeta \in \mathfrak{A}\}. \tag{2.56}$$

Recall that if P and Q are two measures on $(\Omega, \mathcal{F})$, then it is said that Q is *absolutely continuous* with respect to P if $A \in \mathcal{F}$ and $P(A) = 0$ implies that $Q(A) = 0$. The Radon-Nikodym Theorem says that Q is absolutely continuous with respect to P iff there exists a function $\eta : \Omega \to \mathbb{R}_+$ (density function) such that $Q(A) = \int_A \eta dP$ for every $A \in \mathcal{F}$. We also sometimes write $\mathbb{E}_\zeta[Z]$ for the expectation $\mathbb{E}_Q[Z]$ with $dQ = \zeta dP$.

By the above, the result of Theorem 2.5 can be interpreted as follows.

- Let $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$. Then a risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is coherent iff there exists a set $\mathfrak{Q}$ of absolutely continuous with respect to P probability measures such that the set of densities $\left\{\frac{dQ}{dP} : Q \in \mathfrak{Q}\right\}$ forms a bounded set in the dual space $\mathcal{Z}^*$ and the representation (2.55) holds.

Therefore robust stochastic programs of the form (2.1) can be formulated in terms of coherent risk measures (see section 2.5.2). It could be noted that the set $\mathfrak{Q}$ is formed by probability measures absolutely continuous with respect to a specified reference measure, while we didn't make such an assumption for the set $\mathfrak{M}$ in (2.1). We will discuss this later.

Let us consider some examples. The following risk measure is called the *mean-upper-semideviation* risk measure of order $p \in [1, \infty)$:

$$\rho(Z) := \mathbb{E}[Z] + \lambda \left(\mathbb{E}\left[\left[Z - \mathbb{E}[Z]\right]_+^p\right]\right)^{1/p}. \tag{2.57}$$

In the second term of the right hand side of (2.57), the excess of Z over its expectation is penalized. In order for this risk measure to be real valued it is natural to take

$\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$. For any $\lambda \in [0,1]$ this risk measure is coherent and has the dual representation (2.53) with the set

$$\mathfrak{A} = \big\{\zeta' \in \mathcal{Z}^* : \zeta' = 1 + \zeta - \mathbb{E}[\zeta],\ \|\zeta\|_q \leq \lambda,\ \zeta \succeq 0\big\}. \tag{2.58}$$

Note that the above set $\mathfrak{A}$ is a bounded convex closed subset of the dual space $\mathcal{Z}^* = \mathcal{L}_q(\Omega, \mathcal{F}, P)$.

An important example of risk measure is *Value-at-Risk* measure

$$\mathsf{V@R}_\alpha(Z) := \inf\big\{t : \mathsf{Pr}(Z \leq t) \geq 1 - \alpha\big\},\ \ \alpha \in (0,1). \tag{2.59}$$

That is, $\mathsf{V@R}_\alpha(Z) = H^{-1}(1-\alpha)$ is the left side $(1-\alpha)$-quantile of the distribution of Z. Here $H(t) := \mathsf{Pr}(Z \leq t)$ if the *cumulative distribution function* (cdf) of Z and

$$H^{-1}(\gamma) := \inf\big\{t : H(t) \geq \gamma\big\}$$

for $\gamma \in (0,1)$. For $\gamma = 0$ the corresponding left side quantile $H^{-1}(0) = -\infty$, and by the definition $H^{-1}(1) = +\infty$ if $Z(\omega)$ is unbounded from above. The $\mathsf{V@R}_\alpha$ risk measure is not coherent, it satisfies axioms (A1),(A3) and (A4) but is not convex, i.e., it does not possess the subadditivity property (2.51).

An important example of coherent risk measure is the *Average Value-at-Risk* measure

$$\mathsf{AV@R}_\alpha(Z) := \inf_{t \in \mathbb{R}} \big\{t + \alpha^{-1}\mathbb{E}[Z-t]_+\big\},\ \ \alpha \in (0,1]. \tag{2.60}$$

It is natural to take here $\mathcal{Z} := \mathcal{L}_1(\Omega, \mathcal{F}, P)$. This risk measure is also known under the names Expected Shortfall, Expected Tail Loss and Conditional Value-at-Risk; these names are motivated by formulas (2.62) and (2.63) below. It is possible to show that the set of minimizers of the right hand side of (2.60) is formed by $(1-\alpha)$-quantiles of the distribution of Z. In particular $t^* = \mathsf{V@R}_\alpha(Z)$ is such a minimizer. It follows that $\mathsf{AV@R}_\alpha(Z) \geq \mathsf{V@R}_\alpha(Z)$. Also it follows from (2.60) that

$$\mathsf{AV@R}_{\alpha_1}(Z) \geq \mathsf{AV@R}_{\alpha_2}(Z),\ \ \text{for } 0 < \alpha_1 \leq \alpha_2 \leq 1. \tag{2.61}$$

The Average Value-at-Risk can be also written as

$$\mathsf{AV@R}_\alpha(Z) = \frac{1}{\alpha}\int_0^\alpha \mathsf{V@R}_\tau(Z)\, d\tau, \tag{2.62}$$

and, moreover, if the cumulative distribution function $H(t)$ of Z is continuous at $t^* = \mathsf{V@R}_\alpha(Z)$, then

$$\mathsf{AV@R}_\alpha(Z) = \frac{1}{\alpha}\int_{\mathsf{V@R}_\alpha(Z)}^{+\infty} t\, dH(t) = \mathbb{E}\left[Z \middle| Z \geq \mathsf{V@R}_\alpha(Z)\right]. \tag{2.63}$$

The dual representation (2.53) for $\rho(Z) := \mathsf{AV@R}_\alpha(Z)$ holds with the set

$$\mathfrak{A} = \left\{\zeta \in \mathcal{L}_\infty(\Omega, \mathcal{F}, P) : \zeta(\omega) \in [0, \alpha^{-1}] \text{ a.e. } \omega \in \Omega,\ \mathbb{E}[\zeta] = 1\right\}. \tag{2.64}$$

Note that the above set $\mathfrak{A}$ is a bounded weakly* closed subset of the dual space $\mathcal{Z}^* = \mathcal{L}_\infty(\Omega, \mathcal{F}, P)$. If $\alpha = 1$, then the set $\mathfrak{A}$ consists of unique point $\zeta(\omega) \equiv 1$. That is, $\mathsf{AV@R}_1(Z) = \mathbb{E}[Z]$, this can be verified directly from the definition (2.60). We have the following limit

$$\lim_{\alpha \downarrow 0} \mathsf{AV@R}_\alpha(Z) = \operatorname{ess\,sup}(Z). \tag{2.65}$$

In order for the risk measure $\rho(Z) := \operatorname{ess\,sup}(Z)$ to be finite valued it should be considered on the space $\mathcal{Z} := \mathcal{L}_\infty(\Omega, \mathcal{F}, P)$; defined on that space this risk measure is coherent.

In both examples considered above the risk measures are functions of the distribution of the random variable Z. Such risk measures are called law invariant. Recall that two random variables Z and Z' have the same distribution if their cumulative distribution functions coincide, i.e., $\mathsf{Pr}(Z \leq t) = \mathsf{Pr}(Z' \leq t)$ for all $t \in \mathbb{R}$. We write this relation as $Z \overset{\mathcal{D}}{\sim} Z'$.

Definition 2.2 It is said that a risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is *law invariant*, with respect to the reference distribution P, if for any $Z, Z' \in \mathcal{Z}$ such that $Z \overset{\mathcal{D}}{\sim} Z'$ it follows that $\rho(Z) = \rho(Z')$.

Suppose for the moment that the set $\Omega = \{\omega_1, ..., \omega_K\}$ is finite with respective probabilities $p_1, ..., p_K$ such that any partial sums of p_k are different, i.e., $\sum_{k\in\mathcal{I}} p_k = \sum_{k\in\mathcal{J}} p_k$ for $\mathcal{I}, \mathcal{J} \subset \{1, ..., K\}$ only if $\mathcal{I} = \mathcal{J}$. Then $Z, Z' : \Omega \to \mathbb{R}$ have the same distribution only if $Z(\omega) = Z'(\omega)$ for all $\omega \in \Omega$. In that case any risk measure, defined on the space of random variables $Z : \Omega \to \mathbb{R}$, is law invariant. Therefore, for a meaningful discussion of law invariant risk measures it is natural to consider nonatomic probability spaces. It is said that measure P, and hence the space $(\Omega, \mathcal{F}, P)$, is *nonatomic* if any set $A \in \mathcal{F}$ of positive measure $P(A)$ contains a subset $B \in \mathcal{F}$ such that $P(A) > P(B) > 0$.

A natural question is how law invariance can be described in terms of the set $\mathfrak{A}$ in the dual representation (2.53). Let $T : \Omega \to \Omega$ be one-to-one onto mapping, i.e., $T(\omega) = T(\omega')$ iff $\omega = \omega'$ and $T(\Omega) = \Omega$. It is said that T is a *measure-preserving transformation* if image $T(A) = \{T(\omega) : \omega \in A\}$ of any measurable set $A \in \mathcal{F}$ is also measurable and $P(A) = P(T(A))$ (see, e.g., [5, p. 311]). Let us denote by

$$\mathfrak{G} := \{\text{the set of one-to-one onto measure-preserving transformations } T : \Omega \to \Omega\}.$$

We have that if $T \in \mathfrak{G}$, then $T^{-1} \in \mathfrak{G}$; and if $T_1, T_2 \in \mathfrak{G}$, then their composition[7] $T_1 \circ T_2 \in \mathfrak{G}$. That is, $\mathfrak{G}$ forms a group of transformations.

Theorem 2.6 *Suppose that the probability space $(\Omega, \mathcal{F}, P)$ is nonatomic. Then a coherent risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is law invariant iff its dual set $\mathfrak{A}$ is invariant with respect to measure-preserving transformations, i.e., iff for any $\zeta \in \mathfrak{A}$ and any $T \in \mathfrak{G}$ and $\zeta' := \zeta \circ T$ it follows that $\zeta' \in \mathfrak{A}$.*

Proof. Let $T \in \mathfrak{G}$ and $\zeta \in \mathfrak{A}$. Consider $\zeta' := \zeta \circ T$. For $Z \in \mathcal{Z}$ we have

$$\langle Z, \zeta' \rangle = \int_\Omega Z(\omega)\zeta(T(\omega))dP(\omega) = \int_\Omega Z(T^{-1}(\omega))\zeta(\omega)dQ(\omega) = \langle Z', \zeta \rangle \tag{2.66}$$

where $Q = PT^{-1} = P$ and $Z' := Z \circ T^{-1}$. Since T is measure-preserving we have that $Z \overset{\mathcal{D}}{\sim} Z'$ and since ρ is law invariant, it follows that $\rho(Z) = \rho(Z')$. Therefore by (2.54) we obtain that $\zeta' \in \mathfrak{A}$.

Conversely suppose that $\zeta \circ T \in \mathfrak{A}$ for any $\zeta \in \mathfrak{A}$ and any $T \in \mathfrak{G}$. Let Z, Z' be two random variables having the same distribution. Since the probability space $(\Omega, \mathcal{F}, P)$ is nonatomic, there is $T \in \mathfrak{G}$ such that $Z' = Z \circ T$. For $\varepsilon > 0$ let $\zeta \in \mathfrak{A}$ be such that $\rho(Z') \leq \langle Z', \zeta \rangle + \varepsilon$. By (2.66) and since $\zeta' \in \mathfrak{A}$ it follows that

$$\rho(Z') \leq \langle Z', \zeta \rangle + \varepsilon = \langle Z, \zeta' \rangle + \varepsilon \leq \rho(Z) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain that $\rho(Z') \leq \rho(Z)$. The other inequality $\rho(Z') \geq \rho(Z)$ can be obtained in the same way and hence $\rho(Z') = \rho(Z)$. This competes the proof. ∎

A particular example of law invariant coherent risk measure is the Average Value-at-Risk measure $\mathsf{AV@R}_\alpha$. A convex combination $\sum_{i=1}^m \mu_i \mathsf{AV@R}_{\alpha_i}$, with $\alpha_i \in (0, 1]$, $\mu_i \geq 0$, $\sum_{i=1}^m \mu_i = 1$, of Average Value-at-Risk measures is also a law invariant coherent risk measure, and maximum of several law invariant coherent risk measures is again a law invariant coherent risk measure (see Remark 3 on page 18). By a result due[8] to Kusuoka [13], it turns out that any law invariant coherent risk measure can be constructed by the operations of taking convex combinations and maximum from the class of Average Value-at-Risk measures.

[7] Composition $T = T_1 \circ T_2$ of two mappings is the mapping $T(\omega) = T_1(T_2(\omega))$.

[8] Original proof by Kusuoka [13] is for $\mathcal{Z} = \mathcal{L}_\infty(\Omega, \mathcal{F}, P)$ space. For a general discussion see, e.g., [18, Section 2.2.4].

Theorem 2.7 (Kusuoka) *Suppose that the probability space $(\Omega, \mathcal{F}, P)$ is nonatomic and let $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty]$. Then a risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is law invariant and coherent iff there exists a set $\mathfrak{C}$ of probability measures on the interval $(0, 1]$ such that*

$$\rho(Z) = \sup_{\mu \in \mathfrak{C}} \int_0^1 \mathsf{AV@R}_\alpha(Z) d\mu(\alpha), \quad \forall Z \in \mathcal{Z}. \tag{2.67}$$

The representation (2.67) is not unique (e.g., [18, p.63]). We will discuss this further in the next section.

2.5.1 Comonotonicity

Let us discuss now the concept of *comonotonicity* (see, e.g., [9] for a thorough discussion of this concept). A set $A \subset \mathbb{R}^2$ is said to be *comonotonic* if for any $x, y \in A$ it holds that either $x \geq y$ or $y \leq x$, where the inequality $x \geq y$ is understood componentwise. Note that it follows from this definition that comonotonic set cannot have an open subset. Now let X and Y be two random variables. Denote by H_X and H_Y their respective cumulative distribution functions and $H(x, y) := \mathsf{Pr}(X \leq x, Y \leq y)$ their joint cdf. The following properties are equivalent to each other and any one of them can be used as a definition of X and Y to be comonotonic.

(i) Distribution of $(X, Y) \in \mathbb{R}^2$ has a comonotonic support.

(ii) For all $(x, y) \in \mathbb{R}^2$ it holds that $H(x, y) = \min\{H_X(x), H_Y(y)\}$.

(iii) $(X, Y) \overset{\mathcal{D}}{\sim} \left(H_X^{-1}(U), H_Y^{-1}(U)\right)$, where U is a random variable uniformly distributed on the interval $[0, 1]$.

Definition 2.3 It is said that a risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is *comonotonic* if for any two comonotonic random variables $X, Y \in \mathcal{Z}$ it follows that $\rho(X+Y) = \rho(X)+\rho(Y)$.

Let us observe that $\mathsf{V@R}_\alpha$, $\alpha \in (0, 1)$, risk measure is comonotonic. Indeed, let X and Y be comonotonic random variables. By the above property (iii) we can assume that $X = H_X^{-1}(U)$ and $Y = H_Y^{-1}(U)$, with U being a random variable uniformly distributed on the interval $[0, 1]$. Consider function $g(\cdot) := H_X^{-1}(\cdot) + H_Y^{-1}(\cdot)$. Note that $g : \mathbb{R} \to \mathbb{R}$ is a monotonically nondecreasing left-continuous function. Then

$$\begin{aligned}
\mathsf{V@R}_\alpha(X + Y) &= \inf\left\{t : \mathsf{Pr}(g(U) \leq t) \geq 1 - \alpha\right\} \\
&= \inf\left\{t : \mathsf{Pr}(U \leq g^{-1}(t)) \geq 1 - \alpha\right\} \\
&= \inf\left\{t : g^{-1}(t) \geq 1 - \alpha\right\} \\
&= g(1 - \alpha) = H_X^{-1}(1 - \alpha) + H_Y^{-1}(1 - \alpha) \\
&= \mathsf{V@R}_\alpha(X) + \mathsf{V@R}_\alpha(Y).
\end{aligned}$$

By (2.62) it follows that if X and Y are comonotonic, then

$$\begin{array}{rcl} \mathsf{AV@R}_\alpha(X+Y) & = & \frac{1}{\alpha}\int_0^\alpha \mathsf{V@R}_\tau(X+Y)\,d\tau = \frac{1}{\alpha}\int_0^\alpha [\mathsf{V@R}_\tau(X)+\mathsf{V@R}_\tau(Y)]\,d\tau \\ & = & \mathsf{AV@R}_\alpha(X)+\mathsf{AV@R}_\alpha(Y). \end{array}$$

That is, $\mathsf{AV@R}_\alpha$ is comonotonic for all $\alpha \in (0,1]$. If we define $\mathsf{AV@R}_0(\cdot) := \operatorname{ess\,sup}(\cdot)$, then it follows by (2.65) that $\mathsf{AV@R}_0(\cdot)$ is also comonotonic. Consequently if μ is a probability measure on the interval $[0,1]$, then risk measure

$$\rho(Z) = \int_0^1 \mathsf{AV@R}_\alpha(Z)d\mu(\alpha), \tag{2.68}$$

defined on an appropriate space $\mathcal{Z}$, is coherent law invariant and comonotonic. The second part of Kusuoka theorem says that the converse is also true (see, e.g., [18, Proposition 2.49] for a proof).

Theorem 2.8 (Kusuoka) *Suppose that the probability space $(\Omega, \mathcal{F}, P)$ is nonatomic and let $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty]$. Then a risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is law invariant, coherent and comonotonic iff there exists a probability measure μ on the interval $[0,1]$ such that the representation* (2.68) *holds.*

Remark 4 Note that if the space $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$ in representation (2.68) is taken with $p < \infty$, then the measure μ cannot have positive mass at $\alpha = 0$. Otherwise $\rho(Z)$ will be equal to $+\infty$ for any $Z \in \mathcal{Z}$ such that $\operatorname{ess\,sup}(Z) = +\infty$. $\Diamond$

Let the space $(\Omega, \mathcal{F}, P)$ be nonatomic, $\rho : \mathcal{Z} \to \mathbb{R}$ be a coherent law invariant risk measure and $\mathfrak{A}$ be its dual set. By Theorem 2.6 we have that for any $\eta \in \mathfrak{A}$ the set

$$\mathfrak{A}^*(\eta) := \{\eta \circ T : T \in \mathfrak{G}\} \tag{2.69}$$

is a subset of $\mathfrak{A}$.

Definition 2.4 We say that the dual set $\mathfrak{A}$, of a coherent risk measure $\rho : \mathcal{Z} \to \mathbb{R}$, is *generated* by an element $\eta \in \mathfrak{A}$ if it holds that

$$\sup_{\zeta \in \mathfrak{A}^*(\eta)} \langle Z, \zeta \rangle = \rho(Z), \quad \forall Z \in \mathcal{Z}. \tag{2.70}$$

If the dual set $\mathfrak{A}$ is generated by an element $\eta \in \mathfrak{A}$, then $\mathfrak{A}$ coincides with the weak* topological closure of the convex hull of the set $\mathfrak{A}^*(\eta)$ (see Remark 2 and property (i) of Remark 3, page 18).

Example 2 Let the space $(\Omega, \mathcal{F}, P)$ be nonatomic and consider the $\mathsf{AV@R}_\alpha$ measure. Let us show that the corresponding dual set $\mathfrak{A}$ (described in (2.64)) is generated by its element[9] $\eta := \alpha^{-1}\mathbf{1}_A$, where $A \in \mathcal{F}$ is a measurable set such that $P(A) = \alpha$. The set $\mathfrak{A}^*(\eta)$ generated by η can be written as

$$\mathfrak{A}^*(\eta) = \{\alpha^{-1}\mathbf{1}_B : B \in \mathcal{F},\ P(B) = \alpha\}.$$

Let $Z \in \mathcal{Z}$ and

$$t^* := \inf\big\{t : P\{\omega : Z(\omega) \geq t\} \leq \alpha\big\} = \mathsf{V@R}_\alpha(Z).$$

Suppose for the moment that $\mathsf{Pr}(Z = t^*) = 0$, i.e., cumulative distribution function $H(t)$ of Z is continuous at $t = t^*$. Then for $B^* := \{\omega : Z(\omega) \geq t^*\}$ we have that $P(B^*) = \alpha$. It follows that

$$\sup_{\zeta \in \mathfrak{A}^*(\eta)} \langle Z, \zeta \rangle = \sup_{B \in \mathcal{F}} \left\{\alpha^{-1} \int_B Z dP : P(B) = \alpha\right\} = \alpha^{-1} \int_{B^*} Z dP.$$

Moreover, $\int_{B^*} Z dP = \int_{t^*}^{+\infty} t dH(t)$, and hence

$$\sup_{\zeta \in \mathfrak{A}^*(\eta)} \langle Z, \zeta \rangle = \mathsf{AV@R}_\alpha(Z). \tag{2.71}$$

Since the set of random variables $Z \in \mathcal{Z}$ having continuous cdf forms a dense subset of $\mathcal{Z}$, it follows that formula (2.71) holds for all $Z \in \mathcal{Z}$. We obtain that the set $\mathfrak{A}$ is generated by η. $\Diamond$

Theorem 2.9 *Suppose that the probability space $(\Omega, \mathcal{F}, P)$ is nonatomic. Let $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$, and $\rho : \mathcal{Z} \to \mathbb{R}$ be a law invariant coherent risk measure. Then ρ is comonotonic iff its dual set $\mathfrak{A}$ is generated by some element $\eta \in \mathfrak{A}$.*

Proof. Since $(\Omega, \mathcal{F}, P)$ is nonatomic, we can assume without loss of generality that Ω is the interval $[0, 1]$ equipped with its Borel sigma algebra and uniform reference distribution.

Suppose that $\mathfrak{A}$ is generated by some element $\eta \in \mathfrak{A}$. Let $X, Y \in \mathcal{Z}$ be comonotonic variables. By the property (iii) of comonotonic variables we can assume that both functions $X, Y : [0, 1] \to \mathbb{R}$ are monotonically nondecreasing, and hence $X + Y$ is also monotonically nondecreasing. Let $T \in \mathfrak{G}$ be such that $\zeta := \eta \circ T$ is monotonically nondecreasing on the interval [0,1]. Then the maximum in (2.70) is attained

[9] By $\mathbf{1}_A(\cdot)$ we denote indicator function of set A, i.e., $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$, and $\mathbf{1}_A(\omega) = 0$ if $\omega \notin A$.

at ζ for both X and Y and for $X+Y$, i.e., $\rho(X)=\langle X,\zeta\rangle$, $\rho(Y)=\langle Y,\zeta\rangle$ and $\rho(X+Y)=\langle X+Y,\zeta\rangle$. It follows that $\rho(X+Y)=\rho(X)+\rho(Y)$, and hence ρ is comonotonic.

Conversely suppose that ρ is comonotonic. By Theorem 2.8 we have that ρ can be represented in the form (2.68) for some probability measure μ. Note that since we use here the space $\mathcal{Z}=\mathcal{L}_p(\Omega,\mathcal{F},P)$ with $p<\infty$, the measure μ cannot have a positive mass at $\alpha=0$ (see Remark 4). Suppose for the moment that measure μ has finite support[10], i.e., $\mu=\lambda_1\Delta(\alpha_1)+...+\lambda_m\Delta(\alpha_m)$, with $\alpha_i\in(0,1]$, $i=1,...,m$. Let $\mathfrak{A}_i$ be the dual set of risk measure $\mathsf{AV@R}_{\alpha_i}$, $i=1,...,m$. Then $\mathfrak{A}=\sum_{i=1}^m\lambda_i\mathfrak{A}_i$ and

$$\rho(Z)=\sup\left\{\sum_{i=1}^m\lambda_i\langle Z,\zeta_i\rangle:\zeta_i\in\mathfrak{A}_i,\ i=1,...,m\right\},\quad Z\in\mathcal{Z}.$$

By making the transformation $Z\mapsto Z\circ T$ for some $T\in\mathfrak{G}$, we can assume that Z is monotonically nondecreasing on the interval [0,1]. As it was shown in Example 2 the maximum of $\langle Z,\zeta_i\rangle$ over $\zeta_i\in\mathfrak{A}_i$ is attained at $\bar{\zeta}_i(\cdot)=\alpha_i^{-1}\mathbf{1}_{A_i}(\cdot)$, where $A_i:=[1-\alpha_i,1]$. It follows that the maximum of $\langle Z,\zeta\rangle$ over $\zeta\in\mathfrak{A}$ is attained at $\bar{\zeta}(\cdot)=\sum_{i=1}^m\lambda_i\alpha_i^{-1}\mathbf{1}_{A_i}(\cdot)$. For general measure μ in the representation (2.68) and monotonically nondecreasing $Z\in\mathcal{Z}$, the maximum $\langle Z,\zeta\rangle$ over $\zeta\in\mathfrak{A}$ is attained at

$$\bar{\zeta}(t)=\int_0^1\phi_t(\alpha)d\mu(\alpha),\quad t\in[0,1],\tag{2.72}$$

where $\phi_t(\alpha):=\alpha^{-1}\mathbf{1}_{[1-\alpha,1]}(t)$. It follows that the dual set $\mathfrak{A}$ is generated by $\bar{\zeta}$. ∎

Let us observe that equation (2.72) defines measure μ uniquely. Indeed, we can view $\mu(\cdot)$ as a right hand side continuous monotonically nondecreasing on [0,1] function, i.e., view the integral in (2.72) as the Lebesgue-Stieltjes integral. Then for $t\in[0,1)$ using integration by parts we have

$$\begin{array}{rcl}\int_0^1\phi_t(\alpha)d\mu(\alpha) & = & 1-\int_0^1\mu(\alpha)d\phi_t(\alpha)=1-\frac{\mu(1-t)}{1-t}-\int_{1-t}^1\mu(\alpha)d\alpha^{-1}\\ & = & 1-\frac{\mu(\tau)}{\tau}-\int_\tau^1\frac{\mu(\alpha)}{\alpha^2}d\alpha,\end{array}$$

where $\tau=1-t$. Suppose that there are two measures μ_1 and μ_2 which give the same integral in (2.72). Then for the function $\psi(\alpha):=-[\mu_1(\alpha)-\mu_2(\alpha)]/\alpha$ we have the following equation

$$\psi(\tau)+\int_\tau^1\frac{\psi(\alpha)}{\alpha}d\alpha=0,\ \tau\in(0,1].\tag{2.73}$$

[10]By $\Delta(\alpha)$ we denote measure of mass one at the point α. Measure $\Delta(\alpha)$ is often called *Dirac measure*.

It follows that $\psi(\cdot)$ is differentiable on $(0, 1]$ and satisfies the equation

$$\frac{d\psi(\tau)}{d\tau} - \frac{\psi(\tau)}{\tau} = 0. \tag{2.74}$$

The last equation has solutions of the form $\psi(\tau) = c\tau$ for some constant c. Substituting $\psi(\tau) = c\tau$ into (2.73) we obtain that $c = 0$. Consequently we have the following result.

- Suppose that the space $(\Omega, \mathcal{F}, P)$ is nonatomic and $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$. Then for a law invariant comonotonic coherent risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ the corresponding measure μ in the representation (2.68) is defined uniquely.

On the other hand the representation (2.67) clearly is not unique since there are various ways how the dual set $\mathfrak{A}$ can be represented as the topological closure of the convex hull of sets of the form $\mathfrak{A}^*(\eta)$, $\eta \in \mathfrak{A}$.

2.5.2 Risk Averse Optimization

In this section we discuss optimization problems of the form

$$\operatorname*{Min}_{x \in \mathcal{X}} \big\{ f(x) := \rho(F(x, \xi)) \big\}. \tag{2.75}$$

Here $\mathcal{X} \subset \mathbb{R}^n$ is a nonempty set, $F : \mathbb{R}^n \times \Xi \to \mathbb{R}$ is a real valued function and $\rho : \mathcal{Z} \to \mathbb{R}$ is a risk measure defined on space $\mathcal{Z} := \mathcal{L}_p(\Xi, \mathcal{B}, P)$, $p \in [1, \infty)$, where $\Xi \subset \mathbb{R}^d$ is a closed nonempty set equipped with its Borel sigma algebra $\mathcal{B}$ and a reference probability measure P. We assume that for every $x \in \mathcal{X}$ the function $F_x(\cdot) = F(x, \cdot)$ belongs to the space $\mathcal{Z}$ and write $\rho(F(x, \xi))$ for risk measure $\rho(F_x)$ of the random variable $F_x(\xi)$. Of course, for $\rho(\cdot) := \mathbb{E}(\cdot)$ problem (2.75) coincides with the stochastic programming problem (2.2).

By using the dual representation (2.55) of risk measure ρ we can write problem (2.75) in the following minimax form

$$\operatorname*{Min}_{x \in \mathcal{X}} \sup_{Q \in \mathfrak{Q}} \mathbb{E}_Q[F(x, \xi)], \tag{2.76}$$

where $\mathfrak{Q}$ is the respective set of (absolutely continuous with respect to P) probability measures on $(\Xi, \mathcal{B})$. In that form the above problem can be viewed as a robust stochastic problem of the form (2.1). By interchanging the *min* and *max* operators in (2.76) we obtain the following dual of problem (2.76)

$$\operatorname*{Max}_{Q \in \mathfrak{Q}} \inf_{x \in \mathcal{X}} \mathbb{E}_Q[F(x, \xi)]. \tag{2.77}$$

Relations between problems (2.76) and (2.77) are discussed in Theorems 2.3 and 2.4.

Suppose that function $F(x,\xi)$ is given as the optimal value of the second stage problem (2.3). In that case problem (2.75) can be viewed as a risk averse formulation of a two-stage problem. By using equivalent formulation (2.12) of the two-stage minimization part of problem (2.77) we can write the max-min problem (2.77) in the form

$$\begin{array}{lll} \underset{Q\in\mathfrak{Q}}{\text{Max}} & \underset{x,y(\cdot)}{\inf} & \mathbb{E}_Q[g(x,y(\xi),\xi)] \\ & \text{s.t.} & x\in\mathcal{X},\ y(\xi)\in\mathcal{G}(x,\xi) \text{ a.e. } \xi\in\Xi. \end{array} \tag{2.78}$$

Recall that the minimization in (2.78) is performed over $x\in\mathbb{R}^n$ and functions $y(\cdot):\Xi\to\mathbb{R}^m$ in an appropriate functional space.

Let us observe that if the risk measure ρ is coherent, then convexity of the $F(\cdot,\xi)$ is preserved in the composite function $f(x)=\rho(F(x,\xi))$. Indeed, suppose that $F(x,\xi)$ is convex in x for all $\xi\in\Xi$ and ρ is coherent. Then for any $x,y\in\mathbb{R}^n$, $t\in[0,1]$ and $\xi\in\Xi$ we have that $F(tx+(1-t)y,\xi)\leq tF(x,\xi)+(1-t)F(y,\xi)$, and hence by using monotonicity and convexity properties of ρ (axioms (A1) and (A2)) we obtain

$$\begin{array}{rcl} f(tx+(1-t)y) & = & \rho(F(tx+(1-t)y,\xi))\leq\rho(tF(x,\xi)+(1-t)F(y,\xi)) \\ & \leq & t\rho(F(x,\xi))+(1-t)\rho(F(y,\xi))=tf(x)+(1-t)f(y). \end{array}$$

Note that in this derivation the monotonicity property of ρ is essential, and convexity of ρ and $F(\cdot,\xi)$ alone does not guarantee convexity of the composite function $f(\cdot)$.

Another important property of coherent risk measures is that for such measures the interchangeability principle holds (cf., [30, section 6.4]). Suppose that function $F(x,\xi)$ is given as the optimal value of the second stage problem (2.3). As it was pointed out in section 2.2, in the risk neutral case (i.e., when $\rho(\cdot):=\mathbb{E}[\cdot]$) the corresponding two-stage problem can be written as one large problem (2.12). Similar result holds for coherent risk measures. Under mild regularity conditions the risk averse two stage problem (2.75), with the second stage (2.12), can be written as

$$\begin{array}{ll} \underset{x,y(\cdot)}{\text{Min}} & \rho[g(x,y(\xi),\xi)] \\ \text{s.t.} & x\in\mathcal{X},\ y(\xi)\in\mathcal{G}(x,\xi) \text{ a.e. } \xi\in\Xi. \end{array} \tag{2.79}$$

Note that the monotonicity property of ρ is essential for the equivalence of formulations (2.75) and (2.79) of risk averse two-stage problems. Without the monotonicity condition this equivalence is not guaranteed (cf., [32]). By the dual representation (2.55) of risk measure ρ we can rewrite problem (2.79) in the form

$$\begin{array}{lll} \underset{x,y(\cdot)}{\text{Min}} & \underset{Q\in\mathfrak{Q}}{\sup} & \mathbb{E}_Q[g(x,y(\xi),\xi)] \\ \text{s.t.} & & x\in\mathcal{X},\ y(\xi)\in\mathcal{G}(x,\xi) \text{ a.e. } \xi\in\Xi. \end{array} \tag{2.80}$$

Problems (2.78) and (2.80) can be viewed as dual to each other.

Theorem 2.10 *Let $F(x,\xi)$ be the optimal value of the second stage problem* (2.3)*, $\mathcal{Z} := \mathcal{L}_p(\Xi, \mathcal{B}, P)$, with $p \in [1,\infty)$, $\mathfrak{A}$ be a bounded convex set of probability density functions in the dual space $\mathcal{Z}^*$ and $\mathfrak{Q} := \{Q : dQ = \zeta dP,\ \zeta \in \mathfrak{A}\}$. Suppose, further, that problem* (2.76) *has an optimal solution $\bar{x} \in \mathcal{X}$, the set $\mathcal{X}$ is convex and closed, the set Ξ is compact and $F(x,\xi)$ is convex in x for every $\xi \in \Xi$ and continuous in ξ for every $x \in \mathcal{X}$. Then optimal values of problems* (2.78) *and* (2.80) *are equal to each other. Moreover, if the set $\mathfrak{Q}$ is closed in the weak topology, then problem* (2.78) *has an optimal solution $\bar{Q} \in \mathfrak{Q}$.*

Proof. Since problem (2.76) has an optimal solution, its optimal value is finite of course. By Theorem 2.3 we have that, under the specified assumptions, optimal values of problems (2.76) and (2.77) are equal to each other. Moreover, problem (2.77) has an optimal solution provided the set $\mathfrak{Q}$ is closed. Now the minimization part of problem (2.77) is equivalent to the minimization part of problem (2.78), and hence optimal values of problems (2.77) and (2.78) are equal to each other.

Consider the risk measure ρ associated with the set $\mathfrak{A}$ by equation (2.55). Then problem (2.76) can be written in the form (2.75). By an interchangeability principle (cf., [30, Theorem 2.20]) problem (2.75) is equivalent to the problem (2.79), and hence to the problem (2.80). It follows that optimal values of problems (2.78) and (2.80) are equal to each other. ∎

In particular, suppose that the number of scenarios is finite, i.e., the set $\Xi = \{\xi_1, ..., \xi_K\}$ is finite and is equipped with sigma algebra of all its subsets. In that case any mapping $Z : \Xi \to \mathbb{R}$ is measurable and can be identified with K-dimensional vector $(Z(\xi_1), ..., Z(\xi_K))$, and the space $\mathcal{Z}$ of all such mappings can be identified with $\mathbb{R}^K$. Hence for a risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ we write $\rho(Z(\xi_1), ..., Z(\xi_K))$ for $\rho(Z)$. Then problem (2.79) becomes (compare with problem (2.13) in the risk neutral case)

$$\begin{array}{ll} \underset{x, y_1, ..., y_K}{\text{Min}} & \rho\big(g(x, y_1, \xi_1), ..., g(x, y_K, \xi_K)\big) \\ \text{s.t.} & x \in \mathcal{X},\ y_i \in \mathcal{G}(x, \xi_i),\ i = 1, ..., K. \end{array} \tag{2.81}$$

Suppose, further, that the problem is linear, i.e., $\mathcal{X} = \{x : Ax = b,\ x \geq 0\}$, $g(x, y, \xi) = q^{\mathsf{T}} y$, $\mathcal{G}(x, \xi)$ is given in the form (2.6) and there is additional linear term $c^{\mathsf{T}} x$ at the first stage. Then problem (2.81) becomes

$$\begin{array}{ll} \underset{x, y_1, ..., y_K}{\text{Min}} & c^{\mathsf{T}} x + \rho(q^{\mathsf{T}} y_1, ..., q^{\mathsf{T}} y_K) \\ \text{s.t.} & Ax = b,\ x \geq 0, \\ & T_i x + W_i y_i = h_i,\ y_i \geq 0,\ i = 1, ..., K. \end{array} \tag{2.82}$$

For a general (coherent) risk measure it could be difficult to solve the corresponding two-stage risk averse problem even if the number of scenarios is finite, and not

too large, and the problem is linear. For the following class of risk measures the problem can be reduced to a standard stochastic programming problem. Consider risk measure

$$\rho_{\alpha,\lambda}(Z) := (1-\lambda)\mathbb{E}[Z] + \lambda\, \mathsf{AV@R}_\alpha[Z], \tag{2.83}$$

with $\alpha \in (0,1)$ and $\lambda \in [0,1]$ being tuning parameters. As a convex combination of two coherent risk measures, risk measure $\rho_{\alpha,\lambda}$ is coherent.

By using definition (2.60) of $\mathsf{AV@R}_\alpha$ we can write the corresponding risk averse problem (2.75) as

$$\operatorname*{Min}_{x\in\mathcal{X},\, z\in\mathbb{R}} \lambda z + \mathbb{E}\left\{(1-\lambda)F(x,\xi) + \lambda\alpha^{-1}[F(x,\xi)-z]_+\right\}, \tag{2.84}$$

and hence as the following two-stage problem

$$\operatorname*{Min}_{x\in\mathcal{X},\, z\in\mathbb{R}} \lambda z + \mathbb{E}[V(x,z,\xi)], \tag{2.85}$$

where $V(x,z,\xi)$ is the optimal value of the second stage problem

$$\operatorname*{Min}_{y\in\mathcal{G}(x,\xi)} (1-\lambda)g(x,y,\xi) + \lambda\alpha^{-1}[g(x,y,\xi)-z]_+. \tag{2.86}$$

In particular, if the problem is linear, then the second stage problem (2.86) becomes

$$\operatorname*{Min}_{y\geq 0} (1-\lambda)q^{\mathsf{T}}y + \lambda\alpha^{-1}[q^{\mathsf{T}}y - z]_+ \ \text{ s.t. } Tx + Wy = h. \tag{2.87}$$

Problem (2.87) can be written as the linear program

$$\begin{array}{ll} \operatorname*{Min}\limits_{y\in\mathbb{R}^m,\, u\in\mathbb{R}} & (1-\lambda)q^{\mathsf{T}}y + \lambda\alpha^{-1}u \\ \text{s.t.} & q^{\mathsf{T}}y - z \leq u,\ u \geq 0, \\ & Tx + Wy = h,\ y \geq 0, \end{array} \tag{2.88}$$

and hence this linear risk averse problem can be formulated as a linear two-stage stochastic program with (first stage) decision variables $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$, and second stage problem (2.88) with (second stage) decision variables $y \in \mathbb{R}^m$ and $u \in \mathbb{R}$.

Remark 5 We can give the following motivation for considering risk measure $\rho_{\alpha,\lambda}$. Suppose that we want to control the value $Q(x,\xi)$ of the second stage problem (2.3) in the risk neutral formulation. Since we deal with minimization formulations, we want $Q(x,\xi)$ to be "not too large", say less than a chosen level η with a high probability $1-\alpha$. We can write this requirement as the following chance (probabilistic) constraint

$$\mathsf{Pr}\{Q(x,\xi) \leq \eta\} \geq 1-\alpha. \tag{2.89}$$

Equivalently the above constraint (2.89) can be written as

$$\mathsf{V@R}_\alpha[Q(x,\xi)] \leq \eta. \tag{2.90}$$

The difficulty with this chance constraint is that the function $\mathsf{V@R}_\alpha[Q(x,\xi)]$ typically is not convex in x, even if $Q(\cdot,\xi)$ is convex, and is difficult to handle numerically. Therefore it makes sense to replace chance constraint (2.90) by the following conservative approximation

$$\mathsf{AV@R}_\alpha[Q(x,\xi)] \leq \eta. \tag{2.91}$$

Recall that $\mathsf{AV@R}_\alpha(\cdot) \geq \mathsf{V@R}_\alpha(\cdot)$, and hence if a point x satisfies (2.91), then it is also feasible for (2.90). Therefore, indeed, constraint (2.91) is a conservative approximation of (2.90). Introducing additional constraint (2.91) may result in infeasibility of the corresponding two-stage problem even if the original (risk neutral) problem was feasible. Moving the constraint (2.91) into the expected value objective as a penalty term leads to the risk measure $\rho_{\alpha,\lambda}$. $\Diamond$

2.6 Two-Stage Problems with Expectations Constraints

Consider the following two-stage stochastic programming problem

$$\begin{array}{ll} \underset{x,y(\cdot)}{\mathrm{Min}} & \mathbb{E}[G_0(x,y(\xi),\xi)] \\ \text{s.t.} & \mathbb{E}[G_i(x,y(\xi),\xi)] \leq 0,\ i=1,...,k, \\ & x \in \mathcal{X},\ y(\xi) \in \mathcal{G}(x,\xi) \text{ a.e. } \xi \in \Xi. \end{array} \tag{2.92}$$

This formulation is more general than formulation (2.12) since it also involves constraints in the form of expectations. The setting here is similar to the one of section 2. That is, $\mathcal{X} \subset \mathbb{R}^n$ and $\Xi \subset \mathbb{R}^d$ are nonempty closed sets, $G_i : \mathbb{R}^n \times \mathbb{R}^m \times \Xi \to \mathbb{R}$, $i = 0,...,k$, are random functions and $\mathcal{G} : \mathbb{R}^n \times \Xi \rightrightarrows \mathbb{R}^m$ is a (measurable) multifunction. The expectations in (2.92) are taken with respect to a (uniquely) specified probability measure P on $(\Xi, \mathcal{B})$, thus (2.92) is a risk neutral type problem. The optimization in (2.92) is performed over $x \in \mathbb{R}^n$ and functions $y(\cdot) : \Xi \to \mathbb{R}^m$ belonging to a functional space $\mathfrak{Y}$. To be specific we use $\mathfrak{Y} := \mathcal{L}_p(\Xi, \mathcal{B}, P)$, with $p \in [1,\infty]$. We assume that for all $x \in \mathcal{X}$ and $y(\cdot) \in \mathfrak{Y}$ the expectations $\mathbb{E}[G_i(x,y(\xi),\xi)]$, $i = 0,...,k$, are well defined and finite valued.

By performing minimization in (2.92) first with respect to $y(\cdot) \in \mathfrak{Y}$ and then with respect to $x \in \mathcal{X}$, we can write (2.92) as the following two stage problem

$$\underset{x \in \mathcal{X}}{\mathrm{Min}}\ \mathcal{Q}(x), \tag{2.93}$$

where $\mathcal{Q}(x)$ is the optimal value of problem

$$\begin{array}{ll} \underset{y(\cdot)\in\mathcal{Y}}{\text{Min}} & \mathbb{E}[G_0(x, y(\xi), \xi)] \\ \text{s.t.} & \mathbb{E}[G_i(x, y(\xi), \xi)] \leq 0,\ i = 1, ..., k, \\ & y(\xi) \in \mathcal{G}(x, \xi) \text{ a.e. } \xi \in \Xi. \end{array} \tag{2.94}$$

In particular, if distribution of ξ has a finite support $\Xi = \{\xi_1, ..., \xi_N\}$ with respective probabilities $p_1, ..., p_N$, then the second stage problem (2.94) becomes

$$\begin{array}{ll} \underset{y_1,...,y_N}{\text{Min}} & \sum_{j=1}^N p_j G_0(x, y_j, \xi_j) \\ \text{s.t.} & \sum_{j=1}^N p_j G_i(x, y_j, \xi_j) \leq 0,\ i = 1, ..., k, \\ & y_j \in \mathcal{G}(x, \xi_j),\ j = 1, ..., N. \end{array} \tag{2.95}$$

Note that because of the expectation constraints, here the second stage problem is not separable (decomposable) into a sum of individual optimization problems. The Sample Average Approximation (SAA) method can be applied to problem (2.92) in a straightforward way. That is, a sample $\xi^1, ..., \xi^N$ of random vector ξ is generated and the "true" problem (2.92) is approximated by the SAA problem with the second stage problem of the form (2.95) with $\xi_j = \xi^j$ and $p_j = 1/N$, $j = 1, ..., N$.

In some cases the expectation constraints can be moved into the objective. That is, we can write the following (Lagrangian) dual of problem (2.92)

$$\begin{array}{lll} \underset{\lambda\in\mathbb{R}^k_+}{\text{Max}} & \underset{x,y(\cdot)}{\inf} & \mathbb{E}[L(x, y(\xi), \lambda, \xi)] \\ & \text{s.t.} & x \in \mathcal{X},\ y(\xi) \in \mathcal{G}(x, \xi) \text{ a.e. } \xi \in \Xi, \end{array} \tag{2.96}$$

where

$$L(x, y, \lambda, \xi) := G_0(x, y, \xi) + \sum_{i=1}^k \lambda_i G_i(x, y, \xi).$$

The optimal value of problem (2.96) is always less than or equal to the optimal value of problem (2.92), and it is said that there is no duality gap between these problems if their optimal values are equal to each other.

In order to ensure that there is no duality gap between problems (2.92) and (2.96) typically one needs to assume convexity. We say that problem (2.92) is *convex* if the set

$$\mathfrak{S} := \big\{(x, y(\cdot)) : x \in \mathcal{X},\ y(\xi) \in \mathcal{G}(x, \xi) \text{ a.e. } \xi \in \Xi\big\} \tag{2.97}$$

is a convex subset of $\mathbb{R}^n \times \mathcal{Y}$, and functions $G_i(x, y, \xi)$, $i = 0, ..., k$, are convex in $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$. Note that the above convexity of functions $G_i(x, y, \xi)$ implies convexity of

$$\phi_i(x, y(\cdot)) := \mathbb{E}[G_i(x, y(\xi), \xi)],\ \ i = 0, ..., k, \tag{2.98}$$

considered as functions on the space $\mathcal{X} \times \mathfrak{Y}$.

It is said that *Slater condition*, for the problem (2.92), holds if there exists $(\bar{x}, \bar{y}(\cdot)) \in \mathfrak{S}$ such that $\phi_i(\bar{x}, \bar{y}(\cdot)) < 0$, $i = 1, ..., k$. Assuming convexity and Slater condition we have the following result (e.g., [7, Theorem 2.165]).

Proposition 2.1 *Suppose that problem* (2.92) *is convex and has a finite optimal value, Slater condition holds and the functions $\phi_i(x, y(\cdot))$, $i = 0, ..., k$, are real valued continuous on the set $\mathfrak{S}$. Then there is no duality gap between problems* (2.92) *and* (2.96) *and problem* (2.96) *has a nonempty and bounded in $\mathbb{R}^k$ set of optimal solutions.*

In order to apply the above result, continuity of the functions $\phi_i : \mathfrak{S} \to \mathbb{R}$, $i = 0, ..., k$, should be verified with respect to the strong (norm) topology of the space $\mathfrak{Y} = \mathcal{L}_p(\Xi, \mathcal{B}, P)$. If the set Ξ is finite, then the problems becomes finite dimensional and this continuity follows from convexity of $G_i(x, y, \xi)$ in $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$. In general, in order to avoid technical complications one can work in the space $\mathfrak{Y} = \mathcal{L}_\infty(\Xi, \mathcal{B}, P)$, where strong (norm) convergence means uniform convergence and hence this continuity property is relatively easy to verify.

Now the dual problem (2.96) can be written as

$$\operatorname*{Max}_{\lambda \in \mathbb{R}^k_+} \inf_{x \in \mathcal{X}} \mathbb{E}[F(x, \lambda, \xi)], \tag{2.99}$$

where $F(x, \lambda, \xi)$ is the optimal value of the second stage problem:

$$\operatorname*{Min}_{y \in \mathbb{R}^m} L(x, y, \lambda, \xi) \ \text{ s.t. } y \in \mathcal{G}(x, \xi). \tag{2.100}$$

The problem (2.99) can be viewed as a minimax stochastic programming problem. The Sample Average Approximation (SAA) method can be applied to problem (2.99). That is, a sample $\xi^1, ..., \xi^N$ of random vector ξ is generated and the "true" problem (2.99) is approximated by the SAA problem:

$$\operatorname*{Max}_{\lambda \in \mathbb{R}^k_+} \inf_{x \in \mathcal{X}} \hat{f}_N(x, \lambda), \tag{2.101}$$

where $\hat{f}_N(x, \lambda) := N^{-1} \sum_{j=1}^N F(x, \lambda, \xi^j)$. Statistical properties of the SAA method in the minimax setting are discussed in [30, section 5.1.4].

In order to solve the SAA problem (2.101) efficiently we should be able to compute derivatives of the corresponding optimal value functions with respect to the first stage decision variables. Analysis of differentiability properties of such optimal value functions is discussed in details, e.g., in [7, section 4.3]. Let us consider the second stage problem (2.100). To be more specific suppose that the multifunction $\mathcal{G}(x, \xi)$

is given by linear constraints in the form (2.6). That is, $F(x, \lambda, \xi)$ is given by the optimal value of the following second stage problem

$$\operatorname*{Min}_{y \geq 0} L(x, y, \lambda, \xi) \ \text{ s.t. } Tx + Wy = h. \tag{2.102}$$

Note that $L(x, \lambda, y, \xi)$ is linear in λ and hence $F(x, \lambda, \xi)$ is concave in λ. Suppose that problem (2.102) has unique optimal solution $\bar{y} = \bar{y}(x, \lambda, \xi)$. Then $F(x, \lambda, \xi)$ is differentiable in λ and

$$\nabla_\lambda F(x, \lambda, \xi) = \nabla_\lambda L(x, \bar{y}, \lambda, \xi) = (G_1(x, \bar{y}, \xi), ..., G_k(x, \bar{y}, \xi)).$$

Suppose, further, that $L(x, y, \lambda, \xi)$ is convex in (x, y). Then $F(x, \lambda, \xi)$ is convex in x and if, moreover, $L(x, y, \lambda, \xi)$ is differentiable in x, then

$$\nabla_x F(x, \lambda, \xi) = \nabla_x L(x, \bar{y}, \lambda, \xi) - T^\mathsf{T} \mu,$$

where μ is an optimal solution of the corresponding dual of problem (2.102).

Example 3 Consider the following two-stage problem

$$\begin{array}{ll} \operatorname*{Min}\limits_{x \in \mathcal{X}, y(\cdot)} & c^\mathsf{T} x \\ \text{s.t.} & \mathsf{AV@R}_\alpha[G_i(x, y(\xi), \xi)] \leq 0, \ \ i = 1, ..., k, \\ & T(\xi)x + W(\xi)y(\xi) \leq h(\xi) \ \text{ a.e. } \xi \in \Xi, \end{array} \tag{2.103}$$

where $\mathcal{X} \subset \mathbb{R}^n$ is a (nonempty) polyhedral set (defined by linear constraints), $\alpha \in (0, 1)$ and

$$G_i(x, y, \xi) := a_i(\xi)^\mathsf{T} x + b_i(\xi)^\mathsf{T} y - d_i(\xi), \ \ i = 1, ..., k.$$

Compared with the linear two-stage stochastic programming formulation (2.4)–(2.5), the additional constraints involving $\mathsf{AV@R}_\alpha$ risk measures are added in (2.103).

The constraints $\mathsf{AV@R}_\alpha[G_i(x, y(\xi), \xi)] \leq 0$ can be viewed as conservative approximations of the respective constraints $\mathsf{V@R}_\alpha[G_i(x, y(\xi), \xi)] \leq 0$, which in turn are equivalent to the corresponding chance constraints $\mathsf{Pr}\{G_i(x, y(\xi), \xi) \leq 0\} \geq 1 - \alpha$. Therefore problem (2.103) can be considered as a conservative approximation of the problem

$$\begin{array}{ll} \operatorname*{Min}\limits_{x \in \mathcal{X}, y(\cdot)} & c^\mathsf{T} x \\ \text{s.t.} & \mathsf{Pr}\big\{a_i(\xi)^\mathsf{T} x + b_i(\xi)^\mathsf{T} y(\xi) \leq d_i(\xi)\big\} \geq 1 - \alpha, \ i = 1, ..., k, \\ & T(\xi)x + W(\xi)y(\xi) \leq h(\xi) \ \text{ a.e. } \xi \in \Xi. \end{array} \tag{2.104}$$

The above problem (2.104) has two types of linear constraints - some should be satisfied with a high probability and some for almost all realizations of the random data.

By using definition (2.60) of $\mathsf{AV@R}_\alpha$ we can write problem (2.103) as

$$\begin{array}{ll} \underset{x\in\mathcal{X},z\in\mathbb{R},y(\cdot)}{\text{Min}} & c^\mathsf{T}x \\ \text{s.t.} & \mathbb{E}\left\{z+\alpha^{-1}[G_i(x,y(\xi),\xi)-z]_+\right\}\leq 0,\ i=1,...,k, \\ & T(\xi)x+W(\xi)y(\xi)\leq h(\xi)\ \text{ a.e. } \xi\in\Xi. \end{array} \tag{2.105}$$

The above problem (2.105) is convex. Suppose further that $\mathfrak{Y}=\mathcal{L}_\infty(\Xi,\mathcal{B},P)$, the set Ξ is compact and $a_i(\xi),b_i(\xi)$ and $d_i(\xi)$, $i=1,...,k$, are continuous on Ξ. Then the Slater condition, for the problem (2.105), holds and the functions

$$(x,z,y(\cdot))\mapsto \mathbb{E}\left\{z+\alpha^{-1}[G_i(x,y(\xi),\xi)-z]_+\right\},\ i=1,...,k,$$

are continuous on $\mathbb{R}^n\times\mathbb{R}\times\mathfrak{Y}$. Thus by Proposition 2.1 there is no duality gap between problem (2.105) and its dual which can be written as the minimax stochastic problem (2.99), with (x,z) and λ being the first stage decision variables, and with the second stage problem:

$$\begin{array}{ll} \underset{y\in\mathbb{R}^m}{\text{Min}} & c^\mathsf{T}x+\sum_{i=1}^k\lambda_i\left(z+\alpha^{-1}[G_i(x,y,\xi)-z]_+\right) \\ \text{s.t.} & T(\xi)x+W(\xi)y\leq h(\xi). \end{array} \tag{2.106}$$

$\Diamond$

2.7 The Problem of Moments

In some situations, it is reasonable to assume that we have a knowledge about certain moments of the corresponding probability distribution. Denote by $\mathfrak{S}$ the space of finite signed measures on the set $\Xi\subset\mathbb{R}^d$ equipped with its Borel sigma algebra $\mathcal{B}$. Note that $P\in\mathfrak{S}$ is a *probability* measure[11] iff $P\succeq 0$ and $\int_\Xi dP=1$. Consider the following set of probability measures on $(\Xi,\mathcal{B})$

$$\mathfrak{M}:=\left\{P:\int_\Xi\Psi(\xi)dP(\xi)\in\mathcal{S},\ \int_\Xi dP=1,\ P\succeq 0\right\}, \tag{2.107}$$

where $\mathcal{S}\subset\mathbb{R}^q$ is a nonempty closed convex set and $\Psi=(\psi_1,...,\psi_q):\Xi\to\mathbb{R}^q$ is a measurable mapping.

[11] Recall that the notation "$P\succeq 0$" means that P is a nonnegative (not necessarily probability) measure on $(\Xi,\mathcal{B})$. That is, $P(A)\geq 0$ for every $A\in\mathcal{B}$.

The set $\mathfrak{M}$ consists of all probability measures $P \in \mathfrak{S}$ such that the expectation (integral) $\mathbb{E}_P[\psi_i(\xi)]$, $i = 1, ..., q$, of every component of the mapping $\Psi(\xi)$, is well defined and finite valued (i.e., functions $\psi_i(\xi)$ are P-integrable) and

$$\big(\mathbb{E}_P[\psi_1(\xi)], ..., \mathbb{E}_P[\psi_q(\xi)]\big) \in \mathcal{S}. \tag{2.108}$$

In particular, if $\mathcal{S} := \{\mu\}$ is a singleton, then the set $\mathfrak{M}$ is defined by the constraints

$$\mathbb{E}_P[\psi_i(\xi)] = \mu_i, \ i = 1, ..., q, \tag{2.109}$$

over the set of probability measures P on $(\Xi, \mathcal{B})$. We consider a more general situation when the moments $\mathbb{E}_P[\psi_i(\xi)]$ could be known with a certain accuracy summarized in the set $\mathcal{S}$. For example, if each moment $\mathbb{E}_P[\psi_i(\xi)]$ is assumed to belong to a corresponding confidence interval, then the set $\mathcal{S}$ could be defined by a finite number of linear constraints.

Consider now the following so-called *problem of moments* (e.g., [14])

$$\operatorname*{Max}_{P \in \mathfrak{M}} \mathbb{E}_P[\psi_0(\xi)], \tag{2.110}$$

where $\psi_0 : \Xi \to \mathbb{R}$ is a P-integrable for every $P \in \mathfrak{M}$ function. We are going to show that it suffices to solve problem (2.110) for discrete probability measures having a finite support of at most $q + 1$ points. In order to proceed we need the following classical result.

Theorem 2.11 (Richter-Rogosinski) *Let $(\Omega, \mathcal{F})$ be a measurable space, $f_1, ..., f_m$ be measurable on $(\Omega, \mathcal{F})$ real valued functions, and P be a (nonnegative) measure on $(\Omega, \mathcal{F})$ such that $f_1, ..., f_m$ are P-integrable. Suppose that every finite subset of Ω is $\mathcal{F}$-measurable. Then there exists a (nonnegative) measure Q on $(\Omega, \mathcal{F})$ with a finite support of at most m points such that $\int_\Omega f_i dP = \int_\Omega f_i dQ$ for all $i = 1, ..., m$.*

We use this result for the space $(\Xi, \mathcal{B})$. Of course, any finite subset of Ξ is $\mathcal{B}$-measurable. Denote by $\mathfrak{M}^*_\ell$ the subset of $\mathfrak{M}$ of probability measures having a finite support of at most ℓ points. That is, a measure $P \in \mathfrak{M}$ belongs to $\mathfrak{M}^*_\ell$ if it can be represented[12] in the form $P = \sum_{i=1}^{\ell} \alpha_i \Delta(\xi_i)$, where $\xi_i \in \Xi$ and α_i, $i = 1, ..., \ell$, are nonnegative numbers such that $\sum_{i=1}^{\ell} \alpha_i = 1$.

Theorem 2.12 *Problem* (2.110) *is equivalent to the problem*

$$\operatorname*{Max}_{P \in \mathfrak{M}^*_\ell} \mathbb{E}_P[\psi_0(\xi)], \tag{2.111}$$

[12]Recall that $\Delta(\xi)$ denotes measure of mass one at the point ξ.

with $\ell = q + 1$. The equivalence is in the sense that both problems have the same optimal value and if problem (2.110) *has an optimal solution, then it also has an optimal solution $P^* \in \mathfrak{M}^*_{q+1}$. In particular if $q = 0$, i.e., the set $\mathfrak{M}$ consists of all probability measures on $(\Xi, \mathcal{B})$, then problem* (2.110) *is equivalent to the problem of maximization of $\psi_0(\xi)$ over $\xi \in \Xi$.*

Proof. If the set $\mathfrak{M}$ is empty, then its subset $\mathfrak{M}^*_{q+1}$ is also empty, and hence the optimal value of both problems (2.110) and (2.111) is $+\infty$. So suppose that $\mathfrak{M}$ is nonempty and let $P \in \mathfrak{M}$. By Theorem 2.11 there exists $Q \in \mathfrak{M}^*_{q+2}$ such that $\mathbb{E}_P[\psi_0(\xi)] = \mathbb{E}_Q[\psi_0(\xi)]$. It follows that for $\ell = q + 2$ the optimal value of (2.110) is greater than or equal to the optimal value of problem (2.111). Since $\mathfrak{M}_\ell$ is a subset of $\mathfrak{M}$, it follows that for $\ell = q + 2$ the optimal values of problems (2.110) and (2.111) are equal to each other and if problem (2.110) has an optimal solution, then it has an optimal solution in the set $\mathfrak{M}^*_{q+2}$.

Now for $P = \sum_{i=1}^{\ell} \alpha_i \Delta(\xi_i)$ problem (2.111) can be written as

$$\begin{array}{ll} \underset{\substack{\xi_1,\ldots,\xi_\ell \in \Xi \\ b \in \mathcal{S},\, \alpha \geq 0}}{\text{Max}} & \sum\limits_{i=1}^{\ell} \alpha_i \psi_0(\xi_i) \\ \text{s.t.} & \sum\limits_{i=1}^{\ell} \alpha_i \Psi(\xi_i) = b,\ \sum\limits_{i=1}^{\ell} \alpha_i = 1. \end{array} \tag{2.112}$$

For fixed $\xi_1, \ldots, \xi_\ell \in \Xi$ and $b \in \mathcal{S}$, the above is a linear programming problem. Its feasible set is bounded and its optimum is attained at an extreme point of its feasible set which has at most $q + 1$ nonzero components of α. Therefore it suffices to take the maximum over $P \in \mathfrak{M}^*_{q+1}$. ∎

Suppose now that the set Ξ is convex compact. Then by Minkowski Theorem, Ξ is equal to the convex hull of its extreme points. Recall that a point $e \in \Xi$ is said to be *extreme* if there do not exist points $e_1, e_2 \in \Xi$, different from e, such that e belongs to the interval $[e_1, e_2]$. In other words, e is an extreme point of Ξ if whenever $e = te_1 + (1 - t)e_2$ for some $e_1, e_2 \in \Xi$ and $t \in (0, 1)$, then $e_1 = e_2 = e$. We denote by $\text{Ext}(\Xi)$ the set of extreme points of Ξ.

Theorem 2.13 *Suppose that the set Ξ is nonempty convex compact, the mapping $\Psi : \Xi \to \mathbb{R}^q$ is affine, the function $\psi_0 : \Xi \to \mathbb{R}$ is convex continuous and the set $\mathcal{S}$ is nonempty compact. Then the maximum in* (2.110) *is attained at a probability measure of the form $P^* = \sum_{i=1}^{q+1} \alpha_i \Delta(e_i)$, where $e_i \in \text{Ext}(\Xi)$ and $\alpha_i \in [0, 1]$, $i = 1, ..., q + 1$, with $\sum_{i=1}^{q+1} \alpha_i = 1$.*

Proof. By Theorem 2.12 it suffices to perform the maximization over discrete measures with finite support. Since the sets Ξ and $\mathcal{S}$ are compact and $\psi_0(\cdot)$ is continuous it follows by compactness arguments that problem (2.110) has an optimal solution of the form $P^* = \sum_{i=1}^k \alpha_i \Delta(\xi_i)$ for some $\xi_i \in \Xi$ and $\alpha_i \in [0,1]$ such that $\sum_{i=1}^k \alpha_i = 1$. We need to show that the points ξ_i can be chosen to be extreme points of the set Ξ.

Suppose that one of the points ξ_i, say ξ_1, is not an extreme point of Ξ. Since Ξ is equal to the convex hull of $\mathrm{Ext}(\Xi)$, there exist points $e_1, ..., e_m \in \mathrm{Ext}(\Xi)$ and $t_j \in (0,1)$, with $\sum_{j=1}^m t_j = 1$, such that $\xi_1 = \sum_{j=1}^m t_j e_j$. Consider the probability measure

$$P' := \alpha_1 \textstyle\sum_{j=1}^m t_j \Delta(e_j) + \sum_{i=2}^k \alpha_i \Delta(\xi_i).$$

Since $\Psi(\cdot)$ is an affine mapping we have that

$$\begin{array}{rcl} \mathbb{E}_{P'}[\Psi(\xi)] & = & \alpha_1 \sum_{j=1}^m t_j \Psi(e_j) + \sum_{i=2}^k \alpha_i \Psi(\xi_i) \\ & = & \alpha_1 \Psi\big(\sum_{j=1}^m t_j e_j\big) + + \sum_{i=2}^k \alpha_i \Psi(\xi_i) \\ & = & \alpha_1 \Psi(\xi_1) + \sum_{i=2}^k \alpha_i \Psi(\xi_i) = \mathbb{E}_{P^*}[\Psi(\xi)]. \end{array}$$

Therefore P' satisfies the feasibility constraints $\mathbb{E}_P[\Psi(\xi)] \in \mathcal{S}$ as well as P^*. Now by convexity of ψ_0 we can write

$$\begin{array}{rcl} \mathbb{E}_{P'}[\psi_0(\xi)] & = & \alpha_1 \sum_{j=1}^m t_j \psi_0(e_j) + \sum_{i=2}^k \alpha_i \psi_0(\xi_i) \\ & \geq & \alpha_1 \psi_0\big(\sum_{j=1}^m t_j e_j\big) + \sum_{i=2}^k \alpha_i \psi_0(\xi_i) \\ & = & \alpha_1 \psi_0(\xi_1) + \sum_{i=2}^k \alpha_i \psi_0(\xi_i) = \mathbb{E}_{P^*}[\psi_0(\xi)]. \end{array}$$

It follows that there exists an optimal solution with a finite support of a set of extreme points. It remains to note that by Theorem 2.11 this support can be chosen to have no mote than $q+1$ points. ∎

We can view problem (2.110) as an optimization problem over the linear space $\mathfrak{S}$ of finite signed measures, subject to the respective constraints, and hence to compute its (Lagrangian) dual. Assume now that the set $\mathcal{S} := \mu - \mathcal{C}$, where $\mu \in \mathbb{R}^q$ and $\mathcal{C} \subset \mathbb{R}^q$ is a closed convex *cone*. By $\mathcal{C}^*$ we denote polar (negative dual) of $\mathcal{C}$,

$$\mathcal{C}^* := \left\{ y \in \mathbb{R}^q : y^\mathsf{T} z \leq 0,\ \forall z \in \mathcal{C} \right\}.$$

Consider the Lagrangian of problem (2.110):

$$\begin{array}{rcl} L(P, \lambda_0, \lambda) & := & \int_\Xi \psi_0(\xi) dP(\xi) + \lambda_0 \left(1 - \int_\Xi dP(\xi)\right) + \lambda^\mathsf{T} \left(\int_\Xi \Psi(\xi) dP(\xi) - \mu\right) \\ & = & \int_\Xi \left(\psi_0(\xi) - \lambda_0 + \lambda^\mathsf{T} \Psi(\xi)\right) dP(\xi) + \lambda_0 - \lambda^\mathsf{T} \mu. \end{array}$$

We have that

$$\inf_{\lambda_0\in\mathbb{R},\,\lambda\in\mathcal{C}^*} L(P,\lambda_0,\lambda) = \begin{cases} \int_\Xi \psi_0(\xi)dP(\xi), & \text{if } \int_\Xi dP = 1,\ \int_\Xi \Psi(\xi)dP(\xi) \in \mu - \mathcal{C}, \\ -\infty, & \text{otherwise.} \end{cases}$$

Therefore problem (2.110) can be written as

$$\operatorname*{Max}_{P\succeq 0}\ \inf_{\lambda_0\in\mathbb{R},\,\lambda\in\mathcal{C}^*} L(P,\lambda_0,\lambda). \tag{2.113}$$

The corresponding Lagrangian dual is obtained by interchanging *max* and *min* operators in (2.113). Now

$$\sup_{P\succeq 0} L(P,\lambda_0,\lambda) = \begin{cases} \lambda_0 - \lambda^\mathsf{T}\mu, & \text{if } \psi_0(\xi) - \lambda_0 + \lambda^\mathsf{T}\Psi(\xi) \le 0,\ \xi\in\Xi, \\ +\infty, & \text{otherwise.} \end{cases}$$

This can be verified by considering atomic measures $P = \alpha\Delta(\xi)$, $\alpha \ge 0$, $\xi \in \Xi$. Therefore the (Lagrangian) dual of (2.110) is the problem

$$\begin{array}{ll} \operatorname*{Min}\limits_{\lambda_0\in\mathbb{R},\,\lambda\in\mathcal{C}^*} & \lambda_0 - \lambda^\mathsf{T}\mu \\ \text{s.t.} & \psi_0(\xi) - \lambda_0 + \lambda^\mathsf{T}\Psi(\xi) \le 0,\ \xi\in\Xi. \end{array} \tag{2.114}$$

In particular, if $\mathcal{C} = \{0\}$, i.e., $\mathcal{S} = \{\mu\}$ is a singleton, then $\mathcal{C}^* = \mathbb{R}^q$. Problem (2.114) involves infinite number of constraints (unless the set Ξ is finite) and such problems are called semi-infinite programming problems.

We have that the optimal value of the dual problem (2.114) is always greater than or equal to the optimal value of the primal problem (2.110). There are various regularity conditions ensuring that these optimal values are the same, i.e., there is no duality gap between problems (2.110) and (2.114). We can consider the minimax problem (2.113) in the framework of the dual problems (2.40) and (2.41). Note that $L(P,\lambda_0,\lambda)$ is linear and hence convex in (λ_0,λ). Therefore by Theorem 2.3 we have the following result.

Theorem 2.14 *Suppose that the set Ξ is compact, the set $\mathfrak{M}$ is nonempty, and the functions $\psi_i : \Xi \to \mathbb{R}$, $i = 0, 1, ..., q$, are continuous. Then the optimal values of problems* (2.110) *and* (2.114) *are equal to each other and problem* (2.110) *has an optimal solution.*

Also we have that there is no duality gap between problems (2.110) and (2.114) if the semi-infinite programming problem (2.114) has a nonempty and bounded set of optimal solutions (compare with Theorem 2.4).

Consider now the minimax problem (2.1) with the set $\mathfrak{M}$ defined in (2.107). Suppose, as above, that $\mathcal{S} := \mu - \mathcal{C}$. Then for a given $x \in \mathcal{X}$ the corresponding max-problem is a problem of moments with dual of the form (2.114). That is, we can write the following dual of problem (2.1)

$$\begin{array}{ll} \underset{x\in\mathcal{X},\, \lambda_0\in\mathbb{R},\, \lambda\in\mathcal{C}^*}{\text{Min}} & \lambda_0 - \lambda^\mathsf{T}\mu \\ \text{s.t.} & F(x,\xi) - \lambda_0 + \lambda^\mathsf{T}\Psi(\xi) \le 0,\ \xi \in \Xi. \end{array} \tag{2.115}$$

By Theorem 2.14 we have that if the set Ξ is compact, the set $\mathfrak{M}$ is nonempty, the functions $\psi_i : \Xi \to \mathbb{R}$, $i = 1, ..., q$, and $F(x, \cdot)$, $x \in \mathcal{X}$, are continuous, then there is no duality gap between problems (2.1) and (2.115).

Suppose, further, that we are in the setting of two-stage linear programming, i.e., $F(x,\xi) := c^\mathsf{T}x + Q(x,\xi)$, where $Q(x,\xi)$ is the optimal value of the second stage problem (2.4). The inequality constraints of problem (2.115) can be written as $v(x,\lambda) \le \lambda_0$, where

$$v(x,\lambda) := \sup_{\xi\in\Xi} \left\{ F(x,\xi) + \lambda^\mathsf{T}\Psi(\xi) \right\}.$$

Recall that $Q(x,\xi)$ is equal to the optimal value of the dual problem (2.7), provided $W^\mathsf{T}\pi \le q$ has a feasible solution, and hence

$$v(x,\lambda) := \sup_{\xi\in\Xi,\, \pi: W^\mathsf{T}\pi\le q} \left\{ c^\mathsf{T}x + \pi^\mathsf{T}(h - Tx) + \lambda^\mathsf{T}\Psi(\xi) \right\}.$$

Consequently, the dual problem (2.115) can be written as

$$\begin{array}{ll} \underset{x,\, \pi,\, \lambda_0,\, \lambda}{\text{Min}} & \lambda_0 - \lambda^\mathsf{T}\mu \\ \text{s.t.} & Ax = b,\ x \ge 0,\ \lambda \in \mathcal{C}^*, \\ & c^\mathsf{T}x + \pi^\mathsf{T}(h - Tx) + \lambda^\mathsf{T}\Psi(\xi) \le \lambda_0,\ \xi \in \Xi, \\ & W^\mathsf{T}\pi \le q,\ \xi \in \Xi, \end{array} \tag{2.116}$$

with $\xi = (q, T, W, h)$.

2.8 Ambiguous Chance Constraints

Consider a chance constraint of the form

$$P\{C(x,\omega) \le 0\} \ge 1 - \alpha. \tag{2.117}$$

Here P is a probability measure on a measurable space $(\Omega, \mathcal{F})$, $C : \mathbb{R}^n \times \Omega \to \mathbb{R}$ is a random function and $\alpha \in (0,1)$ is a small number representing the specified

level for probability of violating the constraint $C(x,\omega) \leq 0$. It is assumed in this formulation of chance constraint that the probability measure (distribution), with respect to which the corresponding probabilities are calculated, is known. Suppose now that the underlying probability distribution is not known exactly, but rather is assumed to belong to a specified family of probability distributions. Optimization problems involving such constrains are called *ambiguous chance constrained* problems.

For a specified uncertainty set $\mathfrak{M}$ of probability measures on $(\Omega, \mathcal{F})$, the corresponding ambiguous chance constraint defines a feasible set $\mathcal{X} \subset \mathbb{R}^n$ which can be written as

$$\mathcal{X} := \big\{x \in \mathbb{R}^n : P\{C(x,\omega) \leq 0\} \geq 1-\alpha,\ \forall P \in \mathfrak{M}\big\}. \tag{2.118}$$

We have that

$$P\{C(x,\omega) \leq 0\} = 1 - P\{C(x,\omega) > 0\} = 1 - \mathbb{E}_P[\mathbf{1}_{A_x}],$$

where $\mathbf{1}_{A_x}$ is the indicator function of the set

$$A_x := \{\omega \in \Omega : C(x,\omega) > 0\}.$$

Therefore we can write $\mathcal{X}$ as follows

$$\mathcal{X} = \left\{x \in \mathbb{R}^n : \sup_{P\in\mathfrak{M}} \mathbb{E}_P[\mathbf{1}_{A_x}] \leq \alpha\right\}. \tag{2.119}$$

Consider a coherent risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ with $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$, where P is a reference probability measure. Let $\mathfrak{Q}$ be the corresponding set of probability measures in the dual representation (2.55) of ρ. Recall that $\mathfrak{Q}$ is formed by probability measures which are absolutely continuous with respect to the reference measure P. For the uncertainty set $\mathfrak{M} = \mathfrak{Q}$ of probability measures we can write the feasible set $\mathcal{X}$ as

$$\mathcal{X} = \left\{x \in \mathbb{R}^n : \rho\left(\mathbf{1}_{A_x}\right) \leq \alpha\right\}. \tag{2.120}$$

Suppose, further, that the risk measure ρ is law invariant. Then for $A \in \mathcal{F}$ the quantity $\rho(\mathbf{1}_A)$ depends only on $P(A)$. Indeed, if $Z := \mathbf{1}_A$ for some $A \in \mathcal{F}$, then its cdf $H(z) := P(Z \leq z)$ is

$$H(z) = \begin{cases} 0, & \text{if } \ z < 0, \\ 1 - P(A), & \text{if } \ 0 \leq z < 1, \\ 1, & \text{if } \ 1 \leq z, \end{cases}$$

which clearly depends only on $P(A)$.

Definition 2.5 *Let $\mathcal{T} := \{P(A) : A \in \mathcal{F}\}$ and $\rho : \mathcal{Z} \to \mathbb{R}$ be a law invariant coherent risk measure. We associate with ρ function $\varphi_\rho : \mathcal{T} \to \mathbb{R}$ defined as $\varphi_\rho(t) := \rho\left(\mathbf{1}_A\right)$, where $A \in \mathcal{F}$ is any event such that $P(A) = t$.*

The function φ_ρ is well defined because for law invariant risk measure ρ the quantity $\rho(\mathbf{1}_A)$ depends only on the probability $P(A)$ and hence $\rho(\mathbf{1}_A)$ is the same for any $A \in \mathcal{F}$ such that $P(A) = t$ for a given $t \in \mathcal{T}$. Clearly $\mathcal{T}$ is a subset of the interval $[0,1]$, and $0 \in \mathcal{T}$ (since $\emptyset \in \mathcal{F}$) and $1 \in \mathcal{T}$ (since $\Omega \in \mathcal{F}$). If P is a nonatomic measure, then for any $A \in \mathcal{F}$ the set $\{P(B) : B \subset A,\ B \in \mathcal{F}\}$ coincides with the interval $[0, P(A)]$. In particular, if P is nonatomic, then $\mathcal{T} = [0,1]$. Unless stated otherwise we assume in the remainder of this section that the reference measure P is *nonatomic*.

Consider the Average Value-at-Risk measure $\rho(\cdot) := \mathsf{AV@R}_\gamma(\cdot)$, $\gamma \in (0,1]$. By direct calculations it is straightforward to verify that for any $A \in \mathcal{F}$,

$$\mathsf{AV@R}_\gamma(\mathbf{1}_A) = \begin{cases} \gamma^{-1}P(A), & \text{if} \quad P(A) \leq \gamma, \\ 1, & \text{if} \quad P(A) > \gamma. \end{cases}$$

Consequently the corresponding function φ_ρ can be written as

$$\varphi_\rho = \begin{cases} \gamma^{-1}t & \text{if} \quad t \in [0,\gamma], \\ 1 & \text{if} \quad t \in (\gamma, 1]. \end{cases} \tag{2.121}$$

For $\rho(\cdot) := \mathsf{AV@R}_0(\cdot)$, i.e., for $\rho := \operatorname{ess\,sup}(\cdot)$, we have that $\varphi_\rho(t) = 1$ for $t \in (0,1]$, and $\varphi_\rho(0) = 0$. That is, in that case the function $\varphi_\rho(\cdot)$ is discontinuous at 0.

Now let $\rho := \sum_{i=1}^m \lambda_i \rho_i$ be a convex combination of law invariant coherent risk measures ρ_i, $i = 1, ..., m$. For $A \in \mathcal{F}$ we have that $\rho(\mathbf{1}_A) = \sum_{i=1}^m \lambda_i \rho_i(\mathbf{1}_A)$ and hence $\varphi_\rho = \sum_{i=1}^m \lambda_i \varphi_{\rho_i}$. By taking $\rho_i := \mathsf{AV@R}_{\gamma_i}$, with $\gamma_i \in (0,1]$, $i = 1, ..., m$, and using (2.121), we obtain that $\varphi_\rho : [0,1] \to [0,1]$ is a piecewise linear nondecreasing concave function with $\varphi_\rho(0) = 0$ and $\varphi_\rho(1) = 1$. More generally, let μ be a probability measure on $[0,1]$ and $\rho := \int_0^1 \mathsf{AV@R}_\gamma d\mu(\gamma)$. In that case the corresponding function $\varphi_\rho : [0,1] \to \mathbb{R}$ becomes a nondecreasing concave function with $\varphi_\rho(0) = 0$ and $\varphi_\rho(1) = 1$ (it could be discontinuous at $t = 0$ if $\mathcal{Z} = \mathcal{L}_\infty(\Omega, \mathcal{F}, P)$). By employing Kusuoka Theorem this allows to give the following characterization of functions φ_ρ.

Proposition 2.2 *Let $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$, and $\rho : \mathcal{Z} \to \mathbb{R}$ be a law invariant coherent risk measure. Suppose that the reference probability measure P is nonatomic. Then $\varphi_\rho(\cdot)$ is a continuous nondecreasing function defined on the interval $[0,1]$ such that $\varphi_\rho(0) = 0$ and $\varphi_\rho(1) = 1$, and $\varphi_\rho(t) \geq t$ for all $t \in [0,1]$. Moreover, if the risk measure ρ is comonotonic, then the function $\varphi_\rho(\cdot)$ is concave. Conversely, if $\phi : [0,1] \to \mathbb{R}$ is a continuous concave function with $\phi(0) = 0$ and $\phi(1) = 1$, then there exists a law invariant coherent comonotonic risk measure such that $\phi = \varphi_\rho$.*

Proof. If the law invariant coherent risk measure ρ is comonotonic, then by Theorem 2.8 it can be represented in the form (2.68) for some probability measure μ on the

interval [0,1]. Consequently it follows that the corresponding function $\varphi_\rho(\cdot)$ is concave, nondecreasing on [0,1] with $\varphi_\rho(0) = 0$ and $\varphi_\rho(1) = 1$, and hence $\varphi_\rho(t) \geq t$ for all $t \in [0,1]$.

For ρ not necessarily comonotonic we have by Theorem 2.7 that $\rho(\cdot)$ can be written as a maximum of risk measures of the form $\int_0^1 \mathsf{AV@R}_\gamma d\mu(\gamma)$ for some collection of probability measures μ. It follows that ρ is concave and nondecreasing on [0,1] with $\varphi_\rho(0) = 0$ and $\varphi_\rho(1) = 1$.

As far as continuity of $\varphi_\rho(\cdot)$ is concerned we can argue as follows. Let $t_k \in [0,1]$ be a monotonically increasing sequence tending to t^*. Since P is a nonatomic, there exists a sequence $A_1 \subset A_2 \subset \ldots$, of $\mathcal{F}$-measurable sets such that $P(A_k) = t_k$ for all $k \in \mathbb{N}$. It follows that the set $A := \cup_{k=1}^\infty A_k$ is $\mathcal{F}$-measurable and $P(A) = t^*$. Since $\mathbf{1}_{A_k}$ converges (in the norm topology of $\mathcal{Z}$) to $\mathbf{1}_A$, it follows by continuity of ρ that $\rho(\mathbf{1}_{A_k})$ tends to $\rho(\mathbf{1}_A)$, and hence $\varphi_\rho(t_k)$ tends to $\varphi_\rho(t^*)$. In a similar way we have that $\varphi_\rho(t_k) \to \varphi_\rho(t^*)$ for a monotonically decreasing sequence t_k tending to t^*. This shows that φ_ρ is continuous.

Now let $\rho := \sum_{i=1}^m \lambda_i \mathsf{AV@R}_{\gamma_i}$ be a convex combination of Average Value-at-Risk measures. By using formula (2.121) for the corresponding functions φ_{ρ_i} it is not difficult to see that any continuous concave piecewise linear function $\phi : [0,1] \to \mathbb{R}$, with $\phi(0) = 0$ and $\phi(1) = 1$, can be represented as $\phi = \varphi_\rho$ for an appropriate choice of weights λ_i and points $\gamma_i \in (0,1]$. By arguments of passing to the limit we obtain that for any continuous concave function $\phi : [0,1] \to \mathbb{R}$, with $\phi(0) = 0$ and $\phi(1) = 1$, there exists a measure μ on the interval $(0,1]$ such that $\phi = \varphi_\rho$ for $\rho := \int_0^1 \mathsf{AV@R}_\gamma d\mu(\gamma)$. This completes the proof. ∎

Consider the set $\mathcal{X}$ of the form (2.120) with the reference measure P being nonatomic, $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in [1,\infty)$, and $\rho : \mathcal{Z} \to \mathbb{R}$ being a law invariant coherent risk measure. Then by Proposition 2.2, we have that this set $\mathcal{X}$ can be written in the following equivalent form

$$\mathcal{X} = \big\{x : P\{C(x,\omega) \leq 0\} \geq 1 - \alpha^*\big\}, \tag{2.122}$$

where $\alpha^* := \varphi_\rho^{-1}(\alpha)$. That is, $\mathcal{X}$ can be defined by a chance constraint with respect to the reference distribution P and with the respective probability level $1 - \alpha^*$. Since $\varphi_\rho(t) \geq t$, for any $t \in [0,1]$, it follows that $\alpha^* \leq \alpha$. We have the following.

- For a certain class of uncertainty sets $\mathfrak{M}$, the ambiguous chance constrains of the form (2.118) can be reformulated in the (usual) form (2.117) for an appropriate choice of the probability level $1 - \alpha^* \geq 1 - \alpha$.

Of course, the class of such uncertainty sets $\mathfrak{M}$ is somewhat specific, see Theorem 2.6 for a description of such uncertainty sets.

For instance, let $\mathcal{Z} := \mathcal{L}_1(\Omega, \mathcal{F}, P)$ and $\rho(Z) := (1-\beta)\mathbb{E}[Z] + \beta\mathsf{AV@R}_\gamma(Z)$, where $\beta, \gamma \in (0,1)$ and the expectations are taken with respect to the reference distribution P. Then

$$\varphi_\rho(t) = \begin{cases} (1-\beta+\gamma^{-1}\beta)t, & \text{if } t \in [0,\gamma], \\ \beta + (1-\beta)t, & \text{if } t \in (\gamma, 1]. \end{cases} \tag{2.123}$$

It follows that for this risk measure and for $\alpha \leq \beta + (1-\beta)\gamma$,

$$\alpha^* = \frac{\alpha}{1+\beta(\gamma^{-1}-1)}. \tag{2.124}$$

In particular, for $\beta = 1$, i.e., for $\rho = \mathsf{AV@R}_\gamma$, we have that $\alpha^* = \gamma\alpha$.

As another example consider the mean-upper-semideviation risk measure of order p. That is, $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$ and

$$\rho(Z) := \mathbb{E}[Z] + c\left(\mathbb{E}\left[\big[Z - \mathbb{E}[Z]\big]_+^p\right]\right)^{1/p}$$

. We have here that $\rho(\mathbf{1}_A) = P(A) + c[P(A)(1-P(A))^p]^{1/p}$, and hence

$$\varphi_\rho(t) = t + c\, t^{1/p}(1-t), \quad t \in [0,1]. \tag{2.125}$$

In particular, for $p = 1$ we have that $\varphi_\rho(t) = (1+c)t - ct^2$, and hence

$$\alpha^* = \frac{1+c-\sqrt{(1+c)^2 - 4\alpha c}}{2c}. \tag{2.126}$$

Note that for $c > 1$ the above function $\varphi_\rho(\cdot)$ is not monotonically nondecreasing on the interval $[0,1]$. This should be not surprising since for $c > 1$ and nonatomic P, the corresponding mean-upper-semideviation risk measure is not monotone.

2.9 Stochastic Programming with Equilibrium Constraints

In this section we discuss stochastic programming where the second stage problem is given in a form of equilibrium constraints. Consider the following two stage problem. Assume that, at the second stage, there are m players who are supposed to reach a Nash equilibrium. That is, with each player $i \in \{1, ..., m\}$ is associated a set $\mathcal{Y}_i \subset \mathbb{R}^{m_i}$ and a payoff function $f_i : \mathcal{Y} \to \mathbb{R}$, where $\mathcal{Y} := \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_m$. If each player i chooses respective strategy $y_i \in \mathcal{Y}_i$, resulting in the strategy profile $y = (y_1, ..., y_m) \in Y$ of all players, then a player i obtains payoff $f_i(y)$. The payoff of an individual player i depends on his strategy y_i as well as the strategy of the other players $y_{-i} := (y_1, ..., y_{i-1}, y_{i+1}, ..., y_m)$. A Nash equilibrium is reached if no player can

do better by unilaterally changing his strategy. Formally, $\bar{y} \in \mathcal{Y}$ is a Nash equilibrium strategy if

$$\bar{y}_i \in \arg\min_{y_i \in \mathcal{Y}_i} f_i(y_i, \bar{y}_{-i}), \quad i = 1, ..., m \tag{2.127}$$

(with some abuse of notation we write here (y_i, y_{-i}) for the corresponding vector $y \in \mathcal{Y}$).

Suppose now that there is a first stage player (referred to as the authority) that can control the payoff functions of the second stage players. That is, each payoff function depends on a vector $x \in \mathcal{X} \subset \mathbb{R}^n$ decided by the authority. The purpose of the authority is to minimize an overall cost, which is a function of x and of the Nash equilibrium strategy $\bar{y}$. Suppose, further, that the cost and payoff functions depend on a random data vector ξ, whose probability distribution is supported on a set $\Xi \subset \mathbb{R}^d$, and that a realization of random data is not known to the authority at the time its decision should be made. That is, given a first stage decision $x \in \mathcal{X}$, at the second stage after a realization of the random data ξ becomes known, the players reach a Nash equilibrium $\bar{y} = \bar{y}(x, \xi)$, i.e.,

$$\bar{y}_i \in \arg\min_{y_i \in \mathcal{Y}_i} f_i(x, y_i, \bar{y}_{-i}, \xi), \quad i = 1, ..., m. \tag{2.128}$$

Consequently, for a specified cost function $c : \mathcal{X} \times \mathcal{Y} \times \Xi \to \mathbb{R}$, the first stage cost is a function of x and ξ and is given by $c(x, \bar{y}(x, \xi), \xi)$. The goal of the authority can be formulated as minimizing the resulting cost $c(x, \bar{y}(x, \xi), \xi)$ on average. This leads to the following stochastic programming problem

$$\operatorname*{Min}_{x \in \mathcal{X}} \mathbb{E}[c(x, \bar{y}(x, \xi), \xi)], \tag{2.129}$$

where the expectation is taken with respect to the probability distribution of the random vector ξ.

Implicit in the above formulation (2.129) is the assumption that for every $x \in \mathcal{X}$ and almost every (a.e.) ξ, the corresponding Nash equilibrium $\bar{y} = \bar{y}(x, \xi)$ is attained and is unique. In general, let us denote by $\mathfrak{E}(x, \xi)$ the set of Nash equilibrium points $\bar{y}(x, \xi) \in \mathcal{Y}$. This set can be empty if no Nash equilibrium strategy exists, or may contain more than one point. Then we can consider the following two stage stochastic programming problem

$$\operatorname*{Min}_{x \in \mathcal{X}} \mathbb{E}[Q(x, \xi)], \tag{2.130}$$

where $Q(x, \xi)$ is the optimal value of the problem

$$\operatorname*{Min}_{y \in \mathfrak{E}(x, \xi)} c(x, y, \xi). \tag{2.131}$$

By the definition $Q(x,\xi) := +\infty$ if the set $\mathfrak{E}(x,\xi)$ is empty, i.e., the respective Nash equilibrium is not attained.

If $\mathfrak{E}(x,\xi)$ may have more than one point, then from the point of view of the authority the above formulation (2.130)-(2.131) is optimistic since it assumes that the corresponding Nash equilibrium will be attained at a favorable to the authority point giving the minimal cost value. The corresponding pessimistic approach will be to consider the worst case for the authority, that is, to replace the second stage problem (2.131) by the maximization problem

$$\operatorname*{Max}_{y \in \mathfrak{E}(x,\xi)} c(x,y,\xi). \tag{2.132}$$

Of course, if $\mathfrak{E}(x,\xi) = \{\bar{y}(x,\xi)\}$ is a singleton (i.e., the Nash equilibrium is attained and is unique) for all $x \in \mathcal{X}$ and a.e. ξ, then the optimistic formulation (2.130) and (2.131) is the same as the pessimistic formulation (2.130) and (2.132), and both formulations coincide with problem (2.129). Formally, the maximization problem (2.132) has value $-\infty$ if the set $\mathfrak{E}(x,\xi)$ is empty. Therefore, if for some $\bar{x} \in \mathcal{X}$, the respective Nash equilibrium is not attained with positive probability, then $\mathbb{E}[Q(\bar{x},\xi)] = -\infty$, and hence $\bar{x}$ becomes an optimal solution of the corresponding first stage problem. This is problematic, to say the least, and thus models with no equilibria should be avoided.

Suppose that the set $\Xi = \{\xi_1, ..., \xi_K\}$ is finite, i.e., there is a finite number of scenarios $\xi_1, ..., \xi_K$ with respective probabilities $p_1, ..., p_K$. Then the two stage problem (2.130)–(2.131) can be written in the following equivalent form as one large problem

$$\begin{array}{ll} \operatorname*{Min}\limits_{x, y^1, ..., y^K} & \sum\limits_{k=1}^{K} p_k c(x, y^k, \xi_k) \\ \text{subject to} & x \in \mathcal{X},\ y^k \in \mathfrak{E}(x, \xi_k),\ k = 1, ..., K, \end{array} \tag{2.133}$$

by making one copy y^k of the second stage vector for every scenario ξ_k. For general, not necessarily finitely supported, distribution of ξ we can consider $y(\xi)$ as a (measurable) function of ξ and hence to write the following equivalent of the two stage problem (2.130)–(2.131):

$$\begin{array}{ll} \operatorname*{Min}\limits_{x, y(\cdot)} & \mathbb{E}[c(x, y(\xi), \xi)] \\ \text{subject to} & x \in \mathcal{X},\ y(\xi) \in \mathfrak{E}(x,\xi),\ \text{a.e}\ \xi \in \Xi. \end{array} \tag{2.134}$$

In this formulation, the optimization is performed over a finite dimensional vector $x \in \mathbb{R}^n$ and over functions $y(\xi)$ in an appropriate functional space. If $\Xi = \{\xi_1, ..., \xi_K\}$ is finite, then every such function $y(\xi)$ can be associated with a vector $(y(\xi_1), ..., y(\xi_K))$, and hence the formulation (2.133) follows (compare with (2.12) and (2.13)).

Suppose now that the sets $\mathcal{Y}_i$ are convex and closed and the functions $f_i(x, y, \xi)$ are differentiable with respect to y_i, $i = 1, ..., m$. Then the necessary optimality conditions for (2.128), i.e., for $\bar{y}_i$ to be a minimizer of $f_i(x, y_i, \bar{y}_{-i}, \xi)$ over $y_i \in \mathcal{Y}_i$, can be written as

$$-\nabla_{y_i} f_i(x, \bar{y}, \xi) \in \mathcal{N}_{Y_i}(\bar{y}_i),\ i = 1, ..., m, \tag{2.135}$$

where $\mathcal{N}_{Y_i}(\bar{y}_i)$ denotes the normal cone to the set $\mathcal{Y}_i$ at $\bar{y}_i$. Moreover, the above conditions (2.135) are also sufficient if each function $f_i(y)$ is convex in y_i.

Conditions (2.135) can be written as the following *variational inequality*:

$$-F(x, \bar{y}, \xi) \in \mathcal{N}_Y(\bar{y}), \tag{2.136}$$

where

$$F(x, y, \xi) := (\nabla_{y_1} f_1(x, y, \xi), ..., \nabla_{y_m} f_m(x, y, \xi)).$$

Note that $\mathcal{N}_Y(y) = \mathcal{N}_{\mathcal{Y}_1}(y_1) \times \cdots \times \mathcal{N}_{\mathcal{Y}_m}(y_m)$ for $y = (y_1, ..., y_m) \in \mathcal{Y}$ and $\mathcal{N}_{\mathcal{Y}}(y) = \emptyset$ for $y \notin \mathcal{Y}$, and hence conditions (2.135) and (2.136) are equivalent. This motivates to consider two stage stochastic problem with the first stage of the form (2.130) and with the second stage value $Q(x, \xi)$ defined as the optimal value of the following problem:

$$\operatorname*{Min}_{y \in \mathcal{Y}} c(x, y, \xi) \ \text{ subject to } \ -F(x, y, \xi) \in \mathcal{N}_{\mathcal{Y}}(y). \tag{2.137}$$

The two stage stochastic problem (2.130) and (2.137) can be also written in the following equivalent form:

$$\begin{array}{ll} \operatorname*{Min}\limits_{x, y(\cdot)} & \mathbb{E}[c(x, y(\xi), \xi)] \\ \text{subject to} & x \in \mathcal{X},\ -F(x, y(\xi), \xi) \in \mathcal{N}_{\mathcal{Y}}(y(\xi)),\ \text{a.e } \xi \in \Xi. \end{array} \tag{2.138}$$

In particular, if the set $\Xi = \{\xi_1, ..., \xi_K\}$ is finite, then problem (2.138) takes on the form:

$$\begin{array}{ll} \operatorname*{Min}\limits_{x, y^1, ..., y^K} & \sum\limits_{k=1}^{K} p_k c(x, y^k, \xi_k) \\ \text{subject to} & x \in \mathcal{X},\ -F(x, y^k, \xi_k) \in \mathcal{N}_{\mathcal{Y}}(y^k),\ k = 1, ..., K. \end{array} \tag{2.139}$$

Such two stage stochastic problems are called *Stochastic Mathematical Programming with Equilibrium Constraints* (SMPEQ) problems (cf., [16]).

As it was discussed earlier, SMPEQ problems could be conceptually problematic unless the equilibrium solution set $\mathfrak{E}(x, \xi)$ is a singleton for all $x \in \mathcal{X}$ and a.e. ξ. Conditions ensuring existence and/or uniqueness of a solution of a variational inequality, such as (2.136), are well known. A simple sufficient condition for existence of a solution of a variational inequality

$$-F(y) \in \mathcal{N}_{\mathcal{Y}}(y) \tag{2.140}$$

is that the function (mapping) $F(y)$ is continuous and the set $\mathcal{Y}$ is convex and compact.

Indeed, let $\Pi_{\mathcal{Y}}(y) := \arg\min_{z\in\mathcal{Y}} \|y-z\|$ be the metric projection of y onto $\mathcal{Y}$. It is not difficult to show that $\bar{y}$ is a solution of (2.140) iff

$$\Pi_{\mathcal{Y}}(\bar{y} - F(\bar{y})) = \bar{y},$$

i.e., $\bar{y}$ is a fixed point of the mapping $\Phi(y) := \Pi_{\mathcal{Y}}(y - F(y))$. The metric projection $\Pi_{\mathcal{Y}}(\cdot)$ is continuous and since $F(\cdot)$ is continuous, we have that $\Phi(\cdot)$ is continuous. Since $\mathcal{Y}$ is convex and compact, it follows by Brouwer's Fixed Point Theorem that mapping $\Phi : \mathcal{Y} \to Y$ has at least one fixed point.

Suppose, further, that $F(y)$ is strictly monotone, i.e.,

$$(F(y) - F(y'))^{\mathsf{T}}(y - y') > 0, \ \ \forall y, y' \in \mathcal{Y}, \ y \neq y'. \tag{2.141}$$

Then the solution is unique (if it exists).

Indeed, let $\bar{y}$ and $\hat{y}$ be two solutions of variational inequality (2.140). Then $F(\bar{y})^{\mathsf{T}}(\hat{y} - \bar{y}) \geq 0$ and $F(\hat{y})^{\mathsf{T}}(\bar{y} - \hat{y}) \geq 0$. It follows that

$$(F(\bar{y}) - F(\hat{y}))^{\mathsf{T}}(\bar{y} - \hat{y}) \leq 0,$$

which contradicts (2.141), if $\bar{y} \neq \hat{y}$.

Note that if $F(y) = (\nabla_{y_1} f_1(y), ..., \nabla_{y_m} f_m(y))$, i.e., the mapping $F(y)$ corresponds to the Nash equilibrium condition (2.127), then it is strictly monotone if each function $f_i(y_i, y_{-i})$ is strictly convex in y_i.

3 Multistage Problems

3.1 Risk Neutral Formulation

In a generic form a T-stage stochastic programming problem can be written as

$$\begin{array}{ll} \underset{x_1, x_2(\cdot), \ldots, x_T(\cdot)}{\text{Min}} & \mathbb{E}\big[F_1(x_1) + F_2(x_2(\xi_{[2]}), \xi_2) + \ldots + F_T\left(x_T(\xi_{[T]}), \xi_T\right)\big] \\ \text{s.t.} & x_1 \in \mathcal{X}_1, \ x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}), \xi_t), \ t = 2, \ldots, T. \end{array} \tag{3.1}$$

Here $\xi_1, \xi_2, \ldots, \xi_T$ is a random data process, $x_t \in \mathbb{R}^{n_t}$, $t = 1, \ldots, T$, are decision variables, $F_t : \mathbb{R}^{n_t} \times \mathbb{R}^{d_t} \to \mathbb{R}$ are measurable functions and $\mathcal{X}_t : \mathbb{R}^{n_{t-1}} \times \mathbb{R}^{d_t} \rightrightarrows \mathbb{R}^{n_t}$,

$t = 2, \ldots, T$, are measurable closed valued multifunctions. The first stage data, i.e., the vector ξ_1, the function $F_1 : \mathbb{R}^{n_1} \to \mathbb{R}$, and the set $\mathcal{X}_1 \subset \mathbb{R}^{n_1}$ are deterministic. By $\xi_{[t]} := (\xi_1, ..., \xi_t)$ we denote history of the process up to time $t = 1, ..., T$. We use the same notation ξ_t for random vectors and their particular realizations, which of these two meanings will be used in a specific situation will be clear from the context. It is said that the process $\xi_1, ..., \xi_T$ is *stagewise independent* if random vector ξ_{t+1} is independent of $\xi_{[t]} = (\xi_1, ..., \xi_t)$, $t = 1, ..., T-1$.

Optimization in (3.1) is performed over feasible *policies* (also called *decision rules*). A policy is a sequence of (measurable) functions $x_t = x_t(\xi_{[t]})$, $t = 1, \ldots, T$. Each $x_t(\xi_{[t]})$ is a function of the data process $\xi_{[t]}$ up to time t, this ensures the *nonanticipative* property of a considered policy. A policy[13] $x_t(\cdot) : \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_t} \to \mathbb{R}^{n_t}$, $t = 1, \ldots, T$, is said to be *feasible* if it satisfies the feasibility constraints for almost every realization of the random data process, i.e.,

$$x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}), \xi_t), \ t = 2, \ldots, T, \ \text{w.p.1.} \tag{3.2}$$

Since optimization in (3.1) is performed over policies, which are elements of appropriate functional spaces, formulation (3.1) leads to an infinite dimensional optimization problem, unless the data process $\xi_1, \ldots, \xi_T$ has a finite number of realizations. This is a natural extension of the formulation (2.12) of the two-stage problem.

In formulation (3.1) the expectations are taken with respect to a specified probability distribution of the random process $\xi_1, ..., \xi_T$. The optimization is performed on average and does not take into account risk of a possible deviation from the average for a particular realization of the data process. Therefore we refer to formulation (3.1) as *risk neutral*.

The multistage problem is *linear* if the objective functions and the constraint functions are linear, that is

$$\begin{array}{l} F_t(x_t, \xi_t) := c_t^{\mathsf{T}} x_t, \ \ \mathcal{X}_1 := \{x_1 : A_1 x_1 = b_1, \ x_1 \geq 0\}, \\ \mathcal{X}_t(x_{t-1}, \xi_t) := \{x_t : B_t x_{t-1} + A_t x_t = b_t, \ x_t \geq 0\}, \ t = 2, \ldots, T. \end{array} \tag{3.3}$$

Here $\xi_t := (c_t, B_t, A_t, b_t) \in \mathbb{R}^{d_t}$, $t = 2, \ldots, T$, are data vectors, some/all elements of which can be random, and $\xi_1 := (c_1, A_1, b_1)$ is the first stage data which is assumed to be known (nonrandom).

Recall that if X and Y are two random variables, then[14] $\mathbb{E}[X] = \mathbb{E}\{\mathbb{E}[X|Y]\}$, i.e., average of averages is the total average. Therefore we can write the expectation in

[13] In order to distinguish between a function $x_t(\xi_{[t]})$ and a vector $x_t \in \mathbb{R}^{n_t}$ we often write $x_t(\cdot)$ to emphasize that this denotes a function.

[14] By $\mathbb{E}[\cdot|Y]$ or $\mathbb{E}_{|Y}[\cdot]$ we denote the conditional, with respect to Y, expectation operator.

(3.1) as[15]

$$\begin{aligned}
&\mathbb{E}\left[F_1(x_1) + F_2(x_2(\xi_{[2]}), \xi_2) + \ldots + F_{T-1}\left(x_{T-1}(\xi_{[T-1]}), \xi_{T-1}\right) + F_T\left(x_T(\xi_{[T]}), \xi_T\right)\right] \\
&= \mathbb{E}_{|\xi_1}\Big[\cdots \mathbb{E}_{|\xi_{[T-2]}}\big[\mathbb{E}_{|\xi_{[T-1]}}[F_1(x_1) + F_2(x_2(\xi_{[2]}), \xi_2) + \ldots \\
&\quad + F_{T-1}\left(x_{T-1}(\xi_{[T-1]}), \xi_{T-1}\right) + F_T\left(x_T(\xi_{[T]}), \xi_T\right)]\big]\Big] \\
&= F_1(x_1) + \mathbb{E}_{|\xi_1}\Big[F_2(x_2(\xi_{[2]}), \xi_2) + \ldots + \mathbb{E}_{|\xi_{[T-2]}}\big[F_{T-1}\left(x_{T-1}(\xi_{[T-1]}), \xi_{T-1}\right)\big] \\
&\quad + \mathbb{E}_{|\xi_{[T-1]}}\big[F_T\left(x_T(\xi_{[T]}), \xi_T\right)\big]\Big].
\end{aligned} \tag{3.4}$$

This decomposition property of the expectation operator, together with an interchangeability property of the *expectation* and *minimization* operators (see Theorem 2.1), leads to the following equivalent (nested) formulation of the multistage problem (3.1)

$$\operatorname*{Min}_{x_1 \in \mathcal{X}_1} F_1(x_1) + \mathbb{E}\left[\inf_{x_2 \in \mathcal{X}_2(x_1, \xi_2)} F_2(x_2, \xi_2) + \mathbb{E}\Big[\cdots + \mathbb{E}\left[\inf_{x_T \in \mathcal{X}_T(x_{T-1}, \xi_T)} F_T(x_T, \xi_T)\right]\Big]\right], \tag{3.5}$$

and is a basis for deriving the *dynamic programming* equations. That is, going backward in time the so-called *cost-to-go* (or *value*) functions are defined recursively for $t = T, ..., 2$, as follows

$$Q_t\left(x_{t-1}, \xi_{[t]}\right) = \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \left\{F_t(x_t, \xi_t) + \mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right)\right\}, \tag{3.6}$$

where

$$\mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) := \mathbb{E}\left\{Q_{t+1}\left(x_t, \xi_{[t+1]}\right) \middle| \xi_{[t]}\right\}, \tag{3.7}$$

with $\mathcal{Q}_{T+1}(\cdot, \cdot) \equiv 0$ by definition. At the first stage the following problem should be solved

$$\operatorname*{Min}_{x_1 \in \mathcal{X}_1} F_1(x_1) + \mathbb{E}\left[Q_2\left(x_1, \xi_2\right)\right]. \tag{3.8}$$

The optimal value of the first stage problem (3.8) gives the optimal value of the corresponding multistage problem formulated in the form (3.5), or equivalently in the form (3.1).

A policy $\bar{x}_t(\xi_{[t]})$, $t = 1, \ldots, T$, is *optimal* if $\bar{x}_1$ is an optimal solution of the first stage problem (3.8) and for $t = 2, \ldots, T$,

$$\bar{x}_t(\xi_{[t]}) \in \arg\min_{x_t \in \mathcal{X}_t\left(\bar{x}_{t-1}(\xi_{[t-1]}), \xi_t\right)} \left\{F_t(x_t, \xi_t) + \mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right)\right\}, \quad \text{w.p.1.} \tag{3.9}$$

[15] Of course, since ξ_1 is deterministic, $\mathbb{E}_{|\xi_1}[\,\cdot\,] = \mathbb{E}[\,\cdot\,]$. We write it here for the uniformity of notation.

In the dynamic programming formulation the problem is reduced to solving a sequence of finite dimensional problems, indexed by t and depending on $\xi_{[t]}$.

At stage $t = T$ we have

$$Q_T\left(x_{T-1}, \xi_T\right) = \inf_{x_T \in \mathcal{X}_T(x_{T-1}, \xi_T)} F_T(x_T, \xi_T) \tag{3.10}$$

and

$$\mathcal{Q}_T(x_{T-1}, \xi_{[T-1]}) = \mathbb{E}\left[Q_T\left(x_{T-1}, \xi_T\right) \middle| \xi_{[T-1]}\right]. \tag{3.11}$$

Suppose now that the data process is *stagewise independent*. Then ξ_T is independent of $\xi_{[T-1]}$, and hence $\mathcal{Q}_T(x_{T-1}) = \mathbb{E}\left[Q_T\left(x_{T-1}, \xi_T\right)\right]$ does not depend on $\xi_{[T-1]}$. Consequently for $t = T - 1$,

$$Q_{T-1}\left(x_{T-2}, \xi_{T-1}\right) = \inf_{x_{T-1} \in \mathcal{X}_{T-1}(x_{T-2}, \xi_{T-1})} \left\{F_{T-1}(x_{T-1}, \xi_{T-1}) + \mathcal{Q}_T(x_{T-1})\right\}, \tag{3.12}$$

and hence by the stagewise independence it follows that $\mathcal{Q}_{T-1}(x_{T-2})$ is independent of $\xi_{[T-2]}$. And so on, by induction in t, we obtain the following result.

Proposition 3.1 *Suppose that the data process is stagewise independent. Then the (expected value) cost-to-go functions $\mathcal{Q}_t(x_t)$, $t = 2, ..., T$, do not depend on the data process.*

3.1.1 Multistage Linear Programs

Consider the linear case with the corresponding data of the form (3.3). The nested formulation (3.5) of the linear multistage problem can be written as

$$\underset{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}}{\operatorname{Min}} c_1^{\mathsf{T}} x_1 + \mathbb{E}\left[\min_{\substack{B_2 x_1 + A_2 x_2 = b_2 \\ x_2 \geq 0}} c_2^{\mathsf{T}} x_2 + \mathbb{E}\Big[\cdots + \mathbb{E}\left[\min_{\substack{B_T x_{T-1} + A_T x_T = b_T \\ x_T \geq 0}} c_T^{\mathsf{T}} x_T\right]\Big]\right]. \tag{3.13}$$

The dynamic programming equations here take the form

$$Q_t\left(x_{t-1}, \xi_{[t]}\right) = \inf_{x_t}\left\{c_t^{\mathsf{T}} x_t + \mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) : B_t x_{t-1} + A_t x_t = b_t,\ x_t \geq 0\right\}, \tag{3.14}$$

where

$$\mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) := \mathbb{E}\left\{Q_{t+1}\left(x_t, \xi_{[t+1]}\right) \middle| \xi_{[t]}\right\}. \tag{3.15}$$

Proposition 3.2 *In the linear case the cost-to-go function $Q_t\left(x_{t-1}, \xi_{[t]}\right)$, $t = 2, ..., T$, is convex in x_{t-1}.*

Proof. For $t = T$ we have that $Q_T(x_{T-1}, \xi_T)$ is given by the optimal value of the following linear programming problem

$$\underset{x_T \geq 0}{\text{Min}}\, c_T^{\mathsf{T}} x_T \;\text{ s.t. } B_T x_{T-1} + A_T x_T = b_T. \tag{3.16}$$

It is straightforward to verify that this optimal value is convex in x_{T-1}. By similar arguments and induction in $t = T-1, ...,$ the proof can be completed. ∎

The dual of the linear problem (3.16) is the problem

$$\underset{\pi_T}{\text{Max}}\, \pi_T^{\mathsf{T}}(b_T - B_T x_{T-1}) \;\text{ s.t. } A_T^{\mathsf{T}} \pi_T \leq c_T. \tag{3.17}$$

Optimal values of problems (3.16) and (3.17) are equal to each other unless both problems are infeasible. Assuming that $Q_T(x_{T-1}, \xi_T)$ is finite, we can write the subdifferential of $Q_T(\cdot, \xi_T)$ at the point x_{T-1} as

$$\partial Q_T(x_{T-1}, \xi_T) = -B_T^{\mathsf{T}} \mathcal{S}_T(x_{T-1}, \xi_T), \tag{3.18}$$

where $\mathcal{S}_T(x_{T-1}, \xi_T)$ denotes the set of optimal solutions of the dual problem (3.17). Note that since problem (3.17) is linear, its optimal set $\mathcal{S}_T(x_{T-1}, \xi_T)$ is nonempty provided its optimal value is finite.

By convexity of $Q_T(\cdot, \xi_T)$ we have that the corresponding expected value function $\mathcal{Q}_T(\cdot, \xi_{[T-1]})$, defined in (3.11), is also convex. If $\mathcal{Q}_T(\cdot, \xi_{[T-1]})$ is finite valued in a neighborhood of the point x_{T-1}, then its subdifferential can be taken inside the expectation, that is

$$\partial \mathcal{Q}_T\left(x_{T-1}, \xi_{[T-1]}\right) = \mathbb{E}_{|\xi_{[T-1]}}\left[\partial Q_T(x_{T-1}, \xi_T)\right]. \tag{3.19}$$

It follows that $\mathcal{Q}_T(\cdot, \xi_{[T-1]})$ is differentiable at x_{T-1} iff the set $\mathcal{S}_T(x_{T-1}, \xi_T)$ is a singleton, i.e., the problem (3.17) has unique optimal solution w.p.1 with respect to the conditional distribution of ξ_T given $\xi_{[T-1]}$.

For $t = T-1, ...,$ the (Lagrangian) dual of the optimization problem in the right hand side of (3.14), which defines the cost-to-go function $Q_t\left(x_{t-1}, \xi_{[t]}\right)$, is

$$\underset{\pi_t}{\text{Max}} \left\{ \pi_t^{\mathsf{T}}(b_t - B_t x_{t-1}) + \inf_{x_t \geq 0} \left[\left(c_t^{\mathsf{T}} - \pi_t^{\mathsf{T}} A_t\right) x_t + \mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) \right] \right\}. \tag{3.20}$$

In order to ensure that $Q_t\left(x_{t-1}, \xi_{[t]}\right)$ is equal to the optimal value of the problem (3.20), i.e., that there is no duality gap between problem (3.14) and its dual (3.20), there is a need for constraint qualification. For example, the no duality gap property

holds if the set of optimal solutions of the dual problem (3.20), denoted $\mathcal{S}_t\left(x_{t-1}, \xi_{[t]}\right)$, is nonempty and bounded. In that case $Q_t\left(\cdot, \xi_{[t]}\right)$ is continuous at x_{t-1} and the subdifferential

$$\partial Q_t\left(x_{t-1}, \xi_{[t]}\right) = -B_t^{\mathsf{T}} \mathcal{S}_t\left(x_{t-1}, \xi_{[t]}\right). \tag{3.21}$$

Suppose now that the data process is stagewise independent. Then the (expected value) cost-to-go functions $\mathcal{Q}_{t+1}(x_t)$ do not depend on the data (see Proposition 3.1). Suppose, further, that the multistage problem (3.13) has a finite number of scenarios. Then functions $\mathcal{Q}_{t+1}(\cdot)$ are convex piecewise linear, i.e., can be written as maximum of a finite family of affine functions $\alpha_{it} + \beta_{it}^{\mathsf{T}} x_t$, $i \in \mathcal{I}$, with $\mathcal{I}$ being a finite index set. That is,

$$\mathcal{Q}_{t+1}(x_t) = \max_{i \in \mathcal{I}} \left\{\alpha_{it} + \beta_{it}^{\mathsf{T}} x_t\right\}. \tag{3.22}$$

Consequently, the cost-to-go function $Q_t(x_{t-1}, \xi_t)$ is given by the optimal value of the linear program

$$\begin{array}{ll} \underset{x_t \in \mathbb{R}^{n_t},\, z \in \mathbb{R}}{\text{Min}} & c_t^{\mathsf{T}} x_t + z \\ \text{s.t.} & B_t x_{t-1} + A_t x_t = b_t,\ x_t \geq 0, \\ & \alpha_{it} + \beta_{it}^{\mathsf{T}} x_t \leq z,\ i \in \mathcal{I}. \end{array} \tag{3.23}$$

In that case there is no duality gap between problem (3.23) and its dual, and formula (3.21) holds.

3.2 Lagrange Multipliers of Nonanticipativity Constraints

Consider the multistage stochastic problem (3.1). The optimization there is performed over implementable policies satisfying the nonanticipativity condition. That is, at stage t the decision $x_t(\cdot) = x_t(\xi_{[t]})$ is a function of the history of the data process available at time t and does not depend on future observations[16] $\xi_{[t+1,T]} = (\xi_{t+1}, ..., \xi_T)$. We can reformulate this problem by allowing $x_t(\cdot) = x_t(\xi_{[T]})$ to depend on whole data vector $\xi_{[T]} = (\xi_1, ..., \xi_T)$ and then writing the requirement of nonanticipativity in the following form of constraints

$$x_t(\xi_{[T]}) = \mathbb{E}\big[x_t(\xi_{[T]})|\xi_{[t]}\big], \quad t = 1, ..., T-1, \tag{3.24}$$

which should hold w.p.1, i.e., for a.e. $\xi_{[T]}$. The above constraints (3.24) ensure that $x_t(\cdot)$ does not depend on $\xi_{[t+1,T]}$ and is a function $\xi_{[t]}$ alone.

With problem (3.1) and constraints (3.24) is associated the following Lagrangian

$$L(x(\cdot), \lambda(\cdot)) := \mathbb{E}\left\{\sum_{t=1}^T F_t(x_t(\xi_{[T]}), \xi_t) + \sum_{t=1}^{T-1} \lambda_t(\xi_{[T]})^{\mathsf{T}}\big(x_t(\xi_{[T]}) - \mathbb{E}[x_t(\xi_{[T]})|\xi_{[t]}]\big)\right\},$$

[16] We denote by $\xi_{[s,t]} := (\xi_s, ..., \xi_t)$ history of the process from time s to time $t \geq s$. In particular, $\xi_{[1,t]} = \xi_{[t]}$ and $\xi_{[t,t]} = \xi_t$.

where $x(\cdot) = (x_1(\cdot), ..., x_T(\cdot))$ and $\lambda(\cdot) = (\lambda_1(\cdot), ..., \lambda_{T-1}(\cdot))$. Note that the Lagrange multipliers $\lambda_1(\cdot), ..., \lambda_{T-1}(\cdot)$, as well as the decision variables $x_1(\cdot), ..., x_T(\cdot)$, are viewed here as elements of an appropriate functional space. Note also that

$$\mathbb{E}_{|\xi_{[t]}}\big\{x_t(\xi_{[T]}) - \mathbb{E}[x_t(\xi_{[T]})|\xi_{[t]}]\big\} = 0,$$

and hence replacing $\lambda_t(\xi_{[T]})$ with $\lambda_t(\xi_{[T]}) - \mathbb{E}_{|\xi_{[t]}}[\lambda_t(\xi_{[T]})]$ does not change the above Lagrangian.

Therefore by rewriting the Lagrangian as

$$L(x(\cdot), \lambda(\cdot)) := \mathbb{E}\left\{\sum_{t=1}^{T} F_t(x_t(\xi_{[T]}), \xi_t) + \sum_{t=1}^{T-1} \lambda_t(\xi_{[T]})^{\mathsf{T}} x_t(\xi_{[T]})\right\}, \tag{3.25}$$

we can write problem (3.1) in the following minimax form

$$\operatorname*{Min}_{x(\cdot)\in\mathfrak{X}} \sup_{\lambda(\cdot)\in\Lambda} L(x(\cdot), \lambda(\cdot)), \tag{3.26}$$

where

$$\Lambda := \{\lambda(\cdot) : \mathbb{E}[\lambda_t(\xi_{[T]})|\xi_{[t]}] = 0 \text{ w.p.1, } t = 1, ..., T-1\}, \tag{3.27}$$

$$\mathfrak{X} := \big\{x(\cdot) : x_t(\xi_{[T]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[T]}), \xi_t) \text{ w.p.1, } t = 1, ..., T\big\}. \tag{3.28}$$

This leads to the following dual of the problem (3.1):

$$\operatorname*{Max}_{\lambda(\cdot)\in\Lambda} \inf_{x(\cdot)\in\mathfrak{X}} L(x(\cdot), \lambda(\cdot)). \tag{3.29}$$

For the linear multistage problem (3.13) with a finite number of scenarios, both the primal and dual problems are linear programming problems. Consequently, if moreover the primal problem has a finite optimal value, then there is no duality gap between problem (3.13) and its dual (3.29) and both problems have optimal solutions.

3.2.1 The Two Stage Case

Consider the two stage case, i.e., $T = 2$. Then we can write the dual problem (3.29) as

$$\operatorname*{Max}_{\lambda(\cdot):\mathbb{E}[\lambda]=0} \inf_{x_1(\cdot)\in\mathcal{X}_1} \mathbb{E}\big[F_1(x_1(\xi)) + Q(x_1(\xi), \xi) + \lambda(\xi)^{\mathsf{T}} x_1(\xi)\big], \tag{3.30}$$

where $Q(x_1, \xi)$ is the optimal value of the second stage problem. We have that $(\bar{x}_1(\cdot), \bar{\lambda}(\cdot))$ is a saddle point of the problem (3.30) iff $\bar{x}_1(\cdot) \equiv \bar{x}_1$ is an optimal solution of the (first stage) of the primal problem, $\bar{\lambda}(\cdot)$ is an optimal solution of the dual problem and there is no duality gap between these problems. By interchanging the

minimization and expectation operators (see Theorem 2.1) we have that, for a given $\lambda(\cdot)$, an optimal solution $\bar{x}(\cdot)$ of the minimization problem in (3.30) is characterized by

$$\bar{x}_1(\xi) \in \arg\min_{x_1 \in \mathcal{X}_1} \left\{ F_1(x_1) + Q(x_1, \xi) + \lambda(\xi)^{\mathsf{T}} x_1 \right\}. \tag{3.31}$$

Suppose that the problem is convex, i.e., the set $\mathcal{X}_1$ is convex and the functions $F_1(x_1)$ and $Q(x_1, \xi)$ are convex in $x_1 \in \mathbb{R}^{n_1}$. Then $\bar{x}_1 = \bar{x}_1(\xi)$ is a minimizer of $F_1(x_1) + Q(x_1, \xi) + \lambda(\xi) x_1$ over $x_1 \in \mathcal{X}_1$ iff $\bar{x}_1 \in \mathcal{X}_1$ and

$$0 \in \partial F_1(\bar{x}_1) + \partial Q(\bar{x}_1, \xi) + \lambda(\xi) + \mathcal{N}_{\mathcal{X}_1}(\bar{x}_1), \tag{3.32}$$

provided $Q(\cdot, \xi)$ is finite valued in a neighborhood of $\bar{x}_1$. The solution $\bar{x}_1(\xi) \equiv \bar{x}_1$ is constant (does not depend on ξ) if

$$-\lambda(\xi) \in \partial F_1(\bar{x}_1) + \partial Q(\bar{x}_1, \xi) + \mathcal{N}_{\mathcal{X}_1}(\bar{x}_1). \tag{3.33}$$

By taking expectation of both sides of (3.33) and interchanging the expectation and subdifferentiation operators we obtain that (3.33) implies that

$$-\mathbb{E}[\lambda] \in \partial F_1(\bar{x}_1) + \partial \mathcal{Q}(\bar{x}_1) + \mathcal{N}_{\mathcal{X}_1}(\bar{x}_1), \tag{3.34}$$

where $\mathcal{Q}(x_1) := \mathbb{E}[Q(x_1, \xi)]$. We also have that the condition

$$0 \in \partial F_1(\bar{x}_1) + \partial \mathcal{Q}(\bar{x}_1) + \mathcal{N}_{\mathcal{X}_1}(\bar{x}_1) \tag{3.35}$$

is necessary and sufficient for $\bar{x}_1 \in \mathcal{X}_1$ to be an optimal solution of the first stage problem. It follows that if $\bar{x}_1 \in \mathcal{X}_1$ is an optimal solution of the first stage problem, then we can choose a measurable selection $\bar{\lambda}(\xi)$ satisfying (3.33) such that $\mathbb{E}[\bar{\lambda}] = 0$. It follows that $(\bar{x}_1, \bar{\lambda}(\cdot))$ is a saddle point of the problem (3.30) (see [30, section 2.4.3] for details).

- Let $\bar{x}_1$ be an optimal solution of the first stage problem. Suppose that the problem is convex and $\mathcal{Q}(x_1) := \mathbb{E}[Q(x_1, \xi)]$ is finite valued in a neighborhood of $\bar{x}_1$. Then there is no duality gap between the primal and dual problems and a measurable function $\bar{\lambda}(\xi)$ is the corresponding Lagrange multiplier if the following condition holds

$$-\bar{\lambda}(\xi) \in \partial F_1(\bar{x}_1) + \partial Q(\bar{x}_1, \xi) + \mathcal{N}_{\mathcal{X}_1}(\bar{x}_1) \ \text{ and } \ \mathbb{E}[\bar{\lambda}] = 0. \tag{3.36}$$

Suppose, further, that $F_1(\cdot)$ is differentiable at $\bar{x}_1$ and $Q(\cdot, \xi)$ is differentiable at $\bar{x}_1$ w.p.1. Then condition (3.36) becomes

$$-\bar{\lambda}(\xi) \in \nabla F_1(\bar{x}_1) + \nabla Q(\bar{x}_1, \xi) + \mathcal{N}_{\mathcal{X}_1}(\bar{x}_1) \ \text{ and } \ \mathbb{E}[\bar{\lambda}] = 0. \tag{3.37}$$

Even so the Lagrange multiplier may be not defined uniquely if $\bar{x}_1$ is a boundary point of the set $\mathcal{X}_1$ and hence the normal cone $\mathcal{N}_{\mathcal{X}_1}(\bar{x}_1)$ is bigger than $\{0\}$. By (3.35) we have that

$$0 \in \nabla F_1(\bar{x}_1) + \nabla \mathcal{Q}(\bar{x}_1) + \mathcal{N}_{\mathcal{X}_1}(\bar{x}_1). \tag{3.38}$$

Therefore the Lagrange multiplier can be written in the following form

$$\bar{\lambda}(\xi) = \nabla \mathcal{Q}(\bar{x}_1) - \nabla Q(\bar{x}_1, \xi) - \mu(\xi), \tag{3.39}$$

where $\mu(\xi)$ can be any measurable function such that

$$\mu(\xi) \in \nabla F_1(\bar{x}_1) + \nabla \mathcal{Q}(\bar{x}_1) + \mathcal{N}_{\mathcal{X}_1}(\bar{x}_1) \text{ and } \mathbb{E}[\mu] = 0. \tag{3.40}$$

In particular, because of (3.38), we can take $\mu(\cdot) \equiv 0$.

3.3 Conditional Risk Measures

With every law invariant risk measure ρ is associated its conditional analogue. That is, let Z be a random variable and Y be a random vector. Since ρ is law invariant, $\rho(Z)$ is a function of the distribution of $Z \in \mathcal{Z}$. Consider the conditional distribution of Z given $Y = y$, and value of $\rho(\cdot)$, denoted $\rho(Z|Y = y)$, at this conditional distribution. Note that $\rho(Z|Y = y) = \phi(y)$ is a function of y, and hence $\phi(Y)$ is a random variable. We denote this random variable $\phi(Y)$ as $\rho(Z|Y)$ or $\rho_{|Y}(Z)$ and refer to $\rho_{|Y}(\cdot)$ as *conditional risk measure*. Of course, if Z and Y are independent, then the distribution of Z does not depend on Y and hence in that case $\rho_{|Y}(Z) = \rho(Z)$.

To be more precise let us consider the following construction. Let $(X, Y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ be a random vector having probability distribution P supported on (closed) set $\Xi \subset \mathbb{R}^d$, where $d = d_1 + d_2$, equipped with its Borel sigma algebra $\mathcal{B}$. Consider the probability space $(\Xi, \mathcal{B}, P)$, the space $\mathcal{Z} := \mathcal{L}_p(\Xi, \mathcal{B}, P)$ of measurable functions $Z : \Xi \to \mathbb{R}$ having finite p-th order moment, and a law invariant coherent risk measure $\rho : \mathcal{Z} \to \mathbb{R}$. We can view $Z = Z(X, Y)$ as a function of random vector (X, Y), or as a random variable defined on the probability space $(\Xi, \mathcal{B}, P)$.

We can write the dual representation of the conditional risk measure $\rho_{|Y}$ as follows. Given $Y = y$ consider function $Z_y(\cdot) := Z(\cdot, y)$. We can view Z_y as a random variable whose (conditional) distribution is supported on the set $\Xi_y := \{x : (x, y) \in \Xi\}$. For the considered law invariant coherent risk measure ρ we have the corresponding dual representation (2.53) of $\rho(Z_y)$ with the associated dual set $\mathfrak{A} = \mathfrak{A}_y$ of density functions $\zeta : \Xi_y \to \mathbb{R}$. That is

$$\rho_{|Y}(Z) = \sup_{\zeta \in \mathfrak{A}_Y} \mathbb{E}_{|Y}[Z(X, Y)\zeta(X)]. \tag{3.41}$$

For example the conditional analogue of the mean-upper semideviation risk measure (2.57) is

$$\rho_{|Y}(Z) = \mathbb{E}_{|Y}[Z] + \lambda \left(\mathbb{E}_{|Y} \left[\left[Z - \mathbb{E}_{|Y}[Z] \right]_+^p \right] \right)^{1/p}, \quad Z \in \mathcal{L}_p(\Omega, \mathcal{F}, P). \tag{3.42}$$

The conditional analogue of $\mathsf{V@R}_\alpha(Z)$ is the left side $(1-\alpha)$-quantile of the conditional distribution of Z given Y, denoted $\mathsf{V@R}_\alpha(Z|Y)$ or $\mathsf{V@R}_{\alpha|Y}(Z)$. Recall that risk measure $\mathsf{V@R}_\alpha(\cdot)$ is not coherent, it does not posses the subadditivity property (2.51). Nevertheless the conditional analogue $\mathsf{V@R}_{\alpha|Y}(\cdot)$ is well defined.

For $\alpha \in (0, 1]$ the conditional analogue of $\mathsf{AV@R}_\alpha(\cdot)$ is

$$\mathsf{AV@R}_{\alpha|Y}(Z) = \inf_{z \in \mathbb{R}} \left\{ z + \alpha^{-1} \mathbb{E}_{|Y}[Z - z]_+ \right\}, \quad Z \in \mathcal{L}_1(\Omega, \mathcal{F}, P). \tag{3.43}$$

The set of minimizers of the right hand side of (3.43) is given by $(1-\alpha)$-quantiles of the conditional distribution of Z, given Y, and is a function of Y. In particular, $z^* = \mathsf{V@R}_{\alpha|Y}(Z)$ is such a minimizer and hence

$$\mathsf{AV@R}_{\alpha|Y}(Z) = \mathsf{V@R}_{\alpha|Y}(Z) + \alpha^{-1} \mathbb{E}_{|Y} \big[Z - \mathsf{V@R}_{\alpha|Y}(Z) \big]_+. \tag{3.44}$$

By (3.41) we also have the following dual representation

$$\mathsf{AV@R}_{\alpha|Y}(Z) = \sup \big\{ \mathbb{E}_{|Y}[Z(X,Y)\zeta(X)] : 0 \preceq \zeta \preceq \alpha^{-1},\ \mathbb{E}[\zeta] = 1 \big\}. \tag{3.45}$$

There is an alternative, and in a sense equivalent, approach to defining conditional risk measures which is based on an axiomatic method (cf., [20],[24]). Both approaches have advantages and disadvantages. Some properties could be easier seen in one approach than the other. We use here the above approach of conditional distributions since it is more intuitive and seems to better suit the dynamic optimization setting. Note that $\rho_{|Y}$ inherits basic properties of the coherent risk measure ρ - monotonicity, convexity and positive homogeneity. As far as the translation equivariance is concerned, we have that

$$\rho_{|Y}(Z + h(Y)) = \rho_{|Y}(Z) + h(Y) \tag{3.46}$$

for any measurable function $h(y)$.

Since $\rho(Z|Y)$ is a random variable, we can condition it on another random vector W. That is, we can consider the following conditional risk measure $\rho[\rho(Z|Y)|W]$. We refer to this (conditional) risk measure as the composite risk measure and write it as $\rho_{|W} \circ \rho_{|Y}(Z)$. In particular, we can consider the composition $\rho \circ \rho_{|Y}$. The composite risk measure $\rho \circ \rho_{|Y}$ inherits basic properties of ρ. If ρ is a law invariant coherent risk measure, then so is the composite risk measure $\rho \circ \rho_{|Y}$.

The composite risk measures $\rho \circ \rho_{|Y}$ can be quite complicated and difficult to write explicitly (cf., [24, section 5]). In general, the composite risk measure $\rho \circ \rho_{|Y}(\cdot)$ depends on Y, and the equality

$$\rho \circ \rho_{|Y} = \rho \tag{3.47}$$

does **not** necessarily hold. For example, for nonconstant Y equation (3.47) does not hold for $\rho := \mathsf{AV@R}_\alpha$ with $\alpha \in (0,1)$. Of course, if Z and Y are independent, then $\rho(Z|Y) = \rho(Z)$ and hence $\rho \circ \rho_{|Y}(Z) = \rho(Z)$. In particular, (3.47) holds if Y is constant and hence Z is independent of Y for any $Z \in \mathcal{Z}$. Equation (3.47) holds for any Y in at least in two cases, namely for $\rho(\cdot) := \mathbb{E}(\cdot)$ and $\rho(\cdot) := \operatorname{ess\,sup}(\cdot)$.

3.4 Minimax and Risk Averse Multistage Programming

Consider the following minimax extension of the risk neutral formulation (3.1) of multistage stochastic programs:

$$\begin{array}{cl} \underset{x_1, x_2(\cdot), \ldots, x_T(\cdot)}{\operatorname{Min}} & \sup\limits_{P \in \mathfrak{M}} \left\{ \mathbb{E}_P \big[F_1(x_1) + F_2(x_2(\xi_{[2]}), \xi_2) + \ldots + F_T\left(x_T(\xi_{[T]}), \xi_T\right) \big] \right\} \\ \text{s.t.} & x_1 \in \mathcal{X}_1,\ x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}), \xi_t),\ t = 2, \ldots, T. \end{array} \tag{3.48}$$

Here $\mathfrak{M}$ is a set of probability measures associated with vector $(\xi_2, ..., \xi_T) \in \mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_T}$. We assume that probability measures of the set $\mathfrak{M}$ are supported on a closed set $\Xi \subset \mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_T}$, i.e., for every $P \in \mathfrak{M}$ it holds that P-almost surely $(\xi_2, ..., \xi_T) \in \Xi$. Formulation (3.48) can be viewed as an extension of the two-stage minimax problem (2.1) to the multistage setting. As in the risk neutral case the minimization in (3.48) is performed over feasible policies.

We can also view the above minimax formulation from the point of view of risk measures. Let $\bar{P}$ be a (reference) probability measure[17] on the set Ξ equipped with its Borel sigma algebra $\mathcal{B}$ and let $\mathcal{Z} := \mathcal{L}_p(\Xi, \mathcal{B}, \bar{P})$. That is, for $p \in [1, \infty)$ the space $\mathcal{Z}$ consists of measurable functions $Z : \Xi \to \mathbb{R}$ viewed as random variables having finite p-th order moment (with respect to the reference probability measure $\bar{P}$), and for $p = \infty$ this is the space of essentially bounded measurable functions $Z(\xi_{[T]})$. Consider a coherent risk measure $\rho : \mathcal{Z} \to \mathbb{R}$. The corresponding *risk averse*

[17] Unless stated otherwise all expectations and probabilistic statements will be made with respect to the reference measure $\bar{P}$.

multistage problem can be written as

$$\begin{array}{ll} \underset{x_1, x_2(\cdot), \ldots, x_T(\cdot)}{\text{Min}} & \rho\Big[\overbrace{F_1(x_1) + F_2(x_2(\xi_{[2]}), \xi_2) + \ldots + F_T\left(x_T(\xi_{[T]}), \xi_T\right)}^{Z(\xi_{[T]})}\Big] \\ \text{s.t.} & x_1 \in \mathcal{X}_1,\ x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}), \xi_t),\ t = 2, \ldots, T. \end{array} \tag{3.49}$$

The optimization in (3.49) is performed over policies satisfying the feasibility constraints for P-almost every realization of the data process and such that the function (random variable) $Z(\xi_{[T]})$ belongs to the considered space $\mathcal{Z}$.

Using dual representation (2.55) we can write this risk measure as

$$\rho(Z) = \sup_{Q \in \mathfrak{Q}} \mathbb{E}_Q[Z], \quad \forall Z \in \mathcal{Z}, \tag{3.50}$$

where $\mathfrak{Q}$ is a set of absolutely continuous with respect to $\bar{P}$ probability measures on $(\Xi, \mathcal{B})$. Consequently problem (3.49) can be represented in the minimax form (3.48) with $\mathfrak{M} = \mathfrak{Q}$. There is a slight difference between respective formulations (3.48) and (3.49) of robust multistage programs - the set $\mathfrak{Q}$ consists of probability measures on $(\Xi, \mathcal{B})$ which are *absolutely continuous* with respect to the reference measure $\bar{P}$, while we didn't make such assumption for the set $\mathfrak{M}$. However, at this point this is not essential, we will discuss this later.

In order to write dynamic programming equations for problems (3.48) and (3.49) we need a decomposable structure similar to (3.4) for the expectation operator. At every stage $t = 2, ..., T$ of the process we know the past, i.e., we observe a realization $\xi_{[t]}$ of the data process. For observed at stage t realization $\xi_{[t]}$ we need to define what do we optimize in the future stages. From the point of view of the minimax formulation (3.48) we can specify conditional distribution[18] of $\xi_{[t+1,T]}$ given $\xi_{[t]}$ for every probability distribution $P \in \mathfrak{M}$ of $\xi_{[T]} = (\xi_{[t]}, \xi_{[t+1,T]})$.

Consider a linear space $\mathcal{Z}$ of measurable functions $Z(\cdot) : \Xi \to \mathbb{R}$, for example take $\mathcal{Z} := \mathcal{L}_p(\Xi, \mathcal{B}, \bar{P})$, and sequence of spaces $\mathcal{Z}_1 \subset \mathcal{Z}_2 \subset \cdots \subset \mathcal{Z}_T$ with $\mathcal{Z}_t$ being the space of functions $Z \in \mathcal{Z}$ such that $Z(\xi_{[T]})$ does not depend on $\xi_{t+1}, ..., \xi_T$; with some abuse of notation we write such functions as $Z_t(\xi_{[t]})$. In particular, $\mathcal{Z}_T = \mathcal{Z}$ and $\mathcal{Z}_1$ is the space of constants and can be identified with $\mathbb{R}$. It could be noted that functions $Z_t \in \mathcal{Z}_t$ are defined on the set

$$\Xi_t := \left\{\xi_{[t]} \in \mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_t} : \exists\, \xi'_{[T]} \in \Xi \text{ such that } \xi_{[t]} = \xi'_{[t]}\right\},$$

which is the projection of Ξ onto $\mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_t}$.

[18] Recall that $\xi_{[s,t]} := (\xi_s, ..., \xi_t)$ denotes history of the process from time s to time $t \geq s$.

Consider sequence of mappings $\varrho_{t,T}(\cdot) : \mathcal{Z} \to \mathcal{Z}_t$, $t = 1, ..., T-1$, defined as

$$[\varrho_{t,T}(Z)](\xi_{[t]}) := \sup_{P \in \mathfrak{M}} \mathbb{E}_{P|\xi_{[t]}} \left[Z(\xi_{[T]}) \right], \quad Z \in \mathcal{Z}, \tag{3.51}$$

where the notation $\mathbb{E}_{P|\xi_{[t]}}$ means that the expectation is conditional on $\xi_{[t]}$ and with respect to probability distribution P of $\xi_{[T]} = (\xi_{[t]}, \xi_{[t+1,T]})$. We assume that the maximum in the right hand side of (3.51) is finite valued. Restricted to the space $\mathcal{Z}_{t+1} \subset \mathcal{Z}$ the mapping $\varrho_{t,T}$ will be denoted ϱ_t, i.e., $\varrho_t : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$ is given by

$$[\varrho_t(Z_{t+1})](\xi_{[t]}) = \sup_{P \in \mathfrak{M}} \mathbb{E}_{P|\xi_{[t]}} \left[Z_{t+1}(\xi_{[t+1]}) \right], \quad Z_{t+1} \in \mathcal{Z}_{t+1}. \tag{3.52}$$

We also use notation $\varrho_{t,T|\xi_{[t]}}(Z)$ and $\varrho_{t|\xi_{[t]}}(Z_{t+1})$ for $[\varrho_{t,T}(Z)](\xi_{[t]})$ and $[\varrho_t(Z_{t+1})](\xi_{[t]})$, respectively.

Remark 6 Note that, for the risk averse formulation (3.49), mappings $\varrho_{t|\xi_{[t]}}$ are not the same as the respective conditional risk measures $\rho_{t|\xi_{[t]}}$ (discussed in section 3.3), associated with the risk measure ρ. Suppose, for example, that $T = 3$ and let $\rho := \mathsf{AV@R}_\alpha$, $\alpha \in (0, 1)$. Here $\rho : \mathcal{Z} \to \mathbb{R}$ with $\mathcal{Z}$ being the space of random variables $Z = Z(\xi_2, \xi_3)$ having finite first order moment. The dual set of ρ is

$$\mathfrak{A} = \{\zeta(\xi_2, \xi_3) : 0 \preceq \zeta(\xi_2, \xi_3) \preceq \alpha^{-1},\ \mathbb{E}[\zeta] = 1\},$$

and

$$\varrho_{2|\xi_2}(Z) = \sup_{\zeta \in \mathfrak{A}} \mathbb{E}_{|\xi_2}[Z(\xi_2, \xi_3)\zeta(\xi_2, \xi_3)]. \tag{3.53}$$

Suppose, further, that random vectors ξ_2 and ξ_3 are independent. Consider the set $\mathfrak{A}'$ formed by densities $\zeta \in \mathfrak{A}$ which are functions of ξ_3 alone, i.e.,

$$\mathfrak{A}' = \{\zeta(\xi_3) : 0 \preceq \zeta(\xi_3) \preceq \alpha^{-1},\ \mathbb{E}[\zeta] = 1\}.$$

Then

$$\mathsf{AV@R}_{\alpha|\xi_2}(Z) = \sup_{\zeta \in \mathfrak{A}'} \mathbb{E}_{|\xi_2}[Z(\xi_2, \xi_3)\zeta(\xi_3)]. \tag{3.54}$$

Since $\mathfrak{A}'$ is a (strict) subset of $\mathfrak{A}$, it follows that $\varrho_{2|\xi_2}(Z) \geq \mathsf{AV@R}_{\alpha|\xi_2}(Z)$, and the inequality can be strict.

More specifically, let $(X, Y) \in \mathbb{R}^{n+m}$ be a random vector having uniform probability distribution P on a closed convex set $\Xi \subset \mathbb{R}^{n+m}$. Let $\mathfrak{Q}$ be a set of probability distributions supported on Ξ, absolutely continuous with respect to P with respective densities bounded by α^{-1} for some $\alpha \in (0, 1)$, i.e., $\mathfrak{Q}$ is the dual set of measures corresponding to $\mathsf{AV@R}_\alpha$. Denote by $\mathfrak{A}$ the set of respective densities, i.e., $\mathfrak{A}$ is formed by

densities $f : \Xi \to \mathbb{R}_+$, $\int_\Xi f(x,y)dxdy = 1$, such that $f(x,y) \leq \alpha^{-1}$ for all $(x,y) \in \Xi$. Suppose that $\Xi = \Xi_1 \times \Xi_2$, where $\Xi_1 \subset \mathbb{R}^n$ and $\Xi_2 \subset \mathbb{R}^m$ are convex closed sets with nonempty interior. (The following arguments can be pushed through for any convex set Ξ with nonempty interior, we assume that Ξ is given by a direct product of two sets for the sake of simplicity.)

For a random variable $Z = Z(X,Y)$ consider

$$\varrho_{|Y}(Z) := \sup_{Q \in \mathfrak{Q}} \mathbb{E}_{Q|Y}[Z].$$

Given $Y = y$, with $y \in \Xi_2$, and $Q \in \mathfrak{Q}$ with density $f \in \mathfrak{A}$ the conditional distribution of X is defined by the conditional density

$$f_{X|Y}(x|y) = c^{-1} f(x,y),\ x \in \Xi_1,$$

where $c = \int_{\Xi_1} f(x,y)dx$.

Let us observe that for $\bar{y} \in \Xi_2$, the conditional density $f_{X|Y}(\cdot|\bar{y})$ of $Q \in \mathfrak{Q}$ can be any bounded density supported on the set Ξ_1. Indeed, let $g(x)$ be a bounded density function supported on the set Ξ_1. Choose a constant $\kappa > 0$ such that $\kappa g(x) \leq \alpha^{-1}$ for all $x \in \Xi_1$ (such κ exists since $g(\cdot)$ is bounded). Then there exists $f \in \mathfrak{A}$ such that $f(\cdot, \bar{y}) = \kappa g(\cdot)$. Indeed, choose a neighborhood N of $\bar{y}$ and define $f(x,y) := \kappa g(x)$ for $(x,y) \in \Xi_1 \times N$. Choose N small enough such that $P(\Xi_1 \times N) \leq 1 - \alpha$. Then for $y \in \Xi_2 \setminus N$ we can choose $f(x,y) \geq 0$ such that the total integral $\int_\Xi f(x,y)dxdy = 1$.

It follows that

$$\varrho_{|Y}(Z) = \operatorname{ess\,sup}_{|Y}(Z),$$

where $\operatorname{ess\,sup}_{|Y}(Z)$ is the conditional essential sup of Z given Y. Note that for $\alpha = 1$, $\mathsf{AV@R}_1(\cdot) = \mathbb{E}(\cdot)$. In that case $\mathfrak{Q} = \{P\}$ and $\varrho_{|Y}(Z) = \mathbb{E}_{|Y}[Z]$ is given by conditional expectation.

Suppose now that the set $\mathfrak{Q}$ is given by a convex combination $\mathfrak{Q} = \lambda_1 \mathfrak{Q}_1 + ... + \lambda_r \mathfrak{Q}_r$, with $\mathfrak{Q}_i$, $i = 1, ..., r$, being set of probability measures corresponding to $\mathsf{AV@R}_{\alpha_i}$, $\alpha_i \in (0,1)$. Then

$$\sup_{Q \in \mathfrak{Q}} \mathbb{E}_{Q|Y}[Z] = \lambda_1 \sup_{Q \in \mathfrak{Q}_1} \mathbb{E}_{Q|Y}[Z] + ... + \lambda_r \sup_{Q \in \mathfrak{Q}_r} \mathbb{E}_{Q|Y}[Z],$$

and again $\varrho_{|Y}(Z) = \operatorname{ess\,sup}_{|Y}(Z)$. This can be extended to risk measures of the form $\int_0^1 \mathsf{AV@R}_\alpha d\mu(\alpha)$ for a probability measure μ on (0,1) (with zero mass at $\alpha = 0$ and $\alpha = 1$). Now if μ has positive measure λ at $\alpha = 1$, then since $\mathsf{AV@R}_1(\cdot) = \mathbb{E}(\cdot)$, it follows that

$$\varrho_{|Y}(Z) = \lambda \mathbb{E}_{|Y}[Z] + (1 - \lambda) \operatorname{ess\,sup}_{|Y}(Z).$$

$\Diamond$

After observing value $\xi_{[t]}$ of the data process at stage t, it is natural to perform future optimization at later stages using the conditional distributions of $\xi_{[t+1,T]}$ given $\xi_{[t]}$. However, choice of the corresponding objective function is not unique. We consider now the optimization (minimization) with respect to the conditional risk mappings $\varrho_{t,T}$. This leads to considering the composite function

$$\bar{\varrho}(Z) := \varrho_{1,T}(\varrho_{2,T}\dots(\varrho_{T-1,T}(Z))\dots), \quad Z \in \mathcal{Z}, \tag{3.55}$$

denoted

$$\bar{\varrho} = \varrho_{1,T} \circ \varrho_{2,T} \circ \cdots \circ \varrho_{T-1,T}.$$

Recall that mappings $\varrho_{t,T}(\cdot)$ and $\varrho_t(\cdot)$ do coincide on $\mathcal{Z}_{t+1}$, and hence

$$\bar{\varrho} = \varrho_1 \circ \varrho_2 \circ \cdots \circ \varrho_{T-1}$$

as well. Since $\mathcal{Z}_1$ can be identified with $\mathbb{R}$, we can view $\bar{\varrho} : \mathcal{Z} \to \mathbb{R}$ as a real valued function, i.e., as a risk measure. Another possibility will be to use the conditional risk measures $\rho_{t|\xi_{[t]}}$ instead of $\varrho_{t|\xi_{[t]}}$, we will discuss this later (see Remark 8 on page 66).

For the composite risk measure $\bar{\varrho}$ the corresponding risk averse problem can be written in the following nested form similar to (3.5):

$$\begin{aligned} \min_{x_1 \in \mathcal{X}_1} F_1(x_1) \quad & + \varrho_{1|\xi_{[1]}}\Big[\inf_{x_2 \in \mathcal{X}_2(x_1,\xi_2)} F_2(x_2,\xi_2) + \varrho_{2|\xi_{[2]}}\big[\cdots \\ & + \varrho_{T-1|\xi_{[T-1]}}\big[\inf_{x_T \in \mathcal{X}_T(x_{T-1},\xi_T)} F_T(x_T,\xi_T)\big]\Big]. \end{aligned} \tag{3.56}$$

Note that each mapping ϱ_t, $t = 1, ..., T-1$, in (3.56) can be equivalently replaced by the respective mapping $\varrho_{t,T}$.

For the nested formulation (3.56) it is possible to write dynamic programming equations in a way similar to (3.6)–(3.7). That is, for $t = T, ..., 2$,

$$Q_t\left(x_{t-1}, \xi_{[t]}\right) = \inf_{x_t \in \mathcal{X}_t(x_{t-1},\xi_t)} \left\{ F_t(x_t,\xi_t) + \mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) \right\}, \tag{3.57}$$

where

$$\mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) = \varrho_{t|\xi_{[t]}}\left[Q_{t+1}\left(x_t, \xi_{[t+1]}\right)\right], \tag{3.58}$$

with $\mathcal{Q}_{T+1}(\cdot,\cdot) \equiv 0$ by definition.

In order to see a relation between formulation (3.48) (formulation (3.49)) and the corresponding nested formulation (3.56) let us observe the following. For $Z \in \mathcal{Z}$, we can write

$$\mathbb{E}_P[Z(\xi_{[T]})] = \mathbb{E}_{P|\xi_1}\Big[\cdots \mathbb{E}_{P|\xi_{[T-2]}}\big[\mathbb{E}_{P|\xi_{[T-1]}}[Z(\xi_{[T]})]\big] \cdots \Big],$$

and hence for $\rho(\cdot) = \sup_{P\in\mathfrak{M}} \mathbb{E}_P[\,\cdot\,]$ we have

$$\begin{array}{rcl} \rho(Z) & = & \sup\limits_{P\in\mathfrak{M}} \mathbb{E}_{P|\xi_1}\Big[\,\cdots \mathbb{E}_{P|\xi_{[T-2]}}\big[\mathbb{E}_{P|\xi_{[T-1]}}[Z(\xi_{[T]})]\big]\,\cdots\Big] \\ & \le & \sup\limits_{P\in\mathfrak{M}} \mathbb{E}_{P|\xi_1}\Big[\,\cdots \sup\limits_{P\in\mathfrak{M}} \mathbb{E}_{P|\xi_{[T-2]}}\big[\sup\limits_{P\in\mathfrak{M}} \mathbb{E}_{P|\xi_{[T-1]}}[Z(\xi_{[T]})]\big]\,\cdots\Big] \\ & = & \varrho_1 \circ \varrho_2 \circ \cdots \circ \varrho_{T-1}(Z). \end{array} \tag{3.59}$$

We obtain the following result.

Proposition 3.3 *For risk measure $\rho(Z) := \sup_{P\in\mathfrak{M}} \mathbb{E}_P[Z]$ and the corresponding composite risk measure $\bar{\varrho} = \varrho_1 \circ \varrho_2 \circ \cdots \circ \varrho_{T-1}$ the following inequality holds*

$$\rho(Z) \le \bar{\varrho}(Z), \ \ \forall Z \in \mathcal{Z}. \tag{3.60}$$

It follows from (3.60) that the optimal value of the minimax problem (3.48) (risk averse problem (3.49)) is less than or equal to the optimal value of the corresponding nested problem (3.56). The inequality (3.60) can be strict (see, e.g., Example 4 on page 65). That is, risk measure ρ is not necessarily the same as the associated composite risk measure $\bar{\varrho}$, and formulations (3.48) and (3.56) are not necessarily equivalent.

- From the point of view of information at stage t - observed realization $\xi_{[t]}$ of the data process and the corresponding conditional distributions at future stages - the nested formulation (3.56) is *time consistent*. Therefore from this point of view the minimax formulation (3.48) (the risk averse formulation (3.49)), of the considered multistage problem, is time consistent if it is equivalent to the nested formulation (3.56), i.e., the optimal values of problems (3.48) and (3.56) are equal to each other.

Of course, if $\rho(\cdot) = \bar{\varrho}(\cdot)$, then the minimax and nested formulations are equivalent for any (allowable) choice of objective functions and feasibility constraints. That is, $\rho(\cdot) = \bar{\varrho}(\cdot)$ is a sufficient condition for the time consistency in the above sense. Some risk averse formulations are time consistent and some are not, we will discuss this further in the next sections.

Remark 7 By interchanging the *min* and *max* operators we can consider the following dual of the minimax problem (3.48):

$$\begin{array}{lll} \operatorname{Max}\limits_{P\in\mathfrak{M}} & \inf\limits_{x_1, x_2(\cdot),\ldots,x_T(\cdot)} & \mathbb{E}_P\big[F_1(x_1) + F_2(x_2(\xi_{[2]}),\xi_2) + \ldots + F_T\left(x_T(\xi_{[T]}),\xi_T\right)\big] \\ & \text{s.t.} & x_1 \in \mathcal{X}_1,\ x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}),\xi_t),\ t = 2,\ldots,T. \end{array} \tag{3.61}$$

Assuming that the problem is convex, under certain regularity conditions, it is possible to show that there is no duality gap between problems (3.48) and (3.61). For two-stage problems conditions ensuring such "no duality gap" property were given in Theorem 2.10. If the number of scenarios is finite, then the minimization part of problems (3.48) and (3.61) is finite dimensional and hence it is possible to apply results of Theorems 2.3 and 2.4.

If there is no duality gap between problems (3.48) and (3.61) and the dual problem (3.61) has an optimal solution $\bar{P}$, then the minimax problem (3.48) is equivalent to the corresponding risk neutral problem, of the form (3.5), with respect to the probability measure $\bar{P}$ of the data process. However, an optimal solution of the dual problem (3.61) depends on all realizations of the data process and hence this does not resolve the question of time consistency. $\Diamond$

3.4.1 Stagewise Independence

Similar to the risk neutral case, the cost-to-go functions $\mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right)$ do not depend on $\xi_{[t]}$ if the data process is stagewise independent. Here the stagewise independence means that ξ_{t+1} is independent of $\xi_{[t]}$ for every distribution $P \in \mathfrak{M}$ of $\xi_{[T]}$ and $t = 1, ..., T-1$. In terms of the set $\mathfrak{M}$ the stagewise independence means that for $t = 2, ..., T$, there is a set $\mathcal{M}_t$ of probability measures on a (closed) set $\Xi_t \subset \mathbb{R}^{d_t}$, equipped with its Borel sigma algebra $\mathcal{B}_t$, such that

$$\mathfrak{M} = \big\{P = P_2 \times \cdots \times P_T : P_t \in \mathcal{M}_t,\ t = 2, ..., T\big\}. \tag{3.62}$$

Note that here measures $P \in \mathfrak{M}$ are defined on the set $\Xi = \Xi_2 \times \cdots \times \Xi_T$. Note also that the set $\mathfrak{M}$ is not necessarily convex even if all sets $\mathcal{M}_t$, $t = 2, ..., T$, are convex. We will use the following notation for the set $\mathfrak{M}$:

$$\mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_T := \big\{P = P_2 \times \cdots \times P_T : P_t \in \mathcal{M}_t,\ t = 2, ..., T\big\}. \tag{3.63}$$

In the case of stagewise independence equation (3.52) takes the form

$$\varrho_{t|\xi_{[t]}}(Z_{t+1}) = \sup_{P_{t+1} \in \mathcal{M}_{t+1}} \mathbb{E}_{P_{t+1}}\left[Z_{t+1}(\xi_{[t]}, \xi_{t+1})\right], \tag{3.64}$$

where the expectation $\mathbb{E}_{P_{t+1}}\left[Z_{t+1}(\xi_{[t]}, \xi_{t+1})\right]$ is taken with respect to the distribution P_{t+1} of ξ_{t+1} for fixed $\xi_{[t]}$. Furthermore, the function (risk measure) $\rho(\cdot) := \sup_{P \in \mathfrak{M}} \mathbb{E}_P[\,\cdot\,]$ can be written as

$$\begin{array}{rcl} \rho(Z) & = & \sup\limits_{P_2 \in \mathcal{M}_2, ..., P_T \in \mathcal{M}_T} \mathbb{E}_{P_2 \times \cdots \times P_T}[Z(\xi_1, ..., \xi_T)] \\ & = & \sup\limits_{P_2 \in \mathcal{M}_2, ..., P_T \in \mathcal{M}_T} \mathbb{E}_{P_2}\Big[\, \cdots \mathbb{E}_{P_{T-1}}\big[\mathbb{E}_{P_T}[Z(\xi_1, ..., \xi_T)]\big] \cdots \Big], \end{array} \tag{3.65}$$

and the corresponding composite risk measure $\bar{\varrho} = \varrho_1 \circ \varrho_2 \circ \cdots \circ \varrho_{T-1}$ as

$$\bar{\varrho}(Z) = \sup_{P_2 \in \mathcal{M}_2} \mathbb{E}_{P_2}\Big[\cdots \sup_{P_{T-1} \in \mathcal{M}_{T-1}} \mathbb{E}_{P_{T-1}}\big[\sup_{P_T \in \mathcal{M}_T} \mathbb{E}_{P_T}[Z(\xi_1, ..., \xi_T)]\big] \cdots \Big]. \tag{3.66}$$

As the following example shows the inequality (3.60) can be strict even in the case of stagewise independence.

Example 4 Let $T = 3$ and $\mathfrak{M} := \mathcal{M}_2 \otimes \mathcal{M}_3$, with set $\mathcal{M}_2 := \{P\}$ being a singleton and $\mathcal{M}_3 := \{\Delta(\xi) : \xi \in \Xi_3\}$ being a set of probability measures of mass one. That is, the set $\mathfrak{M}$ consists of measures $P \times P_3$, $P_3 \in \mathcal{M}_3$. Then for $Z = Z(\xi_2, \xi_3)$,

$$\rho(Z) = \sup_{P_2 \in \mathcal{M}_2, P_3 \in \mathcal{M}_3} \mathbb{E}_{P_2 \times P_3}[Z(\xi_2, \xi_3)] = \sup_{\xi_3 \in \Xi_3} \mathbb{E}_P[Z(\xi_2, \xi_3)], \tag{3.67}$$

and

$$\bar{\varrho}(Z) = \sup_{P_2 \in \mathcal{M}_2} \mathbb{E}_{P_2}\left[\sup_{P_3 \in \mathcal{M}_3} \mathbb{E}_{P_3}[Z(\xi_2, \xi_3)]\right] = \mathbb{E}_P\left\{ \sup_{\xi_3 \in \Xi_3} Z(\xi_2, \xi_3)\right\}. \tag{3.68}$$

In (3.67) and (3.68) the expectations are taken with respect to the probability distribution P of ξ_2. As it is well known in stochastic programming the inequality

$$\sup_{\xi_3 \in \Xi_3} \mathbb{E}_P[Z(\xi_2, \xi_3)] \leq \mathbb{E}_P\left\{ \sup_{\xi_3 \in \Xi_3} Z(\xi_2, \xi_3)\right\} \tag{3.69}$$

can be strict.

Suppose, for example, that both sets Ξ_2 and Ξ_3 are finite, say $\Xi_2 = \{\xi_2^1, ..., \xi_2^k\}$ and $\Xi_3 = \{\xi_3^1, ..., \xi_3^m\}$, and let $\mathcal{Z}$ be the space of functions $Z : \Xi_2 \times \Xi_3 \to \mathbb{R}$. Denote $Z_{ij} := Z(\xi_2^i, \xi_3^j)$, $i = 1, ..., k$, $j = 1, ..., m$. Let $p_1, ..., p_k$ be (positive) probabilities, associated with points of Ξ_2, defining measure P. Then the inequality (3.69) can be written as

$$\max_{1 \leq j \leq m}\left\{ \sum_{i=1}^k p_i Z_{ij}\right\} \leq \sum_{i=1}^k p_i \max_{1 \leq j \leq m}\{Z_{ij}\}. \tag{3.70}$$

For a given $Z \in \mathcal{Z}$ the maximum in the left hand side of (3.70) is attained at some $j^* \in \{1, ..., m\}$, independent of i, while the maximum and the right hand side of (3.70) is attained at some point $j^\star(i) \in \{1, ..., m\}$ which is a function of i. For $k > 1$ and $m > 1$ the inequality (3.70) is strict, i.e., $\rho(Z) < \bar{\varrho}(Z)$ for some $Z \in \mathcal{Z}$. $\Diamond$

Proposition 3.4 *Let $\rho(Z) := \sup_{P\in\mathfrak{M}} \mathbb{E}_P[Z]$ and suppose that the stagewise independence condition holds, i.e., the set $\mathfrak{M}$ is given in the form* (3.62)*. Then $\rho(\cdot) = \bar{\varrho}(\cdot)$ if the interchageability property*

$$\mathbb{E}_{P_2\times\cdots\times P_t}\left\{\sup_{P_{t+1}\in\mathcal{M}_{t+1}} \mathbb{E}_{P_{t+1}}\left[Z_{t+1}(\xi_{[t]},\xi_{t+1})\right]\right\} = \sup_{P_{t+1}\in\mathcal{M}_{t+1}} \mathbb{E}_{P_2\times\cdots\times P_{t+1}}\left[Z_{t+1}(\xi_{[t]},\xi_{t+1})\right] \tag{3.71}$$

holds for all $Z\in\mathcal{Z}$ and $t = 2,...,T-1$.

Proof. Suppose that condition (3.71) holds. Then

$$\begin{aligned}
\rho(Z) &= \sup_{P_2\in\mathcal{M}_2}\cdots\sup_{P_T\in\mathcal{M}_T}\mathbb{E}_{P_2}\Big[\cdots\mathbb{E}_{P_{T-1}}\big[\mathbb{E}_{P_T}[Z(\xi_1,...,\xi_T)]\big]\cdots\Big] \\
&= \sup_{P_2\in\mathcal{M}_2}\cdots\sup_{P_{T-1}\in\mathcal{M}_{T-1}}\mathbb{E}_{P_2}\Big[\cdots\mathbb{E}_{P_{T-1}}\big[\sup_{P_T\in\mathcal{M}_T}\mathbb{E}_{P_T}[Z(\xi_1,...,\xi_T)]\big]\cdots\Big] \\
&= \sup_{P_2\in\mathcal{M}_2}\mathbb{E}_{P_2}\Big[\cdots\sup_{P_{T-1}\in\mathcal{M}_{T-1}}\mathbb{E}_{P_{T-1}}\big[\sup_{P_T\in\mathcal{M}_T}\mathbb{E}_{P_T}[Z(\xi_1,...,\xi_T)]\big]\cdots\Big],
\end{aligned} \tag{3.72}$$

and hence $\rho(Z) = \bar{\varrho}(Z)$. ∎

The requirement for (3.71) to hold for all $Z\in\mathcal{Z}$ is rather exceptional. Of course, this holds if the sets $\mathcal{M}_t$, $t = 2,...,T$, are singletons. Another case where this holds if $\mathcal{M}_t$ is the set of all probability measures on Ξ_t, $t = 2,...,T$. We will discuss this case in the next section.

By (3.64) and using induction in $t = T,...,$ we obtain that in the considered stagewise independent case the dynamic programming equations (3.57)–(3.58) take the form

$$Q_t(x_{t-1},\xi_t) = \inf_{x_t\in\mathcal{X}_t(x_{t-1},\xi_t)}\big\{F_t(x_t,\xi_t) + \mathcal{Q}_{t+1}(x_t)\big\}, \tag{3.73}$$

for $t = T,...,2$, where

$$\mathcal{Q}_{t+1}(x_t) = \sup_{P_{t+1}\in\mathcal{M}_{t+1}}\mathbb{E}_{P_{t+1}}\left[Q_{t+1}(x_t,\xi_{t+1})\right], \tag{3.74}$$

with $\mathcal{Q}_{T+1}(\cdot,\cdot)\equiv 0$ by definition.

Remark 8 Let ρ_t, $t = 2,...,T$, be a coherent risk measure defined on a space of random variables $Z_t : \Xi_t\to\mathbb{R}$. Furthermore, let $\mathcal{M}_t := \mathfrak{Q}_t$, $t = 2,...,T$, with $\mathfrak{Q}_t$ being the set of probability measures associated with the dual set $\mathfrak{A}_t$ of ρ_t (see (2.56)). Then $\varrho_{t|\xi_{[t]}}$, defined in (3.64), coincides with the conditional risk measure $\rho_{t|\xi_{[t]}}$ associated

with the risk measure ρ_t. Consequently equations (3.73)–(3.74) represent dynamic programming equations for the nested formulation of the multistage program of the form (3.56) with $\varrho_{t|\xi_{[t]}}$ replaced by $\rho_{t|\xi_{[t]}}$ as well. That is, replacing $\varrho_{t|\xi_{[t]}}$ with $\rho_{t|\xi_{[t]}}$ in (3.56) we obtain an equivalent multistage problem. $\Diamond$

3.5 Robust Multistage Programming

Let $\mathfrak{M}$ be the set of *all* probability measures on $(\Xi, \mathcal{B})$. Then for computing the maximum in $\rho(\cdot) = \sup_{P\in\mathfrak{M}} \mathbb{E}_P[\,\cdot\,]$ it suffices to perform the maximization with respect to measures of mass one at a point of the set Ξ, and hence the minimax formulation (3.48) can be written as

$$\begin{array}{cc} \underset{x_1, x_2(\cdot),\ldots,x_T(\cdot)}{\text{Min}} & \sup\limits_{(\xi_2,\ldots,\xi_T)\in\Xi} \left\{F_1(x_1) + F_2(x_2(\xi_{[2]}),\xi_2) + \ldots + F_T\left(x_T(\xi_{[T]}),\xi_T\right)\right\} \\ \text{s.t.} & x_1 \in \mathcal{X}_1,\ x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}),\xi_t),\ t = 2,\ldots,T. \end{array} \tag{3.75}$$

The above worst-case formulation (3.75) of multistage programs can be considered in the framework of robust optimization, where it is called *adjustable* since decisions $x_t(\xi_{[t]})$, $t = 2, ..., T$, are adjusted to the observed data (cf., [3]).

We can use the $\rho(\cdot) := \operatorname{ess\,sup}(\cdot)$ risk measure in order to write problem (3.75) in the form (3.49). However, this will result in replacing the "sup" by the "ess sup" operator in (3.75), which is not natural from the point of view of robust optimization. Therefore we are going now to analyze the worst-case formulation (3.75) directly.

Let us consider the following construction. Denote by $\mathcal{Z}_t$, $t = 2, ..., T$, the linear space of bounded real valued functions $Z : \mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_t} \to \mathbb{R}$, with $\mathcal{Z}_1 \equiv \mathbb{R}$ (i.e., $\mathcal{Z}_1$ is the space of constants). For $1 \le s < t \le T$ consider the mapping $\varrho_{s,t} : \mathcal{Z}_t \to \mathcal{Z}_s$ defined as follows

$$[\varrho_{s,t}(Z)](\xi_{[s]}) = \sup_{(\xi'_2,\ldots,\xi'_T)\in\Xi} \left\{Z(\xi'_{[t]}) : \xi'_{[s]} = \xi_{[s]}\right\},\ \ Z \in \mathcal{Z}_t. \tag{3.76}$$

In particular, $\varrho_{1,t} : \mathcal{Z}_t \to \mathbb{R}$ is

$$\varrho_{1,t}(Z) = \sup_{(\xi_2,\ldots,\xi_T)\in\Xi} Z(\xi_{[t]}). \tag{3.77}$$

Note that the objective function in the right hand side of (3.76) does not depend on $\xi'_{t+1}, ..., \xi'_T$ and the maximization can be performed over the set Ξ_t (instead of Ξ), where Ξ_t is the projection of Ξ onto $\mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_t}$, i.e.,

$$\Xi_t = \left\{\xi_{[t]} : \exists\, \xi'_{[T]} \in \Xi \text{ such that } \xi_{[t]} = \xi'_{[t]}\right\}. \tag{3.78}$$

For $t = T, ..., 2$, consider the following dynamic programming equations

$$Q_t(x_{t-1}, \xi_{[t]}) = \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \Big\{ F_t(x_t, \xi_t) + \mathcal{Q}_{t+1}(x_t, \xi_{[t]}) \Big\}, \tag{3.79}$$

where

$$\mathcal{Q}_{t+1}(x_t, \xi_{[t]}) = \varrho_{t,t+1}\big[Q_{t+1}(x_t, \xi_{[t+1]}) \big], \tag{3.80}$$

with $\mathcal{Q}_{T+1}(\cdot, \cdot) \equiv 0$ by definition. At the first stage we need to solve the problem

$$\operatorname*{Min}_{x_1 \in \mathcal{X}_1} F_1(x_1) + \mathcal{Q}_2(x_1). \tag{3.81}$$

We are going to establish a connection between these dynamic equations and multistage robust problem (3.75). The mapping $\varrho_{t,t+1}$ in the right hand side of (3.80) is applied to the function $Q_{t+1}(x_t, \cdot)$ for given (fixed) x_t. That is, the *cost-to-go functions*, defined in (3.80), can be written as

$$\mathcal{Q}_{t+1}(x_t, \xi_{[t]}) = \sup_{(\xi'_2, ..., \xi'_T) \in \Xi} \big\{ Q_{t+1}\big(x_t, \xi'_{[t+1]}\big) : \xi'_{[t]} = \xi_{[t]} \big\}. \tag{3.82}$$

Of course, in order for the function $\mathcal{Q}_{t+1}(x_t, \xi_{[t]})$ to be real valued we need to impose some boundedness conditions ensuring that the maximum in the right hand side of (3.82) is finite.

It immediately follows from the definition (3.76) that for $1 \leq r < s < t \leq T$, the composite mapping $\varrho_{r,s} \circ \varrho_{s,t} : \mathcal{Z}_t \to \mathcal{Z}_r$ coincides with mapping $\varrho_{r,t} : \mathcal{Z}_t \to \mathcal{Z}_r$, i.e.,

$$\varrho_{r,s} \circ \varrho_{s,t} = \varrho_{r,t}. \tag{3.83}$$

We also will need the following interchangeability property. Let A and B two (abstract) nonempty sets, $A \ni x \mapsto \mathfrak{B}(x) \subset B$ be a multifunction (point-to-set mapping) and $h : A \times B \to \mathbb{R}$ be a real valued function. Consider the min-max problem

$$\operatorname*{Max}_{x \in A} \inf_{y \in \mathfrak{B}(x)} h(x, y). \tag{3.84}$$

Let $\mathcal{Y}$ be the space of mappings $y(\cdot) : A \to B$ such that $y(x) \in \mathfrak{B}(x)$ for all $x \in A$, and consider problem

$$\operatorname*{Min}_{y(\cdot) \in \mathcal{Y}} \sup_{x \in A} h(x, y(x)). \tag{3.85}$$

Proposition 3.5 *Suppose that* $\inf_{y \in \mathfrak{B}(x)} h(x, y)$ *is finite for every* $x \in A$. *Then the optimal values of problems* (3.84) *and* (3.85) *are equal to each other. Moreover,* $\bar{y}(\cdot) \in \mathcal{Y}$ *is an optimal solution of problem* (3.85) *if*

$$\bar{y}(x) \in \arg\min_{y \in \mathfrak{B}(x)} h(x, y), \ \forall x \in A. \tag{3.86}$$

Proof. For any $y(\cdot) \in \mathcal{Y}$ we have that $h(x, y(x)) \geq \inf_{y \in \mathfrak{B}(x)} h(x, y)$ for any $x \in A$, and hence

$$\sup_{x \in A} h(x, y(x)) \geq \sup_{x \in A} \inf_{y \in \mathfrak{B}(x)} h(x, y).$$

It follows that the optimal value of problem (3.84) is less than or equal to the optimal value of problem (3.85).

Conversely, for a chosen $\varepsilon > 0$ let $\bar{y}(\cdot) \in \mathcal{Y}$ be such that

$$\inf_{y \in \mathfrak{B}(x)} h(x, y) \geq h(x, \bar{y}(x)) - \varepsilon, \quad x \in A. \tag{3.87}$$

Such mapping exists since it is assumed that $\inf_{y \in \mathfrak{B}(x)} h(x, y)$ is finite (in particular the set $\mathfrak{B}(x)$ is nonempty) for every $x \in A$. It follows that

$$\sup_{x \in A} \inf_{y \in \mathfrak{B}(x)} h(x, y) \geq \sup_{x \in A} h(x, \bar{y}(x)) - \varepsilon, \tag{3.88}$$

and hence

$$\sup_{x \in A} \inf_{y \in \mathfrak{B}(x)} h(x, y) \geq \inf_{y(\cdot) \in \mathcal{Y}} \sup_{x \in A} h(x, y(x)) - \varepsilon. \tag{3.89}$$

Since $\varepsilon > 0$ is arbitrary, it follows that the optimal value of problem (3.85) is less than or equal to the optimal value of problem (3.84).

Moreover, $\bar{y}(\cdot)$ is an optimal solution of (3.85) iff $\varepsilon = 0$ in (3.88). In turn this holds if $\varepsilon = 0$ in (3.84), i.e., if (3.86) holds. ∎

Suppose for the moment that $\mathfrak{B}(x) = B$ for all $x \in A$ and that problem (3.85) attains its maximal value at a constant mapping $y(x) \equiv \bar{y}$. Then

$$\sup_{x \in A} \inf_{y \in B} h(x, y) = \inf_{y \in B} \sup_{x \in A} h(x, y). \tag{3.90}$$

Moreover, if $\bar{x} \in A$ is an optimal solution of problem (3.84), then $(\bar{x}, \bar{y})$ is a saddle point of problem (3.84). Conversely, if $(\bar{x}, \bar{y})$ is a saddle point of problem (3.84), then $\bar{x}$ is an optimal solution of problem (3.84) and $y(\cdot) \equiv \bar{y}$ is an optimal solution of problem (3.85).

Consider now the multistage problem (3.75). Recall that the minimization is performed over policies satisfying the feasibility constraints. For fixed (feasible) decisions $x_1, x_2(\cdot), ..., x_{T-1}(\cdot)$, let us consider minimization with respect to $x_T(\cdot)$. Assuming that the cost-to-go functions are finite valued, by Proposition 3.5 we can interchange the corresponding minimization and maximization in (3.75). This results in the problem

$$\begin{aligned} \operatorname*{Min}_{x_1, x_2(\cdot), \ldots, x_{T-1}(\cdot)} \quad & \sup_{\xi_{[T]} \in \Xi} \Big[F_1(x_1) + \cdots + F_{T-1}(x_{T-1}, \xi_{T-1}) + \underbrace{\inf_{x_T \in \mathcal{X}_T(x_{T-1}, \xi_T)} F_T(x_T, \xi_T)}_{Q_T(x_{T-1}, \xi_T)} \Big] \\ \text{s.t.} \quad & x_1 \in \mathcal{X}_1, \ x_t \in \mathcal{X}_t(x_{t-1}, \xi_t), \ t = 2, \ldots, T-1. \end{aligned} \tag{3.91}$$

Performing maximization in (3.91) with respect to ξ_T we can write (3.91) as

$$\begin{aligned} \operatorname*{Min}_{x_1, x_2(\cdot), \ldots, x_{T-1}(\cdot)} \quad & \sup_{\xi_{[T]} \in \Xi} \Big[F_1(x_1) + \cdots + F_{T-1}(x_{T-1}, \xi_{T-1}) + \underbrace{\sup_{\xi_{[T]} \in \Xi} Q_T(x_{T-1}, \xi_T)}_{\mathcal{Q}_T(x_{T-1}, \xi_{[T-1]})} \Big] \\ \text{s.t.} \quad & x_1 \in \mathcal{X}_1, \ x_t \in \mathcal{X}_t(x_{t-1}, \xi_t), \ t = 2, \ldots, T-1. \end{aligned} \tag{3.92}$$

Note that the objective function in (3.92) does not depend on ξ_T and the maximization can be performed over $\xi_{[T-1]} \in \Xi_{T-1}$ instead of $\xi_{[T]} \in \Xi$.

Next we can proceed to minimization in (3.92) with respect to $x_{T-1}(\cdot)$. Again using the interchangeability property we obtain

$$\begin{aligned} \operatorname*{Min}_{x_1, x_2(\cdot), \ldots, x_{T-2}(\cdot)} \quad & \sup_{\xi_{[T-1]} \in \Xi_{T-1}} \Big[F_1(x_1) + \ldots \\ & + \underbrace{\inf_{x_{T-1} \in \mathcal{X}_{T-1}(x_{T-2}, \xi_{T-1})} \big\{ F_{T-1}(x_{T-1}, \xi_{T-1}) + \mathcal{Q}_T(x_{T-1}, \xi_{[T-1]}) \big\}}_{Q_{T-1}(x_{T-2}, \xi_{[T-1]})} \Big] \\ \text{s.t.} \quad & x_1 \in \mathcal{X}_1, \ x_t \in \mathcal{X}_t(x_{t-1}, \xi_t), \ t = 2, \ldots, T-2. \end{aligned}$$

Furthermore, by taking maximum in the above problem with respect to ξ_{T-1} we obtain

$$\begin{aligned} \operatorname*{Min}_{x_1, x_2(\cdot), \ldots, x_{T-2}(\cdot)} \quad & \sup_{\xi_{[T-2]} \in \Xi_{T-2}} \Big[F_1(x_1) + \cdots + F_{T-2}(x_{T-2}, \xi_{T-2}) + \mathcal{Q}_{T-1}(x_{T-2}, \xi_{[T-2]}) \Big] \\ \text{s.t.} \quad & x_1 \in \mathcal{X}_1, \ x_t \in \mathcal{X}_t(x_{t-1}, \xi_t), \ t = 2, \ldots, T-2. \end{aligned} \tag{3.93}$$

Continuing this process backwards in time we derive dynamic equations (3.79)–(3.80). This gives the following result.

Proposition 3.6 *Suppose that the cost-to-go functions in dynamic equations* (3.79)–(3.81) *are finite valued. Then the optimal value of problem* (3.75) *is equal to the optimal value of problem* (3.81). *Moreover, a policy* $\bar{x}_t(\xi_{[t]})$, $t = 1, ..., T$, *is optimal for the problem* (3.75) *if*

$$\bar{x}_t(\xi_{[t]}) \in \arg \min_{x_t \in \mathcal{X}_t(\bar{x}_{t-1}(\xi_{[t-1]}), \xi_t)} \Big\{ F_t(x_t, \xi_t) + \mathcal{Q}_{t+1}(x_t, \xi_{[t]}) \Big\}, \ t = 2, ..., T, \tag{3.94}$$

and $\bar{x}_1$ is an optimal solution of the first stage problem (3.81).

- This shows that for the worst-case formulation (3.75) of multistage programs the minimax and nested formulations are equivalent and formulation (3.75) is time consistent.

Consider now the case of stagewise independence. That is, suppose that the uncertainty set is the direct product of nonempty sets $\Xi_t \subset \mathbb{R}^{d_t}$, $t = 2, ..., T$, i.e., $\Xi = \Xi_2 \times \cdots \times \Xi_T$. In that case the max-mapping $\varrho_{s,t}$ takes the form

$$[\varrho_{s,t}(Z)](\xi_{[s]}) = \sup_{\xi'_{s+1} \in \Xi_{s+1}, ..., \xi'_t \in \Xi_t} \left\{ Z(\xi_2, ..., \xi_s, \xi'_{s+1}, ..., \xi'_t) \right\}. \tag{3.95}$$

Consequently, in that case dynamic equations (3.79)–(3.80) become

$$Q_t(x_{t-1}, \xi_t) = \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \Big\{ F_t(x_t, \xi_t) + \mathcal{Q}_{t+1}(x_t) \Big\}, \ t = 2, ..., T, \tag{3.96}$$

with cost-to-go functions

$$\mathcal{Q}_{t+1}(x_t) = \sup_{\xi_{t+1} \in \Xi_{t+1}} Q_{t+1}(x_t, \xi_{t+1}) \tag{3.97}$$

independent of the data process.

3.6 Dynamic Problem of Moments

Let us consider the problem of moments in the following multistage setting. Let $\Xi_t \subset \mathbb{R}^{d_t}$, $\mu_t \in \mathbb{R}^{q_t}$ and $\Psi_t : \Xi_t \to \mathbb{R}^{q_t}$ be a measurable mapping, $t = 2, ..., T$. Define $\mathcal{M}_t$ to be the set of probability measures P_t on $(\Xi_t, \mathcal{B}_t)$ satisfying the following moment conditions

$$\mathbb{E}_{P_t}[\Psi_t(\xi_t)] = \mu_t, \ \ t = 2, ..., T, \tag{3.98}$$

and let $\mathfrak{M} := \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_T$. Of course, this construction maintains the stagewise independence condition.

In this setting the minimax and nested formulations are not necessarily equivalent. In order to see this consider the following example.

Example 5 Let $T = 3$ and the set Ξ_2 be finite. Then for $t = 2$ the moment constraints (3.98) take a form of linear equations for the respective probabilities associated with points of the set Ξ_2 (compare with (2.112)). By an appropriate choice, the moment constraints define a unique probability measure on Ξ_2. If, furthermore, the

set $\mathcal{M}_3$ consists of all probability measures on $\Xi_3 \subset \mathbb{R}^{d_3}$, then this becomes a case considered in Example 4. This shows that the corresponding inequality (3.60) can be strict in this example. If, on the other hand, we assume that the set Ξ_3 is also finite and the respective moment constraints define a unique probability measure on Ξ_3, i.e., both sets $\mathcal{M}_2 = \{P_2\}$ and $\mathcal{M}_3 = \{P_3\}$ are singletons, then of course $\rho(\cdot) = \bar{\varrho}(\cdot)$. This shows how fragile can be the time consistency property. $\Diamond$

For the respective *nested* formulation we can write the dynamic programming equations (see (3.73)–(3.74)):

$$Q_t\left(x_{t-1}, \xi_t\right) = \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \left\{ F_t(x_t, \xi_t) + \mathcal{Q}_{t+1}\left(x_t\right) \right\}, \tag{3.99}$$

where

$$\mathcal{Q}_{t+1}\left(x_t\right) = \sup_{P_{t+1} \in \mathcal{M}_{t+1}} \mathbb{E}_{P_{t+1}}\left[Q_{t+1}\left(x_t, \xi_{t+1}\right)\right]. \tag{3.100}$$

By the Richter - Rogosinski Theorem the maximum in the right hand side of (3.100) is attained at a probability measure supported on at most $q_{t+1}+1$ points of Ξ_{t+1} (see Theorem 2.12).

Example 6 Consider the linear multistage setting with the data in the form (3.3) and with only right hand side vectors b_t being uncertain. Suppose that $b_t \in \Xi_t$, where $\Xi_t \subset \mathbb{R}^{d_t}$ is a bounded convex polyhedral set, $t = 2, ..., T$. Suppose, further, that means $\mu_t \in \Xi_t$ of vectors b_t are known. That is, let $\mathcal{M}_t$ be the set of probability measures P_t on Ξ_t with given mean $\mathbb{E}_{P_t}[b_t] = \mu_t$, $t = 2, ..., T$, and $\mathfrak{M} := \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_T$. Since Ξ_t is bounded polyhedral, the set $\mathrm{Ext}(\Xi_t)$ of its extreme points is finite and Ξ_t is equal to the convex hull of $\mathrm{Ext}(\Xi_t)$.

The cost-to-go functions here are given by dynamic equations

$$Q_t\left(x_{t-1}, b_t\right) = \inf_{x_t} \left\{ c_t^{\mathsf{T}} x_t + \mathcal{Q}_{t+1}\left(x_t\right) : A_t x_t = b_t - B_t x_{t-1},\ x_t \geq 0 \right\}, \tag{3.101}$$

with $\mathcal{Q}_{t+1}(\cdot)$ of the form (3.100) and $\mathcal{Q}_{T+1}(\cdot) \equiv 0$. It follows that functions $\mathcal{Q}_{t+1}(\cdot)$ and $Q_t\left(\cdot, \cdot\right)$ are convex. Consequently by Theorem 2.13 the maximum of $\mathbb{E}_{P_t}\left[Q_t\left(x_{t-1}, b_t\right)\right]$ over $P_t \in \mathcal{M}_t$ is attained at a probability measure P_t^* with a finite support consisting of at most $d_t + 1$ points of $\mathrm{Ext}(\Xi_t)$.

Suppose now that

$$\Xi_t := \left\{ \xi \in \mathbb{R}^{d_t} : \textstyle\sum_{i=1}^{d_t} \xi_i \leq 1,\ \xi \geq 0 \right\},\ \ t = 2, ..., T.$$

Then $\mathrm{Ext}(\Xi_t) = \{0, e_1, ..., e_{d_t}\}$, where e_i are coordinate vectors of $\mathbb{R}^{d_t}$. Consequently $P_t^* = \alpha_{0t}\Delta(0) + \alpha_{1t}\Delta(e_1) + ... + \alpha_{d_t t}\Delta(e_{d_t})$ with α_{it} satisfying the system of equations $\alpha_{it} = \mu_{it}$, $i = 1, ..., d_t$, $\alpha_{0t} = 1 - \sum_{i=1}^{d_t} \mu_{it}$. That is, the probability measure

satisfying the feasibility (moment) constraints and supported on the set $\mathrm{Ext}(\Xi_t)$ is defined uniquely. In particular, if mean μ_t is an interior point of the set Ξ_t, then all probabilities α_{it}, $i = 0, ..., d_t$, are positive. It follows that in this example the nested formulation can be reduced to solving the corresponding multistage problem with respect to unique probability measure $P^* = P_2^* \times \cdots \times P_T^*$, and hence the minimax and nested formulations are equivalent.

On the other hand let each Ξ_t be a box, say $\Xi_t := \{\xi \in \mathbb{R}^{d_t} : |\xi_i| \leq 1,\ i = 1, ..., d_t\}$. Then the set of extreme points of Ξ_t has 2^{d_t} elements. Since for $d_t > 1$ the set Ξ_t has more extreme points than $d_t + 1$, the corresponding probability measure P_t^* may be not uniquely defined and there is no guarantee of equivalence of the minimax and nested formulations. $\Diamond$

3.7 Dynamics of Average Value-at-Risk Measures

Let us consider the Average Value-at-Risk measure $\rho(\cdot) := \mathsf{AV@R}_\alpha(\cdot)$, with $\mathcal{Z} := \mathcal{L}_1(\Xi, \mathcal{B}, P)$, $\alpha \in (0, 1)$ and P being a reference probability measure on the set $\Xi \subset \mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_T}$. Consider also the respective conditional risk measures $\rho_{t|\xi_{[t]}}(\cdot) = \mathsf{AV@R}_{\alpha|\xi_{[t]}}(\cdot)$. We have the following upper bound for the nested Average Value-at-Risk measure.

Proposition 3.7 *For $Z \in \mathcal{Z}$ and $\alpha \in [0, 1]$ it holds that*

$$\mathsf{AV@R}_{\alpha|\xi_1}\Big[\cdots \mathsf{AV@R}_{\alpha|\xi_{[T-2]}}\big[\mathsf{AV@R}_{\alpha|\xi_{[T-1]}}[Z]\big]\cdots\Big] \leq \mathsf{AV@R}_{\alpha^T}[Z]. \tag{3.102}$$

Proof. Let $\alpha \in (0, 1]$ and consider partition $\xi = (X, Y)$. Since the maximum in the dual representation (2.53) is attained, we can write

$$\mathsf{AV@R}_{\alpha|Y}(Z) = \mathbb{E}_{|Y}[Z(X, Y)\zeta_Y(X)] \tag{3.103}$$

for some $\zeta_Y \in \mathfrak{A}_Y$. Thus

$$\begin{aligned}
\mathsf{AV@R}_\alpha(\mathsf{AV@R}_{\alpha|Y}(Z)) &= \sup\Big\{\mathbb{E}[\zeta_1(Y)\mathsf{AV@R}_{\alpha|Y}(Z)] : 0 \preceq \zeta_1 \preceq \alpha^{-1},\ \mathbb{E}[\zeta_1] = 1\Big\} \\
&= \sup\Big\{\mathbb{E}\big[\zeta_1(Y)\mathbb{E}_{|Y}[Z(X, Y)\zeta_Y(X)]\big] : 0 \preceq \zeta_1 \preceq \alpha^{-1},\ \mathbb{E}[\zeta_1] = 1\Big\} \\
&= \sup\Big\{\mathbb{E}\big[\mathbb{E}_{|Y}[Z(X, Y)\zeta_1(Y)\zeta_Y(X)]\big] : 0 \preceq \zeta_1 \preceq \alpha^{-1},\ \mathbb{E}[\zeta_1] = 1\Big\} \\
&= \sup\Big\{\mathbb{E}\big[Z(X, Y)\zeta_1(Y)\zeta_Y(X)\big] : 0 \preceq \zeta_1 \preceq \alpha^{-1},\ \mathbb{E}[\zeta_1] = 1\Big\}.
\end{aligned} \tag{3.104}$$

We also have that

$$\mathbb{E}[\zeta_1(Y)\zeta_Y(X)] = \mathbb{E}\left[\zeta_1(Y)\mathbb{E}_{|Y}[\zeta_Y(X)]\right] = 1$$

and $0 \preceq \zeta_1(Y)\zeta_Y(X) \preceq \alpha^{-2}$. It follows that the last maximum in (3.104) is less than or equal to

$$\sup\Big\{\mathbb{E}[Z(X,Y)\zeta(X,Y)] : 0 \preceq \zeta \preceq \alpha^{-2},\ \mathbb{E}[\zeta] = 1\Big\} = \mathsf{AV@R}_{\alpha^2}(Z).$$

This proves the inequality (3.102) for $T = 2$. For $\alpha \in (0, 1]$ the proof can be completed now by induction. For $\alpha = 0$ the left and right hand sides of (3.102) are equal to each other. ∎

For the risk measure $\rho(\cdot) := \mathsf{AV@R}_\alpha(\cdot)$ the corresponding multistage problem (3.49) can be written as

$$\begin{array}{ll} \underset{x(\cdot),\, z}{\mathrm{Min}} & \mathbb{E}\Big\{z + \alpha^{-1}\big[F_1(x_1) + F_2(x_2(\xi_{[2]}), \xi_2) + \ldots + F_T\left(x_T(\xi_{[T]}), \xi_T\right) - z\big]_+\Big\} \\ \text{s.t.} & x_1 \in \mathcal{X}_1,\ x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}), \xi_t),\ t = 2, \ldots, T, \end{array} \tag{3.105}$$

where $x(\cdot) = (x_1, x_2(\cdot), \ldots, x_T(\cdot))$. If the multistage problem is linear and the number of scenarios (realizations of the data process) is finite, then it is possible to write problem (3.105) as a large linear programming problem.

As far as dynamic equations are concerned let us observe that at the last stage $t = T$ we would need to solve problem conditional on z and decisions up to stage $t = T - 1$. Therefore dynamic equations cannot be written in an obvious way and formulation (3.105) is not time consistent. The corresponding nested formulation, of course, is time consistent. It is interesting to observe that in extreme cases of $\alpha = 1$ (when $\rho(\cdot) = \mathbb{E}(\cdot)$) and $\alpha = 0$ (when $\rho(\cdot) = \operatorname{ess\,sup}(\cdot)$) the minimax and nested formulations are equivalent.

Consider now risk measure

$$\rho_{\alpha,\lambda}(Z) := (1 - \lambda)\mathbb{E}[Z] + \lambda\, \mathsf{AV@R}_\alpha[Z], \tag{3.106}$$

with $\alpha \in (0, 1)$ and $\lambda \in [0, 1]$. This risk measure was discussed in section 2.5.2 with respect to risk averse formulation of two-stage stochastic programming. In the linear case it was possible to formulate the corresponding risk averse two-stage problem as a standard linear two-stage stochastic program by introducing one additional decision variable (see (2.84)–(2.88)). Similar procedure can be extended to the nested formulation of multistage programs.

Conditional analogues of $\rho_{\alpha,\lambda}$ are

$$\rho_{t|\xi_{[t-1]}}(\cdot) := (1 - \lambda_t)\mathbb{E}_{|\xi_{[t-1]}}(\cdot) + \lambda_t \mathsf{AV@R}_{\alpha_t|\xi_{[t-1]}}(\cdot), \tag{3.107}$$

with $\lambda_t \in [0,1]$ and $\alpha_t \in (0,1)$ being chosen parameters. Consider the following *nested* formulation of *linear* multistage programs with the data of the form (3.3):

$$\min_{\substack{A_1x_1=b_1\\ x_1\geq 0}} c_1^{\mathsf{T}}x_1 + \rho_{2|\xi_1}\left[\min_{\substack{B_2x_1+A_2x_2=b_2\\ x_2\geq 0}} c_2^{\mathsf{T}}x_2 + \cdots + \rho_{T|\xi_{[T-1]}}\left[\min_{\substack{B_Tx_{T-1}+A_Tx_T=b_T\\ x_T\geq 0}} c_T^{\mathsf{T}}x_T\right]\right]. \tag{3.108}$$

An intuitive motivation of this formulation is that at t-th stage of the process one tries to control an upper limit of the corresponding cost-to-go function $Q_{t+1}(x_t, \xi_{t+1})$ for different realizations of the data process (see Remark 5 on page 30).

The corresponding dynamic programming equations are

$$Q_t\left(x_{t-1}, \xi_{[t]}\right) = \inf_{x_t\in\mathbb{R}^{n_t}} \left\{c_t^{\mathsf{T}}x_t + \mathcal{Q}_{t+1}(x_t, \xi_{[t]}) : B_tx_{t-1} + A_tx_t = b_t,\ x_t \geq 0\right\}, \tag{3.109}$$

with

$$\mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right) := \rho_{t+1|\xi_{[t]}}\left[Q_{t+1}\left(x_t, \xi_{[t+1]}\right)\right]. \tag{3.110}$$

At the first stage problem

$$\min_{x_1\in\mathbb{R}^{n_1}} c_1^{\mathsf{T}}x_1 + \mathcal{Q}_2(x_1) \ \text{ s.t. } A_1x_1 = b_1,\ x_1 \geq 0, \tag{3.111}$$

should be solved. As it was pointed out before if the stagewise independence condition holds, then the cost-to-go functions $\mathcal{Q}_{t+1}(x_t)$ do not depend on the data process.

By the definition (2.60) of $\mathsf{AV@R}_\alpha$ it follows that $\mathcal{Q}_{t+1}\left(x_t, \xi_{[t]}\right)$ is equal to the optimal value of the problem

$$\min_{u_t} \mathbb{E}_{|\xi_{[t]}}\left\{(1-\lambda_{t+1})Q_{t+1}\left(x_t, \xi_{[t+1]}\right) + \lambda_{t+1}\big(u_t + \alpha_{t+1}^{-1}[Q_{t+1}\left(x_t, \xi_{[t+1]}\right) - u_t]_+\big)\right\}. \tag{3.112}$$

Thus we can write the dynamic programming equations (3.109)–(3.110) as follows. At the last stage $t = T$ we have that $Q_T(x_{T-1}, \xi_T)$ is equal to the optimal value of problem

$$\min_{x_T\in\mathbb{R}^{n_T}} c_T^{\mathsf{T}}x_T \ \text{ s.t. } B_Tx_{T-1} + A_Tx_T = b_T,\ x_T \geq 0, \tag{3.113}$$

and $\mathcal{Q}_T\left(x_{T-1}\xi_{[T-1]}\right)$ is equal to the optimal value of problem

$$\min_{u_{T-1}} \mathbb{E}_{|\xi_{[T-1]}}\left\{(1-\lambda_T)Q_T\left(x_{T-1}, \xi_T\right) + \lambda_T u_{T-1} + \lambda_T\alpha_T^{-1}[Q_T\left(x_{T-1}, \xi_T\right) - u_{T-1}]_+\right\}. \tag{3.114}$$

At stage $t = T-1$ we have that $Q_{T-1}(x_{T-2}, \xi_{[T-1]})$ is equal to the optimal value of problem

$$\begin{array}{ll} \min\limits_{x_{T-1}\in\mathbb{R}^{n_{T-1}}} & c_{T-1}^{\mathsf{T}}x_{T-1} + \mathcal{Q}_T(x_{T-1}, \xi_{[T-1]}) \\ \text{s.t.} & B_{T-1}x_{T-2} + A_{T-1}x_{T-1} = b_{T-1},\ x_{T-1} \geq 0. \end{array} \tag{3.115}$$

By using (3.114) and (3.115) we can write that $Q_{T-1}(x_{T-2}, \xi_{[T-1]})$ is equal to the optimal value of problem

$$\begin{array}{ll} \underset{x_{T-1}\in\mathbb{R}^{n_{T-1}}, u_{T-1}\in\mathbb{R}}{\text{Min}} & c_{T-1}^{\mathsf{T}} x_{T-1} + \lambda_T u_{T-1} + \mathcal{V}_T(x_{T-1}, u_{T-1}, \xi_{[T-1]}) \\ \text{s.t.} & B_{T-1}x_{T-2} + A_{T-1}x_{T-1} = b_{T-1},\ x_{T-1} \geq 0, \end{array} \tag{3.116}$$

where $\mathcal{V}_T(x_{T-1}, u_{T-1}, \xi_{[T-1]})$ is equal to the following conditional expectation

$$\mathbb{E}_{|\xi_{[T-1]}}\left\{(1-\lambda_T)Q_T\left(x_{T-1}, \xi_T\right) + \lambda_T \alpha_T^{-1}[Q_T\left(x_{T-1}, \xi_T\right) - u_{T-1}]_+\right\}. \tag{3.117}$$

By continuing this process backward we can write dynamic programming equations (3.109)–(3.110) for $t = T, ..., 2$ as

$$\begin{array}{rl} Q_t\left(x_{t-1}, \xi_{[t]}\right) = \underset{x_t\in\mathbb{R}^{n_t}, u_t\in\mathbb{R}}{\inf} & \big\{c_t^{\mathsf{T}} x_t + \lambda_{t+1}u_t + \mathcal{V}_{t+1}(x_t, u_t, \xi_{[t]}) : \\ & B_t x_{t-1} + A_t x_t = b_t,\ x_t \geq 0\big\}, \end{array} \tag{3.118}$$

where

$$\begin{array}{rl} \mathcal{V}_{t+1}\left(x_t, u_t, \xi_{[t]}\right) = \mathbb{E}_{|\xi_{[t]}} & \big\{(1-\lambda_{t+1})Q_{t+1}\left(x_t, \xi_{[t+1]}\right) \\ & +\lambda_{t+1}\alpha_{t+1}^{-1}\left[Q_{t+1}\left(x_t, \xi_{[t+1]}\right) - u_t\right]_+\big\}, \end{array} \tag{3.119}$$

with $\mathcal{V}_{T+1}(\cdot) \equiv 0$ and $\lambda_{T+1} := 0$. At the first stage problem

$$\underset{x_1\in\mathbb{R}^{n_1}, u_1\in\mathbb{R}}{\text{Min}}\ c_1^{\mathsf{T}} x_1 + \lambda_2 u_1 + \mathcal{V}_2(x_1, u_1)\ \text{ s.t. } A_1 x_1 = b_1,\ x_1 \geq 0, \tag{3.120}$$

should be solved. Note that in this formulation decision variables at t-th stage are $x_t \in \mathbb{R}^{n_t}$ and $u_t \in \mathbb{R}$. Note also that functions $\mathcal{V}_{t+1}\left(x_t, u_t, \xi_{[t]}\right)$ are convex in (x_t, u_t).

4 Inventory Model

4.1 The Newsvendor Problem

The classical newsvendor (also called newsboy) problem is the following. A newsvendor has to decide about quantity x of newspapers which he purchases from a distributor at the beginning of a day at the cost of c per unit. He can sell a newspaper at the price s per unit and unsold newspapers can be returned to the vendor at the price of r per unit. It is assumed that $0 \leq r < c < s$. If the demand d, i.e., the quantity of newspapers which he is able to sell at a particular day, turns out to be greater than or equal to the order quantity x, then he makes the profit $sx - cx = (s-c)x$, while if

d is less than x, his profit is $sd + r(x-d) - cx = (r-c)x + (s-r)d$. Thus the profit is a function of x and d and is given by

$$G(x,d) = \begin{cases} (s-c)x, & \text{if} \quad x \leq d, \\ (r-c)x + (s-r)d, & \text{if} \quad x > d. \end{cases}, \tag{4.1}$$

or equivalently

$$G(x,d) = \min\big\{(s-c)x, (r-c)x + (s-r)d\big\}. \tag{4.2}$$

The objective is to maximize the profit as a function of the quantity (decision variable) $x \geq 0$.

Closely related to this is the following inventory problem. Suppose that a company has to decide about order quantity x of a certain product to satisfy demand d. The cost of ordering is $c > 0$ per unit. If the demand d is larger than x, then the company makes an additional order for the unit price $b \geq 0$. The cost of this is equal to $b(d-x)$ if $d > x$, and is zero otherwise. On the other hand, if $d < x$, then holding cost of $h(x-d) \geq 0$ is incurred. The total cost is then equal to

$$F(x,d) = cx + b[d-x]_+ + h[x-d]_+ = \max\big\{(c-b)x + bd, (c+h)x - hd\big\}. \tag{4.3}$$

We assume that $b > c$, i.e., the back order penalty cost is *larger* than the ordering cost. The objective is to minimize the total cost $F(x,d)$, with x being the decision variable. Unless stated otherwise in the following we deal with the inventory model (see, e.g., Zipkin [34] for a thorough discussion of the inventory model).

In both problems one has to make a decision before knowing realization of the demand d, i.e., the decision should be made in conditions of uncertainty. There are several ways how the uncertainty can be modeled. One approach is to specify an uncertainty set, say interval $[l,u] \subset \mathbb{R}_+$, of possible realizations of the demand d and to be prepared for the worst possible scenario. This leads to the following worst case formulation of the inventory problem

$$\operatorname*{Min}_{x \geq 0} \left\{ \psi(x) := \max_{d \in [l,u]} F(x,d) \right\}. \tag{4.4}$$

Similar worst case formulation can be written for the newsvendor problem.

Since $F(x,d)$ is convex in d, we have that $\psi(x) = \max\{F(x,l), F(x,u)\}$. The function $\psi(x)$ is a piecewise linear convex function. Recalling that $b > c$, it is straightforward to verify that the optimal solution of problem (4.4) is attained at the point

$$x^* = \frac{hl + bu}{h + b}. \tag{4.5}$$

Of course, if $l = u$ (i.e., the demand is known), then the best (optimal) decision is to order the known quantity of the demand. On the other hand, if the uncertainty interval is large, then the "worst case" solution could be quite conservative. For example, if the back order cost (per unit) b is much bigger than the holding cost h, then x^* is practically equal to the largest possible realization u of the demand.

An alternative approach is to view the demand as a random variable and to perform an optimization *on average*. For the inventory model the corresponding optimization problem can be written as follows

$$\operatorname*{Min}_{x \geq 0} \big\{ f(x) := \mathbb{E}[F(x, D)] \big\}. \tag{4.6}$$

The expectation $\mathbb{E}[F(x, D)]$ is taken here with respect to a specified probability distribution of the random demand D. Suppose that random variable D has a finite first order moment, i.e., $\mathbb{E}|D| < \infty$, and hence the expectation $f(x)$ is well defined and finite values. Since $F(x, d)$ is convex in x, it follows that the function $f(x)$ is convex. We have that for $x \neq d$,

$$\frac{\partial F(x, d)}{\partial x} = \begin{cases} c - b & \text{if} \quad x < d, \\ c + h & \text{if} \quad x > d \end{cases}$$

It follows that if D has a continuous distribution, and hence for any x probability of the event "$D = x$" is zero, then the expectation function $f(x)$ is differentiable and

$$f'(x) = c - b\,\mathsf{Pr}(x < D) + h\,\mathsf{Pr}(x > D) = c - b + (b + h)H(x), \tag{4.7}$$

where $H(x) := \mathsf{Pr}(D \leq x)$ is the cumulative distribution function (cdf) of the random variable D. By the optimality condition $f'(x) = 0$ (recall that the function $f(\cdot)$ is convex), we obtain that an optimal solution of problem (4.6) satisfies equation $H(x) = (b - c)/(b + h)$. Recall that for $\kappa \in (0, 1)$, the left and right side κ-quantiles of the distribution of D are defined as $\inf\{t : H(t) \geq \kappa\}$ and $\sup\{t : H(t) \leq \kappa\}$, respectively. In particular, the left side κ-quantile is denoted $H^{-1}(\kappa)$. We obtain that the set of optimal solutions of problem (4.6) is given by the interval of κ-quantiles with $\kappa := (b - c)/(b + h)$. In fact this holds for general (not necessarily continuous) distributions of D.

What could be a possible justification for formulation (4.6) of the inventory problem? If the same operation is repeated many times, for an *independent identically distributed* (iid) sequence $D_1, ..., $ of realizations of the random variable D, then by the Law of Large Numbers (LLN) we have that $n^{-1} \sum_{i=1}^{n} F(x, D_i)$ converges with probability (w.p.1) to $f(x)$. Indeed, in such a situation

$$\bar{x} = H^{-1}\left(\tfrac{b-c}{b+h}\right) \tag{4.8}$$

gives an optimal decision *on average*. This formulation, however, has several deficiencies. The optimal solution $\bar{x}$, given in (4.8), depends on the specified probability distribution of the demand. This distribution may be not known and could be estimated at best if a historical data is available, it could change with time, etc. Moreover, for a particular realization of the demand, the value $F(x, D)$ could be quite different from the respective expectation $f(x)$, and formulation (4.6) does not take into account an involved risk of everyday operations.

Suppose now that we have a partial information about probability distribution of D. That is, we can specify a family $\mathfrak{M}$ of probability measures on $\mathbb{R}_+$ and consider the following worst case distribution problem

$$\operatorname*{Min}_{x \geq 0} \left\{ \phi(x) := \sup_{P \in \mathfrak{M}} \mathbb{E}_P[F(x, D)] \right\}, \tag{4.9}$$

where the notation $\mathbb{E}_P[F(x, D)]$ emphasizes that the expectation is taken with respect to the probability distribution P of the random variable D. Let us discuss some examples.

Example 7 Let $\mathfrak{M}$ be the set of all probability distributions supported on a given interval $[l, u] \subset \mathbb{R}_+$. Then (see Theorem 2.12) the maximum in (4.9) is attained at an atomic measure supported on a single point of the interval $[l, u]$ (Dirac measure), and hence problem (4.9) becomes the (deterministic) worst case problem (4.4). $\diamondsuit$

Example 8 Suppose now that in addition to the lower and upper bounds of the demand we know its mean (expected value) $\mu = \mathbb{E}[D]$, of course $\mu \in [l, u]$. That is, $\mathfrak{M}$ is the set of probability distributions supported on the interval $[l, u]$ and having mean μ.

Since the function $F(x, d)$ is convex and continuous in d, we have by Theorem 2.13 that for any x the worst probability measure in (4.9) is the measure supported on points l and u. The corresponding probabilities are uniquely defined to be $(u - \mu)/(u - l)$ and $(\mu - l)/(u - l)$, respectively. Therefore problem (4.9) is reduced to problem (4.6) with the respective cdf

$$H(t) = \begin{cases} 0 & \text{if} \quad t < l, \\ \frac{u-\mu}{u-l} & \text{if} \quad l \leq t < u, \\ 1 & \text{if} \quad u \leq t, \end{cases}$$

and hence by (4.8) the optimal solution of (4.9) is

$$\tilde{x} = \begin{cases} l & \text{if} \quad \frac{b-c}{b+h} < \frac{u-\mu}{u-l}, \\ u & \text{if} \quad \frac{b-c}{b+h} > \frac{u-\mu}{u-l}. \end{cases} \tag{4.10}$$

If $\frac{b-c}{b+h} = \frac{u-\mu}{u-l}$, then the set of optimal solutions of (4.9) coincides with the interval $[l, u]$.

Suppose now that the mean μ is not known exactly but is estimated to be in a subinterval $[\alpha, \beta]$ of the interval $[l, u]$. That is, the set $\mathfrak{M}$ is defined as the set of probability measures on the interval $[l, u]$ having mean μ restricted to a given interval $[\alpha, \beta]$. By Theorem 2.13 it will suffice to solve problem (4.9) for probability measures supported on points l and u, and with mean equal either α or β. That is, problem (4.9) can be reduced to solving the following problem

$$\operatorname*{Min}_{x \geq 0} \left\{ \max \left[(u-\alpha)F(x,l) + (\alpha - l)F(x,u), (u-\beta)F(x,l) + (\beta - l)F(x,u) \right] \right\}, \tag{4.11}$$

up to the factor $(u-l)^{-1}$. $\Diamond$

Example 9 (unimodal distributions) Let now $\mathfrak{M}$ be the set unimodal distributions on the interval $[l, u]$ with given mode $\mu \in [l, u]$. Recall that a distribution on the interval $[l, u]$ is said to be *unimodal*, with mode μ, if its cumulative distribution function is convex on the interval $[l, \mu]$ and concave on the interval $[\mu, u]$. Equivalently, the distribution is unimodal if it is a mixture of the distribution concentrated at the single point μ and a distribution with density function that is nondecreasing on $[l, \mu)$ and nonincreasing on $(\mu, u]$ (the density function could be discontinuous, in particular at μ). By a result due to Khintchine we have that a distribution is unimodal on $[u, l]$ with mode μ iff it is the distribution of the random variable $D = \mu + UZ$, where U and Z are independent random variables, U is uniformly distributed on the interval $[0, 1]$ and the distribution of Z is arbitrary on the interval $[l - \mu, u - \mu]$. For $P \in \mathfrak{M}$ we can write then

$$\mathbb{E}_P[F(x, D)] = \mathbb{E}_Z \left\{ \mathbb{E}_{D|Z}[F(x, D)] \right\} = \mathbb{E}_{P'}[G(x, Z)], \tag{4.12}$$

where P' is the probability distribution of Z and

$$G(x, z) := \mathbb{E}[F(x, D)|Z = z] = \mathbb{E}[F(x, \mu + Uz)].$$

Let $\mathfrak{P}$ be the set of probability distributions on the interval $[l - \mu, u - \mu]$. Since $F(x, \mu + Uz)$ is convex in z, it follows that $G(x, z)$ is convex in z. Consequently the maximum of $\mathbb{E}_P[G(x, Z)]$ over $P \in \mathfrak{P}$ is attained at a measure concentrated at a single point either $l - \mu$ or $u - \mu$. Translated back into the set $\mathfrak{M}$ of unimodal distributions of D this means that the maximum of $\mathbb{E}_P[F(x, D)]$ over $P \in \mathfrak{M}$ is attained at either uniform distribution on the interval $[l, \mu]$ or uniform distribution on the interval $[\mu, u]$ (if $\mu = l$, then the corresponding distribution is reduced to

the distribution concentrated at the single point l, and similarly if $\mu = u$). The corresponding worst case distribution problem becomes

$$\operatorname*{Min}_{x\geq 0} \left\{\gamma(x) := \max\left[G(x, l-\mu), G(x, u-\mu)\right]\right\}. \tag{4.13}$$

Since $F(x, \mu+Uz)$, and hence $G(x, z)$, is convex in x it follows that the function $\gamma(x)$ is convex.

The function $G(x, z)$ can be computed in a closed form. Suppose, for example, that $l = 0$ and $\mu = u$, i.e., the considered set $\mathfrak{M}$ of distributions of D consists of distributions having nondecreasing density on the interval $[0, u]$ and may be a positive probability of $D = u$. We have then that for $x \in [0, u]$, $G(x, 0) = F(x, u) = cx + b(u - x)$ and

$$G(x, -u) = \int_0^1 F(x, tu)dt = \tfrac{1}{2}u^{-1}(b+h)x^2 - (b-c)x + \tfrac{1}{2}bu.$$

It follows that the optimal solution of minimax problem (4.13), and hence of (4.9), is

$$\hat{x} = u\sqrt{\frac{b}{b+h}}. \tag{4.14}$$

$\Diamond$

4.2 Multistage Inventory Problem

Consider the following minimax multistage formulation of inventory model

$$\begin{array}{ll} \operatorname*{Min}\limits_{x_t \geq y_t} \sup\limits_{P\in\mathfrak{M}} & \mathbb{E}_P\left[\sum_{t=1}^T c_t(x_t - y_t) + \psi_t(x_t, D_t)\right] \\ \text{s.t.} & y_{t+1} = x_t - D_t,\ t = 1, ..., T-1. \end{array} \tag{4.15}$$

Here y_1 is a given initial inventory level, c_t, b_t, h_t are the ordering, backorder penalty, and holding costs per unit, respectively, at time t, and

$$\psi_t(x_t, d_t) := b_t[d_t - x_t]_+ + h_t[x_t - d_t]_+.$$

We assume that $b_t > c_t > 0$ and $h_t \geq 0$, $t = 1, ..., T$, and that $\mathfrak{M}$ is a set of probability measures (distributions) of the demand process vector $D = (D_1, ..., D_T) \in \mathbb{R}_+^T$. The minimization in (4.15) is performed over (nonanticipative) policies of the form $x_1, x_2(D_{[1]}), ..., x_T(D_{[T-1]})$ satisfying the feasibility constraints of (4.15) for almost every realization of the demand process $(D_1, ..., D_T)$. As before, $D_{[t]} := (D_1, ..., D_t)$ denotes history of the process up to time t.

If $\mathfrak{M} = \{P\}$ consists of a single distribution, then (4.15) becomes a standard (risk neutral) formulation of the multistage inventory model. Let us consider some examples.

4.2.1 Robust Multistage Inventory Problem

Suppose that $\mathfrak{M}$ is the set of all probability distributions supported on a given compact (i.e., bounded and closed) nonempty set $\mathfrak{D} \subset \mathbb{R}^T_+$. This is a particular case of the setting considered in section 3.5. In that case the maximum in (4.15), with respect to $P \in \mathfrak{M}$, is attained at a distribution concentrated at a single point of $\mathfrak{D}$, and hence (4.15) can be written as the following minimax problem

$$\begin{array}{cc} \underset{x_t \geq y_t}{\text{Min}} \sup\limits_{d_{[T]} \in \mathfrak{D}} & \left\{ \sum\limits_{t=1}^{T} c_t(x_t - y_t) + \psi_t(x_t, d_t) \right\} \\ \text{s.t.} & y_{t+1} = x_t - d_t,\ t = 1, ..., T-1. \end{array} \tag{4.16}$$

Again the minimization in (4.16) is performed over (nonanticipative) policies $x_1, x_2(d_{[1]}), ..., x_T(d_{[T-1]})$ satisfying the feasibility constraints. The above problem (4.16) can be viewed as a robust formulation of the inventory model with the uncertainty set $\mathfrak{D}$. As it wa shown in section 3.5 the minimax problem (4.16) is equivalent to the corresponding nested formulation and is time consistent.

The dynamic programming equations for problem (4.16) can be written as follows. At the last stage $t = T$, for given (observed) inventory level y_T and given (observed) demand values $(d_1, ..., d_{T-1})$, we need to solve the problem:

$$\underset{x_T \geq y_T}{\text{Min}} \left\{ c_T(x_T - y_T) + \sup_{(d_1, ..., d_T) \in \mathfrak{D}} \psi_T(x_T, d_T) \right\}. \tag{4.17}$$

The optimal value of problem (4.17) depends on y_T and $d_{[T-1]}$ and is denoted $Q_T(y_T, d_{[T-1]})$. Continuing in this way, for $t = T-1, \ldots, 2$, the corresponding cost-to-go functions $Q_t(y_t, d_{[t-1]})$ are given as optimal values of the respective problems:

$$\underset{x_t \geq y_t}{\text{Min}} \left\{ c_t(x_t - y_t) + \sup_{d'_{[T]} \in \mathfrak{D}} \left[\psi_t(x_t, d'_t) + Q_{t+1}\big(x_t - d'_t, d'_{[t]}\big) : d'_{[t-1]} = d_{[t-1]} \right] \right\}. \tag{4.18}$$

Finally, at the first stage we need to solve problem

$$\underset{x_1 \geq y_1}{\text{Min}}\, c_1(x_1 - y_1) + \sup_{d_{[T]} \in \mathfrak{D}} \big[\psi_1(x_1, d_1) + Q_2\left(x_1 - d_1, d_1\right) \big]. \tag{4.19}$$

Let us observe that the cost-to-go function $Q_T(y_T, d_{[T-1]})$ is convex in y_T. Indeed, the function $\varphi(x_T) := \sup_{d_{[T]} \in \mathfrak{D}} \psi_T(x_T, d_T)$ is given by maximum of convex functions, and hence is convex. It follows that the function $c_T(x_T - y_T) + \varphi(x_T) + \delta(y_T - x_T)$ is convex jointly in x_T and y_T (here $\delta(t) = 0$ if $t \leq 0$, and $\delta(t) = +\infty$ if $t > 0$). It

remains to note that the optimal value of problem (4.17) is equal to the minimum of this function over $x_T \in \mathbb{R}$. It is straightforward then to verify by induction in $t = T, ...,$ that each cost-to-go function $Q_t(y_t, d_{[t-1]})$ is convex in y_t. It also could be noted that the set $\mathfrak{D}$ in (4.18) can be replaced by projection of $\mathfrak{D}$ onto $\mathbb{R}^t$, that is by the set

$$\mathfrak{D}_t := \left\{ d_{[t]} : \exists\, d'_{[T]} \in \mathfrak{D} \text{ such that } d_{[t]} = d'_{[t]} \right\}.$$

Suppose now that the uncertainty set $\mathfrak{D}$ is given by the direct product $\mathfrak{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_T$ of (finite) intervals $\mathcal{D}_t := [l_t, u_t] \subset \mathbb{R}_+$, $t = 1, ..., T$. This implies, of course, that the stagewise independence condition holds in that setting. Then the cost-to-go function at the last stage is

$$Q_T(y_T) = \inf_{x_T \geq y_T} \left\{ c_T(x_T - y_T) + \sup_{d_T \in \mathcal{D}_T} \psi_T(x_T, d_T) \right\}. \tag{4.20}$$

And so on for $t = T - 1, ..., 2$, dynamic programming equations (4.18) can be written as

$$Q_t(y_t) = \inf_{x_t \geq y_t} \left\{ c_t(x_t - y_t) + \sup_{d_t \in \mathcal{D}_t} \left[\psi_t(x_t, d_t) + Q_{t+1}\left(x_t - d_t\right) \right] \right\}. \tag{4.21}$$

Note that here the cost-to-go function $Q_t(y_t)$, $t = 2, \ldots, T$, is independent of $d_{[t-1]}$, and is convex.

The basestock policy for the above problem is defined as $\bar{x}_t := \max\{y_t, x_t^*\}$, where x_t^* is an optimal solution of

$$\operatorname*{Min}_{x_t \in \mathbb{R}} \left\{ c_t x_t + \sup_{d_t \in \mathcal{D}_t} \left[\psi_t(x_t, d_t) + Q_{t+1}\left(x_t - d_t\right) \right] \right\}, \tag{4.22}$$

and $y_t = \bar{x}_{t-1} - d_{t-1}$, $t = 2, ..., T$, with y_1 being given (if problem (4.22) has more than one optimal solution, we can take the smallest one). By convexity of cost-to-go functions we have that the basestock policy $\bar{x}_t = \bar{x}_t(d_{[t-1]})$ satisfies the dynamic programming equations (4.21) and hence is optimal.

It could be noted that since function $\psi_t(x_t, d_t) + Q_{t+1}\left(x_t - d_t\right)$ is convex in d_t, the corresponding maximum in (4.21) is attained either at $d_t = l_t$ or $d_t = u_t$. Therefore the uncertainty set $\mathfrak{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_T$ in the minimax (robust) formulation (4.16) can be replaced by the set $\{l_1, u_1\} \times \cdots \times \{l_T, u_T\}$ having 2^T elements. That is, problem (4.16) can be formulated as a minimax problem with a finite number $N = 2^T$ of scenarios.

Suppose now that the additional (linear) constraint $a^{\mathsf{T}} d \leq b$ is added to the definition of the uncertainty set $\mathfrak{D}$, i.e.,

$$\mathfrak{D} := (\mathcal{D}_1 \times \cdots \times \mathcal{D}_T) \cap \{d \in \mathbb{R}^T : a^{\mathsf{T}} d \leq b\},$$

for some $a \in \mathbb{R}^T$ and $b \in \mathbb{R}$. Suppose that the set $\mathfrak{D}$ is nonempty. Then dynamic equations (4.18) take the form, for $t = T, ..., 2$,

$$Q_t(y_t, d_{[t-1]}) = \inf_{x_t \geq y_t} \Big\{ c_t(x_t - y_t) + \sup_{\substack{d_t \in \mathcal{D}_t, ..., d_T \in \mathcal{D}_T \\ a_1 D_1 + ... + a_{t-1} d_{t-1} + a_t d_t + ... + a_T d_T \leq b}} \big[\psi_t(x_t, d_t) + Q_{t+1}\left(x_t - d_t, d_{[t]}\right) \big] \Big\}, \tag{4.23}$$

with $Q_{T+1}(\cdot, \cdot) \equiv 0$ by definition.

In that case the cost-to-go function $Q_t(y_t, d_{[t-1]})$ depends only on y_t and $W_{t-1} := a_1 d_1 + ... + a_{t-1} d_{t-1}$, and in these variables equations (4.23) can be written as

$$Q_t(y_t, W_{t-1}) = \inf_{x_t \geq y_t} \Big\{ c_t(x_t - y_t) + \sup_{\substack{d_t \in \mathcal{D}_t, ..., d_T \in \mathcal{D}_T \\ a_t d_t + ... + a_T d_T \leq b - W_{t-1}}} \big[\psi_t(x_t, d_t) + Q_{t+1}\left(x_t - d_t, W_{t-1} + a_t d_t\right) \big] \Big\}. \tag{4.24}$$

Note that the cost-to-go functions $Q_t(y_t, W_{t-1})$ are defined only for such W_{t-1} that the constraints in the maximization problem in the right hand side of (4.24) are feasible. We see that adding just one linear constraint significantly complicates the problem.

4.2.2 Inventory Problem with Moment Constraints

Suppose that in addition to the uncertainty set $\mathfrak{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_T$, given by the direct product of (finite) intervals $\mathcal{D}_t := [l_t, u_t] \subset \mathbb{R}_+$, we know respective means $\mathbb{E}[D_t]$. That is, let $\mathcal{M}_t$ be the set of probability distributions supported on the interval $[l_t, u_t]$ and having given mean $\mu_t \in [l_t, u_t]$, $t = 1, ..., T$. Let

$$\mathfrak{M} := \{P = P_1 \times \cdots \times P_T : P_t \in \mathcal{M}_t,\ t = 1, ..., T\}$$

consists of probability distributions with independent components from respective sets $\mathcal{M}_t$. This implies the stagewise condition.

The corresponding cost-to-go functions are given by the following dynamic equations, $t = T, ..., 2$,

$$Q_t(y_t) = \inf_{x_t \geq y_t} \left\{ c_t(x_t - y_t) + \sup_{P \in \mathcal{M}_t} \mathbb{E}_P \big[\psi_t(x_t, D_t) + Q_{t+1}\left(x_t - D_t\right) \big] \right\}, \tag{4.25}$$

where $Q_{T+1}(\cdot) \equiv 0$.

It is straightforward to verify by induction that the functions $Q_t(\cdot)$ are convex, and hence by Theorem 2.13 we have here that the maximum in (4.25), over probability

measures $P \in \mathcal{M}_t$, is attained at the probability measure supported on points l_t and u_t with respective probabilities $p_t = (u_t - \mu_t)/(u_t - l_t)$ and $1 - p_t = (\mu_t - l_t)/(u_t - l_t)$. Therefore the respective problem (4.15) is reduced here to the corresponding problem with single probability distribution of the demand process with the random variables D_t being independent of each other and having discrete distribution $\mathsf{Pr}(D_t = l_t) = p_t$ and $\mathsf{Pr}(D_t = u_t) = 1 - p_t$, $t = 1, ..., T$. It follows that the minimax and nested formulations here are equivalent and the problem is time consistent.

5 Computational Approaches to Multistage Stochastic Programming

5.1 Sample Average Approximations of Multistage Problems

Consider the (risk neutral) formulation (3.5) of multistage stochastic programming problems. It is assumed there that the probability distribution of the data process $\xi_1, ..., \xi_T$ is known, or better to say is specified at the modeling stage of the considered problem. A particular realization of the random process $\xi_1, ..., \xi_T$ (recall that ξ_1 is deterministic) is called *scenario*. If the number of scenarios (realizations of the data process) is finite, then problem (3.5) can be written as one large finite dimensional deterministic problem. In particular, if the problem is linear, say of the form (3.13), then this becomes a large linear programming problem.

By generating a sample of the random data process we can construct a *Sample Average Approximation* of the "true" problem (3.5). To this end the Monte Carlo sampling approach can be employed in the following way. First, a random sample $\xi_2^1, ..., \xi_2^{N_1}$ of N_1 realizations of the random vector ξ_2 is generated. For each ξ_2^j, $j = 1, ..., N_1$, a random sample of size N_2 of ξ_3, according to the distribution of ξ_3 conditional on $\xi_2 = \xi_2^j$, is generated and so forth for later stages. That is, at stage $t = 1, ..., T-1$, given a generated realization $\xi_{[t]}$ of the random process up to time t, N_t realizations of ξ_{t+1} are generated according the distribution of ξ_{t+1} conditional on $\xi_{[t]}$. Here, conditional on $\xi_{[t]}$, the samples of ξ_{t+1} are generated independently of each other. We refer to this procedure as the *conditional sampling*. In that way the true distribution of the random data process is discretized, with every generated path of the process taken with equal probability. We refer to each generated path as scenario and to the collection of all scenarios as *scenario tree*. Note that the total number of scenarios $\mathcal{N} = \prod_{t=1}^{T-1} N_t$, and hence the probability of each generated scenario is $1/\mathcal{N}$.

It could be noted that this construction of the scenario tree does not inherits a possible Markovian structure of the data process. In particular, when the data process is stagewise independent, the constructed scenario tree does not possess the stagewise

independence property. If the original process is stagewise independent it is possible to proceed in the following alternative way. Independent of each other random samples $\xi_t^1, ..., \xi_t^{N_{t-1}}$ of respective ξ_t, $t = 2, ..., T$, are generated and the corresponding scenario tree is constructed by connecting every ancestor node at stage $t-1$ with the same set of children nodes $\xi_t^1, ..., \xi_t^{N_{t-1}}$. In that way stagewise independence is preserved in the scenario tree generated by conditional sampling. We refer to this sampling scheme as the *identical conditional sampling*. Denote by ϑ^* and $\hat{\vartheta}_{\mathcal{N}}$ the optimal values of the true problem (3.5) and the constructed SAA problem, respectively. We have that on average $\hat{\vartheta}_{\mathcal{N}}$ is less than or equal to ϑ^*, i.e.,

$$\vartheta^* \geq \mathbb{E}\big[\hat{\vartheta}_{\mathcal{N}}\big]. \tag{5.1}$$

The inequality (5.1) holds for the conditional sampling, discussed above, and for the identical conditional sampling in case of stagewise independence.

If we measure computational complexity, of the true problem, in terms of the number of scenarios required to approximate true distribution of the random data process with a reasonable accuracy, the conclusion is rather pessimistic. In order for the optimal value and solutions of the SAA problem to converge to their true counterparts all sample sizes $N_1, ..., N_{T-1}$ should tend to infinity. Furthermore, available estimates of the sample sizes required for a first stage solution of the SAA problem to be ε-optimal for the true problem, with a given confidence (probability), sums up to a number of scenarios which grows as $O(\varepsilon^{-2(T-1)})$ with decrease of the error level $\varepsilon > 0$ (cf., [28],[30, section 5.8.2]). This indicates that from the point of view of the number of scenarios, complexity of multistage programming problems grows exponentially with increase of the number of stages. From an applications point of view the multistage programming is too important to be dismissed that easily. We discuss below some possible approaches to solve specific classes of multistage stochastic programs.

5.2 Stochastic Dual Dynamic Programming Method

In this section we deal with the linear multistage stochastic programming problem (3.13). The dynamic programming equations for that problem are formulated in (3.14)–(3.15). There are several difficulties in trying to solve these equations numerically. We assume in the subsequent analysis that the data process is *stagewise independent* (see section 5.2.1 for a discussion of the stagewise independence condition). From the point of view of dynamic programming this is a significant simplification since then the cost-to-go functions $Q_{t+1}(x_t, \xi_{t+1})$ do not depend on $\xi_{[t]}$, $t = 1, ..., T-1$, and their expectations $\mathcal{Q}_{t+1}(x_t) = \mathbb{E}[Q_{t+1}(x_t, \xi_{t+1})]$ do not depend on the data process. Yet we still face two basic problems, namely how to compute the expectations

$\mathbb{E}[Q_{t+1}(x_t, \xi_{t+1})]$ and how to represent functions $\mathcal{Q}_{t+1}(x_t)$ in a numerically accessible way. If the dimension n_t of x_t is small, then one can accurately represent $\mathcal{Q}_{t+1}(x_t)$ by making a discretization of the domain of x_t in the space $\mathbb{R}^{n_t}$. However, the number of discretization points required to represent $\mathcal{Q}_{t+1}(x_t)$ in a reasonably accurate way grows exponentially with increase of the dimension n_t and this approach becomes impractical, say, for $n_t \geq 4$. This is the so-called "curse of dimensionality" problem well known in dynamic programming.

In order to resolve these two problems one needs to compromise on some type of approximations. As far as computing the expectations is concerned we use Monte Carlo sampling techniques. That is, an SAA problem is constructed by employing the *identical conditional sampling* approach based on independently generated samples

$$\xi_t^j = (c_{tj}, A_{tj}, B_{tj}, b_{tj}), \quad j = 1, ..., N_{t-1}, \tag{5.2}$$

of ξ_t, $t = 2, ..., T$. Recall that in that way the stagewise independence is preserved in the constructed SAA problem. If, for example, we use the same sample size $N_t = N$, $t = 1, ..., T-1$, at all stages, then N should be of order $O(\varepsilon^{-2})$ for the first stage solution of the SAA problem to be ε-optimal for the true problem with a given confidence (probability) close to one. In that case the total number of scenarios $\mathcal{N} = N^{T-1}$ quickly becomes astronomically large with increase of the number of stages T even for a moderate values of N, say $N = 50$. This makes a scenarios based approach practically inapplicable, say, for $T > 4$. So we pursue an approach of approximately solving the dynamic programming equations.

There are various ways how the dynamic programming equations can be approximated. We discuss below the so-called *Stochastic Dual Dynamic Programming* (SDDP) method, originated in Pereira and Pinto [17], applied to the SAA problem. Of course, it shouldn't be forgotten that we really want to solve the "true" problem and a constructed SAA problem is just an approximation. For the SAA problem the dynamic programming equations take the form[19]

$$Q_{tj}(x_{t-1}) = \inf_{x_t \in \mathbb{R}^{n_t}} \left\{ c_{tj}^{\mathsf{T}} x_t + \mathcal{Q}_{t+1}(x_t) : B_{tj} x_{t-1} + A_{tj} x_t = b_{tj},\ x_t \geq 0 \right\}, \tag{5.3}$$

for $j = 1, ..., N_{t-1}$, with

$$\mathcal{Q}_{t+1}(x_t) = \frac{1}{N_t} \sum_{j=1}^{N_t} Q_{t+1,j}(x_t), \tag{5.4}$$

[19]Compared with previous notation we denote here by $Q_{tj}(x_{t-1})$ value $Q_t(x_{t-1}, \xi_t^j)$ of the cost-to-go function.

$t = T, ..., 2$ and $Q_{T+1}(\cdot) \equiv 0$. The optimal value of the SAA problem is given by the optimal value of the first stage problem

$$\underset{x_1 \in \mathbb{R}^{n_1}}{\text{Min}} \; c_1^{\mathsf{T}} x_1 + \mathcal{Q}_2(x_1) \;\; \text{s.t. } A_1 x_1 = b_1, \; x_1 \geq 0. \tag{5.5}$$

Note again that because the stagewise independence is preserved in the constructed SAA problem, the cost-to-go functions $\mathcal{Q}_{t+1}\big(x_t\big)$ do not depend on the (sampled) data process. Note also that the cost-to-go functions $\mathcal{Q}_{t+1}\big(x_t\big)$ are convex, and since the number of scenarios of the SAA problem is finite are piecewise linear.

The basic idea of the SDDP approach is to approximate the cost-to-go functions $\mathcal{Q}_{t+1}\big(x_t\big)$ by supporting hyperplanes. This idea forms a basis of various approaches to solving linear multistage programs. What distinguishes the SDDP method is a specific way how the supporting hyperplanes are constructed. The SDDP algorithm consists of the backward and forward steps. In the subsequent analysis we distinguish between cutting and supporting planes (hyperplanes) of a given convex function $Q : \mathbb{R}^n \to \mathbb{R}$. We say that an affine function $\ell(x) = \alpha + \beta^{\mathsf{T}} x$ is a *cutting plane*, of $Q(x)$, if $Q(x) \geq \ell(x)$ for all $x \in \mathbb{R}^n$. Note that cutting plane $\ell(x)$ can be strictly smaller than $Q(x)$ for all $x \in \mathbb{R}^n$. If, moreover, $Q(\bar{x}) = \ell(\bar{x})$ for some $\bar{x} \in \mathbb{R}^n$, it is said that $\ell(x)$ is a *supporting plane* of $Q(x)$. This supporting plane is given by $\ell(x) = Q(\bar{x}) + g^{\mathsf{T}}(x - \bar{x})$ for some subgradient $g \in \partial Q(\bar{x})$.

A *backward* step of the SDDP algorithm, applied to the SAA problem, can be described as follows. Let $\bar{x}_t \in \mathbb{R}^{n_t}$ be a trial decision point at stage $t = 1, ..., T-1$ (it is possible to use more than one trial point at every stage, how these trial points are constructed will be discussed in the forward step of the algorithm described below), and let

$$\mathfrak{Q}_t(x_{t-1}) = \max_{k \in \mathcal{I}_t} \left\{ \alpha_{tk} + \beta_{tk}^{\mathsf{T}} x_{t-1} \right\}, \; t = 2, ..., T, \tag{5.6}$$

be a current approximation of the cost-to-go function $\mathcal{Q}_t(\cdot)$, given by the maximum of a (finite) collection of its cutting planes. At stage $t = T$ we solve the problem

$$\underset{x_T}{\text{Min}} \, c_{Tj}^{\mathsf{T}} x_T \;\; \text{s.t. } B_{Tj} x_{T-1} + A_{Tj} x_T = b_{Tj}, \; x_T \geq 0, \tag{5.7}$$

for $x_{T-1} = \bar{x}_{T-1}$ and $j = 1, ..., N_{T-1}$. Note that the optimal value of problem (5.7) is equal to $Q_{Tj}\big(x_{T-1}\big)$.

Let $\tilde{x}_{Tj}$ be an optimal solution of problem (5.7) and $\tilde{\pi}_{Tj}$ be an optimal solution of its dual

$$\underset{\pi_T}{\text{Max}} \, \pi_T^{\mathsf{T}} \left(b_{Tj} - B_{Tj} x_{T-1} \right) \;\; \text{s.t. } \; A_{Tj}^{\mathsf{T}} \pi_T \leq c_{Tj}, \tag{5.8}$$

for $x_{T-1} = \bar{x}_{T-1}$ and $j = 1, ..., N_{T-1}$. Then

$$\ell_T(x_{T-1}) := \mathcal{Q}_T\big(\bar{x}_{T-1}\big) + g_T^{\mathsf{T}}\big(x_{T-1} - \bar{x}_{T-1}\big), \tag{5.9}$$

where

$$\mathcal{Q}_T\big(\bar{x}_{T-1}\big) = \frac{1}{N_{T-1}} \sum_{j=1}^{N_{T-1}} c_{Tj}^{\mathsf{T}} \tilde{x}_{Tj} \ \text{ and } \ g_T = -\frac{1}{N_{T-1}} \sum_{j=1}^{N_{T-1}} B_{Tj}^{\mathsf{T}} \tilde{\pi}_{Tj}, \tag{5.10}$$

is a supporting plane for $\mathcal{Q}_T(\cdot)$ at $\bar{x}_{T-1}$. This supporting plane is added to the collection of supporting planes of $\mathfrak{Q}_T(\cdot)$, i.e., $\mathfrak{Q}_T(\cdot)$ is replaced by $\max\big\{\mathfrak{Q}_T(\cdot), \ell_T(\cdot)\big\}$.

Now going one stage back let us recall that $Q_{T-1,j}(\bar{x}_{T-2})$ is equal to the optimal value of problem

$$\operatorname*{Min}_{x_{T-1}} c_{T-1,j}^{\mathsf{T}} x_{T-1} + \mathcal{Q}_T(x_{T-1}) \ \text{ s.t. } B_{T-1,j}\bar{x}_{T-2} + A_{T-1,j}x_{T-1} = b_{T-1,j}, \ x_{T-1} \geq 0. \tag{5.11}$$

However, function $\mathcal{Q}_T(\cdot)$ is not available. Therefore we replace it by $\mathfrak{Q}_T(\cdot)$ and hence consider problem

$$\operatorname*{Min}_{x_{T-1}} c_{T-1,j}^{\mathsf{T}} x_{T-1} + \mathfrak{Q}_T(x_{T-1}) \ \text{ s.t. } B_{T-1,j}\bar{x}_{T-2} + A_{T-1,j}x_{T-1} = b_{T-1,j}, \ x_{T-1} \geq 0. \tag{5.12}$$

Recall that $\mathfrak{Q}_T(\cdot)$ is given in the form (5.6) by maximum of affine functions. Therefore we can write problem (5.12) as the following linear programming problem

$$\begin{array}{ll} \operatorname*{Min}\limits_{x_{T-1},\theta} & c_{T-1,j}^{\mathsf{T}} x_{T-1} + \theta \\ \text{s.t.} & B_{T-1,j}\bar{x}_{T-2} + A_{T-1,j}x_{T-1} = b_{T-1,j}, \ x_{T-1} \geq 0, \\ & \theta \geq \alpha_{Tk} + \beta_{Tk}^{\mathsf{T}} x_{T-1}, \ k \in \mathcal{I}_T. \end{array} \tag{5.13}$$

Consider the optimal value, denoted $\underline{Q}_{T-1,j}(\bar{x}_{T-2})$, of problem (5.12) (of problem (5.13)), and let $\tilde{\pi}_{T-1,j}$ be the (partial) vector of an optimal solution of the dual of problem (5.13) corresponding to the constraint $B_{T-1,j}\bar{x}_{T-2} + A_{T-1,j}x_{T-1} = b_{T-1,j}$. Furthermore, let

$$\underline{\mathcal{Q}}_{T-1}\big(x_{T-2}\big) := \frac{1}{N_{T-2}} \sum_{j=1}^{N_{T-2}} \underline{Q}_{T-1,j}(x_{T-2})$$

and

$$g_{T-1} := -\frac{1}{N_{T-2}} \sum_{j=1}^{N_{T-2}} B_{T-1,j}^{\mathsf{T}} \tilde{\pi}_{T-1,j}.$$

Then

$$\ell_{T-1}(x_{T-2}) := \underline{\mathcal{Q}}_{T-1}\big(\bar{x}_{T-2}\big) + g_{T-1}^{\mathsf{T}}\big(x_{T-2} - \bar{x}_{T-2}\big) \tag{5.14}$$

is a supporting plane for $\underline{\mathcal{Q}}_{T-1}(x_{T-2})$ at $x_{T-2} = \bar{x}_{T-2}$. Consequently the approximation $\mathfrak{Q}_{T-1}(\cdot)$ is updated by replacing it with $\max\big\{\mathfrak{Q}_{T-1}(\cdot), \ell_{T-1}(\cdot)\big\}$.

This process is continued backwards until at the first stage the following problem is solved

$$\operatorname*{Min}_{x_1} c_1^{\mathsf{T}} x_1 + \mathfrak{Q}_2(x_1) \ \text{ s.t. } A_1 x_1 = b_1,\ x_1 \geq 0. \tag{5.15}$$

Of course, the above backward step can be performed simultaneously for several values of trial decisions $\bar{x}_t$, $t = 1, ..., T-1$. It could be also noted that starting from $t = T-1, ...,$ values $\underline{\mathcal{Q}}_t(x_{t-1})$ could be strictly smaller than $\mathcal{Q}_t(x_{t-1})$ for some/all x_{t-1}, and the constructed planes are supporting planes for $\underline{\mathcal{Q}}_t(\cdot)$ but could be only cutting planes for $\mathcal{Q}_t(\cdot)$.

The computed approximations $\mathfrak{Q}_2(\cdot), ..., \mathfrak{Q}_T(\cdot)$ (with $\mathfrak{Q}_{T+1}(\cdot) \equiv 0$ by definition) and a feasible first stage solution[20] $\bar{x}_1$ can be used for constructing an implementable policy as follows. For a realization

$$\xi_t = (c_t, A_t, B_t, b_t),\ t = 2, ..., T,$$

of the data process, decisions $\bar{x}_t$, $t = 1, ..., T$, are computed recursively going forward with $\bar{x}_1$ being the chosen feasible solution of the first stage problem (5.15), and $\bar{x}_t$ being an optimal solution of

$$\operatorname*{Min}_{x_t} c_t^{\mathsf{T}} x_t + \mathfrak{Q}_{t+1}(x_t) \ \text{ s.t. } A_t x_t = b_t - B_t \bar{x}_{t-1},\ x_t \geq 0, \tag{5.16}$$

for $t = 2, ..., T$. These optimal solutions can be used as trial decisions in the backward step of the algorithm. Note that $\bar{x}_t$ is a function of $\bar{x}_{t-1}$ and ξ_t, i.e., $\bar{x}_t$ is a function of $\xi_{[t]} = (\xi_1, ..., \xi_t)$, for $t = 2, ..., T$. That is, policy $\bar{x}_t = \bar{x}_t(\xi_{[t]})$ is nonanticipative and by the construction satisfies the feasibility constraints for every realization of the data process. Thus this policy is implementable and feasible for the true problem. If we restrict the data process to the generated sample, i.e., we consider only realizations $\xi_2, ..., \xi_T$ of the data process drawn from scenarios of the SAA problem, then $\bar{x}_t = \bar{x}_t(\xi_{[t]})$ becomes an implementable and feasible policy for the corresponding SAA problem.

Since the policy $\bar{x}_t = \bar{x}_t(\xi_{[t]})$ is feasible, the expectation

$$\mathbb{E}\left[\sum_{t=1}^{T} c_t^{\mathsf{T}} \bar{x}_t(\xi_{[t]})\right] \tag{5.17}$$

gives an upper bound for the optimal value of the corresponding multistage problem. That is, if we take this expectation over the true probability distribution of the

[20] Note that by the construction the first stage solution computed in a backward step is feasible, i.e., satisfies the constraints $A_1 x_1 = b_1$, $x_1 \geq 0$.

random data process, then the above expectation (5.17) gives an upper bound for the optimal value of the true problem. On the other hand, if we restrict the data process to scenarios of the SAA problem, each with equal probability $1/\mathcal{N}$, then the expectation (5.17) gives an upper bound for the optimal value of the SAA problem conditional on the sample used in construction of the SAA problem.

The *forward* step of the SDDP algorithm consists in generating M random realizations (scenarios) of the data process and computing the respective optimal values

$$\vartheta_j := \sum_{t=1}^{T} c_{tj}^{\mathsf{T}} \bar{x}_{tj}, \quad j = 1, ..., M.$$

That is, ϑ_j is the value of the corresponding policy for the realization $\xi_1, \xi_2^j, ..., \xi_T^j$ of the data process. As such, ϑ_j is an unbiased estimate of expected value of that policy, i.e., $\mathbb{E}[\vartheta_j] = \mathbb{E}\left[\sum_{t=1}^{T} c_t^{\mathsf{T}} \bar{x}_t(\xi_{[t]})\right]$. The forward step has two functions. First, some (all) of computed solutions $\bar{x}_{tj}$ can be used as trial points in the next iteration of the backward step of the algorithm. Second, these solutions can be employed for constructing a statistical upper bound for the optimal value of the corresponding multistage program (true or SAA depending on from what distribution the sample scenarios were generated).

Consider the average (sample mean) $\tilde{\vartheta}_M := M^{-1} \sum_{j=1}^{M} \vartheta_j$ and standard error

$$\tilde{\sigma}_M := \sqrt{\frac{1}{M-1} \sum_{j=1}^{M} (\vartheta_j - \tilde{\vartheta}_M)^2}$$

of the computed values ϑ_j. Since ϑ_j is an unbiased estimate of the expected value of the constructed policy, we have that $\tilde{\vartheta}_M$ is also an unbiased estimate of the expected value of that policy. By invoking the Central Limit Theorem we can say that $\tilde{\vartheta}_M$ has an approximately normal distribution provided that M is reasonably large. This leads to the following (approximate) $(1-\alpha)$-confidence upper bound for the value of that policy

$$\mathfrak{u}_{\alpha,M} := \tilde{\vartheta}_M + z_\alpha \frac{\tilde{\sigma}_M}{\sqrt{M}}. \tag{5.18}$$

Here $1 - \alpha \in (0, 1)$ is a chosen confidence level and $z_\alpha = \Phi^{-1}(1-\alpha)$, where $\Phi(\cdot)$ is the cdf of standard normal distribution. For example, for $\alpha = 0.05$ the corresponding critical value $z_{0.05} = 1.64$. That is, with probability approximately $1-\alpha$ the expected value of the constructed policy is less than the upper bound $\mathfrak{u}_{\alpha,M}$. Since the expected value (5.17) of the constructed policy is bigger than or equal to the optimal value of

the considered multistage problem, we have that $\mathfrak{u}_{\alpha,M}$ also gives an upper bound for the optimal value of the multistage problem with confidence at least $1-\alpha$. Note that the upper bound $\mathfrak{u}_{\alpha,M}$ can be used for the SAA or the true problem depending on from what distribution the sampled scenarios were generated.

Since $\mathfrak{Q}_t(\cdot)$ is the maximum of cutting planes of the cost-to-go function $\mathcal{Q}_t(\cdot)$ we have that

$$\mathcal{Q}_t(\cdot) \geq \mathfrak{Q}_t(\cdot),\ t = 2, ..., T. \tag{5.19}$$

Therefore the optimal value of problem (5.15), computed at a backward step of the algorithm, gives a lower bound for the considered SAA problem, i.e., is less that or equal to $\hat{\vartheta}_{\mathcal{N}}$. This lower bound is deterministic (i.e., is not based on sampling) if applied to the corresponding SAA problem. As far as the true problem is concerned, recall that $\vartheta^* \geq \mathbb{E}[\hat{\vartheta}_{\mathcal{N}}]$. Therefore on average this is also a lower bound for the optimal value of the true problem. On the other hand, the upper bound $\mathfrak{u}_{\alpha,M}$ is a function of generated scenarios and thus is stochastic even for considered (fixed) SAA problem. This upper bound may vary for different sets of random samples, in particular from one iteration to the next of the forward step of the algorithm.

5.2.1 The SDDP Method without Stagewise Independence

In the above development of the SDDP algorithm it was essential that the data process is stagewise independent. There are various situations where this condition of stagewise independence can be maintained by a suitable transformation. Consider the (general) multistage stochastic programming problem (3.5). Suppose that the data process satisfies the equations

$$\xi_t = h_t(\xi_{t-1}, \epsilon_t),\ t = 2, ..., T, \tag{5.20}$$

where $\epsilon_t \in \mathbb{R}^{l_t}$, $t = 2, ..., T$, is a sequence of independent random vectors and $h_t : \mathbb{R}^{d_{t-1}} \times \mathbb{R}^{l_t} \to \mathbb{R}^{d_t}$ are given functions. Then we can write problem (3.5) in the following form

$$\min_{x_1 \in \mathcal{X}_1} F_1(x_1) + \mathbb{E}\left[\inf_{y_2 \in \mathcal{Y}_2(y_1, \epsilon_2)} F_2(y_2) + \mathbb{E}\Big[\cdots + \mathbb{E}\left[\inf_{y_T \in \mathcal{Y}_T(y_{T-1}, \epsilon_T)} F_T(y_T)\right]\Big]\right]. \tag{5.21}$$

where $y_t := (x_t, \xi_t)$ and

$$\mathcal{Y}_t(y_{t-1}, \epsilon_t) := \big\{(x_t, \xi_t) : x_t \in \mathcal{X}_t\big(x_{t-1}, h_t(\xi_{t-1}, \epsilon_t)\big),\ \xi_t = h_t(\xi_{t-1}, \epsilon_t)\big\}, \tag{5.22}$$

$t = 2, ..., T$. In the above formulation (5.21), y_t are new decision variables.

Example 10 Consider the linear multistage stochastic programming problem (3.13). Suppose that parameters of the data process ξ_t other than b_t are stagewise independent (in particular are deterministic) and random vectors b_t, $t = 2, ..., T$, form a first order autoregressive process, i.e., $b_t = \mu + \Phi b_{t-1} + \epsilon_t$, with appropriate matrix Φ, vector μ and error vectors $\epsilon_2, ..., \epsilon_T$ being independent of each other. Then the feasibility equations of problem (3.13) can be written as

$$b_t - \Phi b_{t-1} - \mu = \epsilon_t, \ B_t x_{t-1} - \Phi b_{t-1} - \mu + A_t x_t = \epsilon_t, \ x_t \geq 0, \ \ t = 2, ..., T. \quad (5.23)$$

Therefore by replacing x_t with (x_t, b_t) and data process with $(c_t, A_t, B_t, \epsilon_t)$, $t = 2, ..., T$, we transform the problem into a linear multistage stochastic program with stagewise independent data process. $\Diamond$

In the above Example 10 the transformation (5.23) preserved the linear, and hence convexity, structure of the corresponding new problem. Of course, it can happen that the new formulation (5.21) does not inherit convex structure of the original problem, i.e., the corresponding cost-to-go functions (see below) may be not convex in y_t.

Similar to (3.6)–(3.7) we can write the corresponding dynamic programming equations for the new problem (5.21):

$$Q_t(y_{t-1}, \epsilon_t) = \inf_{y_t \in \mathcal{Y}_t(y_{t-1}, \epsilon_t)} \big\{ F_t(y_t) + \mathcal{Q}_{t+1}(y_t) \big\}, \quad (5.24)$$

where $y_t = (x_t, \xi_t)$, multifunctions $\mathcal{Y}_t(y_{t-1}, \epsilon_t)$ defined in (5.22) and

$$\mathcal{Q}_{t+1}(y_t) := \mathbb{E}\left\{ Q_{t+1}(y_t, \epsilon_{t+1}) \right\} \quad (5.25)$$

with $\mathcal{Q}_{T+1}(\cdot) \equiv 0$.

As another case suppose that the data process $\xi_2, ..., \xi_T$ of the multistage problem (3.5) is Markovian. That is, the conditional distribution of ξ_{t+1}, given $\xi_{[t]} = (\xi_1, ..., \xi_t)$, does not depend on $(\xi_1, ..., \xi_{t-1})$, i.e., is the same as the conditional distribution of ξ_{t+1}, given ξ_t, $t = 1, ..., T-1$. Then the corresponding dynamic programming equations (3.6)–(3.7) take the form

$$Q_t(x_{t-1}, \xi_t) = \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \big\{ F_t(x_t, \xi_t) + \mathcal{Q}_{t+1}(x_t, \xi_t) \big\}, \quad (5.26)$$

where

$$\mathcal{Q}_{t+1}(x_t, \xi_t) := \mathbb{E}\left\{ Q_{t+1}(x_t, \xi_{t+1}) \,\middle|\, \xi_t \right\}. \quad (5.27)$$

Here the cost-to-go functions $Q_t(x_{t-1}, \xi_t)$ and $\mathcal{Q}_{t+1}(x_t, \xi_t)$ depend on ξ_t, but not on $(\xi_1, ..., \xi_{t-1})$.

Suppose, further, that the process $\xi_2, ..., \xi_T$ can be reasonably approximated by a discretization so that it becomes a (possibly nonhomogeneous) *Markov chain*. That is, at stage $t = 2, ..., T$, the data process can take values $\xi_t^1, ..., \xi_t^{K_t}$, with specified probabilities of going from state $\xi_t^{j_t}$ (at stage t, $j_t = 1, ..., K_t$), to state $\xi_{t+1}^{j_{t+1}}$ (at stage $t+1$).

If the cost-to-go functions are convex in decision variables x_t (e.g., the multistage problem is linear) and the numbers K_t are reasonably small, then we still can proceed with the SDDP method by constructing cuts, and hence piecewise linear approximations, of the cost-to-go functions $\mathcal{Q}_{t+1}\left(\cdot, \xi_t^{j_t}\right)$, $j_t = 1, ..., K_t$. Note that in the backward step of the algorithm the cuts for $\mathcal{Q}_{t+1}\big(x_t, \xi_t^{j_t}\big)$, with respect to x_t, should be constructed separately for every $j_t = 1, ..., K_t$. Therefore computational complexity of backward steps of the SDDP algorithm grows more or less linearly with increase of the numbers K_t.

If in the forward steps of the algorithm the sample paths are generated from the original (may be continuous) distribution of the data process (rather than its Markov chain discretization), then the corresponding feasibility constraints should be restored by an appropriate projection (cf., [8]).

5.2.2 Convergence Properties of the SDDP Algorithm

One run (iteration) of the backward step of the SDDP algorithm involves solving $T-1$ linear programming problems of the form (5.13). For a given SAA problem, the number of decision variables of the linear programs (5.13) is constant and the number of constraints slowly grows with increase of the number of cutting planes from one iteration to the next. Therefore for a fixed number of iterations, complexity of one run of the backward step grows slightly faster than linearly with increase of the number of stages, and similarly for the forward step of the algorithm. The overall computational complexity of the SDDP algorithm is proportional to the number of iterations, which in turn depends on an applied stopping criterion.

One possible approach to stop the iterations, for considered SAA problem, is the following. The gap between the value of the policy associated with the computed approximations $\mathfrak{Q}_2(\cdot), ..., \mathfrak{Q}_T(\cdot)$ and first stage solution $\bar{x}_1$, can be estimated by taking the difference between the upper bound $\mathfrak{u}_{\alpha,M}$ and the lower bound given by the optimal value of problem (5.15). The algorithm can be stopped when this difference becomes smaller than a specified precision value $\varepsilon > 0$. This will give a guarantee that the SAA problem is solved with accuracy ε and confidence $1-\alpha$. Unfortunately what often happens for larger problems with a large number of stages, is that after

a certain number of iterations both the upper and lower bounds stabilize[21] and the decrease in the gap between these bounds becomes insignificant with increase of the number of iterations. In that case there is no point to continue the iterations even if the estimated gap didn't reach the specified accuracy level. So another stopping criterion is to stop the iterations when the lower bound starts to stabilize.

In that respect it is informative to consider the SDDP algorithm applied to the two stage linear stochastic programming problem (2.4)–(2.5). The first stage (2.4) of that problem can be written as

$$\operatorname*{Min}_{x\in\mathcal{X}} c^{\mathsf{T}}x + \mathcal{Q}(x), \tag{5.28}$$

where $\mathcal{X} := \{x : Ax = b,\ x \geq 0\}$ and $\mathcal{Q}(x) = \mathbb{E}[Q(x,\xi)]$ with $Q(x,\xi)$ being optimal value of the second stage problem (2.5). Let us assume that: (i) the set $\mathcal{X}$ is nonempty and bounded, (ii) the relatively complete recourse holds, i.e., $Q(x,\xi) < +\infty$ for every $x \in \mathcal{X}$ w.p.1, (iii) the feasible set $\{\pi : W^{\mathsf{T}}\pi \leq q\}$ of the dual (2.7), of the second stage problem, is nonempty and hence $Q(x,\xi) > -\infty$ for all x and ξ.

Let now (5.28) be an SAA problem based on a sample $\xi^1, ..., \xi^N$ of the random data vector ξ. In that case $\mathcal{Q}(x) = N^{-1}\sum_{j=1}^{N} Q(x,\xi^j)$ and under the above assumptions (i)–(iii), the function $\mathcal{Q}(x)$ is convex, finite valued and piecewise linear. The backward step of the SDDP algorithm, applied to this SAA problem, becomes the classical Kelley's cutting plane algorithm, [11]. That is, let at k-th iteration, $\mathfrak{Q}^k(\cdot)$ be the corresponding approximation of $\mathcal{Q}(\cdot)$ given by maximum of supporting planes of $\mathcal{Q}(\cdot)$. At the next iteration the backward step solves the problem

$$\operatorname*{Min}_{x\in\mathcal{X}} c^{\mathsf{T}}x + \mathfrak{Q}^k(x), \tag{5.29}$$

and hence compute its optimal value v^{k+1}, an optimal solution x^{k+1} of (5.29) and a subgradient $g^{k+1} \in \partial\mathcal{Q}(x^{k+1})$. Consequently the supporting plane $\ell(x) := \mathcal{Q}(x^{k+1}) + (g^{k+1})^{\mathsf{T}}(x - x^{k+1})$ is added to the collection of cutting (supporting) planes of $\mathcal{Q}(\cdot)$. Note that for the two stage SAA problem, the backward step of the SDDP algorithm does not involve any sampling and the forward step of the algorithm is redundant. In the stochastic programming literature this cutting planes algorithm is often called the L-shape method (see, e.g., [6, section 5.1]).

Since the function $\mathcal{Q}(\cdot)$ is piecewise linear, it is not difficult to show that Kelley's algorithm converges in a finite number of iterations. Arguments of that type are based on the observation that $\mathcal{Q}(\cdot)$ is the maximum of a finite number of its supporting planes. However, the number of supporting planes can be very large and these

[21] Recall that the lower bound is monotonically increasing since no cuts are discarded, while the upper bound $\mathfrak{u}_{\alpha,M}$ is stochastic and varies from one iteration to the next.

arguments do not give an idea about rate of convergence of the algorithm. So let us look at the following proof of convergence which uses only convexity of the function $\mathcal{Q}(\cdot)$ (cf., [22, p.160]).

Denote $f(x) := c^{\mathsf{T}}x + \mathcal{Q}(x)$ and by f^* the optimal value of problem (5.28). Let x^k, $k = 1, ...,$ be a sequence of iteration points generated by the algorithm, $\gamma^k = c + g^k$ be the corresponding subgradients used in construction of the supporting planes and v^k be the optimal value of the problem (5.29). Note that since $x^k \in \mathcal{X}$ we have that $f(x^k) \geq f^*$, and since $\mathfrak{Q}^k(\cdot) \leq \mathcal{Q}(\cdot)$ we have that $v^k \leq f^*$ for all k. Note also that since $\mathcal{Q}(\cdot)$ is piecewise linear and $\mathcal{X}$ is bounded, the subgradients of $f(\cdot)$ are bounded on $\mathcal{X}$, i.e., there is a constant C such that[22] $\|\gamma\| \leq C$ for all $\gamma \in \partial f(x)$ and $x \in \mathcal{X}$ (such boundedness of gradients holds for a general convex function provided the function is finite valued on a neighborhood of the set $\mathcal{X}$). Choose the precision level $\varepsilon > 0$ and denote $\mathfrak{K}_\varepsilon := \{k : f(x^k) - f^* \geq \varepsilon\}$. For $k < k'$ we have

$$f(x^k) + (\gamma^k)^{\mathsf{T}}\big(x^{k'} - x^k\big) \leq v^{k'} \leq f^*,$$

and hence

$$f(x^k) - f^* \leq (\gamma^k)^{\mathsf{T}}\big(x^k - x^{k'}\big) \leq \|\gamma^k\|\,\|x^k - x^{k'}\| \leq C\|x^k - x^{k'}\|.$$

This implies that for any $k, k' \in \mathfrak{K}_\varepsilon$ the following inequality holds

$$\|x^k - x^{k'}\| \geq \varepsilon/(2C). \tag{5.30}$$

Since the set $\mathcal{X}$ is bounded, it follows that the set $\mathfrak{K}_\varepsilon$ is finite. That is, after a finite number of iterations $f(x^k) - f^*$ becomes less than ε, i.e., the algorithm reaches the precision ε in a finite number of iterations.

For $\eta > 0$ denote by $\mathfrak{N}(\mathcal{X}, \eta)$ the maximal number of points in the set $\mathcal{X}$ such that the distance between any two of these points is not less than η. The inequality (5.30) implies that $\mathfrak{N}(\mathcal{X}, \eta)$, with $\eta = \varepsilon/(2C)$, gives an upper bound for the number of iterations required to obtain an ε-optimal solution of problem (5.28) by Kelley's algorithm. Unfortunately, for a given η and $\mathcal{X} \subset \mathbb{R}^n$ say being a ball of fixed diameter, the number $\mathfrak{N}(\mathcal{X}, \eta)$, although is finite, grows exponentially with increase of the dimension n. Worst case analysis of Kelley's algorithm is discussed in [15, pp. 158-160], with the following example of a convex problem

$$\operatorname*{Min}_{x \in \mathbb{R}^{n+1}} f(x) \;\text{ s.t. } \|x\| \leq 1, \tag{5.31}$$

where $f(x) := \max\{x_1^2 + ... + x_n^2, |x_{n+1}|\}$. It is shown there that Kelley's algorithm applied to problem (5.31) with starting point $x_0 := (0, ..., 0, 1)$, requires at least

[22] Unless stated otherwise we assume that the considered norm $\|\cdot\|$ is Euclidean.

$\frac{\ln(\varepsilon^{-1})}{2\ln 2}\left(\frac{2}{\sqrt{3}}\right)^{n-1}$ calls of the oracle to obtain an ε-optimal solution, i.e., the number of required iterations grows exponentially with increase of the dimension n of the problem. It was also observed empirically that Kelley's algorithm could behave quite poorly in practice.

The above analysis suggests quite a pessimistic view on Kelley's algorithm for convex problems of large dimension n. Unfortunately it is not clear how more efficient, bundle type algorithms, can be extended to a multistage setting. On the other hand, from the number-of-scenarios point of view complexity of multistage SAA problems grows very fast with increase of the number of stages even for problems with relatively small dimensions of the involved decision variables. It is possible to show that under mild regularity conditions the SDDP algorithm converges (w.p.1) with increase of the number of iterations. The available proofs of convergence ([19],[31]) are based on arguments that since the number of scenarios is finite, eventually the piecewise linear cost-to-go functions $\mathcal{Q}_t(\cdot)$ will be reconstructed and hence (w.p.1) an optimal policy will be computed in a finite number of iterations. Unfortunately, these proofs don't give an indication of how many iterations will be needed in order to solve the problem with a given accuracy. The analysis of two stage problems indicates that the SDDP method could give reasonable results for problems with a not too large number of decision variables; this seems to be confirmed by numerical experiments.

5.2.3 Risk Averse Implementations of the SDDP Method

Let us look again at the linear multistage stochastic programming problem (3.13). In that formulation the expected value $\mathbb{E}\left[\sum_{t=1}^{T} c_t^{\mathsf{T}} x_t\right]$ of the total cost is minimized subject to the feasibility constraints. That is, the total cost is optimized (minimized) *on average*. Since the costs $c_t^{\mathsf{T}} x_t = c_t^{\mathsf{T}} x_t(\xi_{[t]})$, $t = 2, ..., T$, are functions of the random data process, they are random and hence are subject to random perturbations. For a particular realization of the random process these costs could be much bigger than their average (i.e., expectation) values. Recall that we referred to the formulation (3.13) as *risk neutral* as opposed to *risk averse* approaches. The goal of a risk averse approach is to avoid large values of the costs for some possible realizations of the data process. One such approach will be to maintain constraints $c_t^{\mathsf{T}} x_t \leq \nu_t$, $t = 1, ..., T$, for chosen upper levels ν_t and *all* possible realizations of the data process. However, trying to enforce these upper limits under any circumstances could be unrealistic and infeasible. One may try to relax these constraints by enforcing them with a high (close to one) probability. However, introducing such *chance constraints* can still result in infeasibility and moreover is difficult to handle numerically. So we consider here penalization approaches. That is, at every stage the cost is penalized while

exceeding a specified upper limit.

In a simple form this leads to the following risk averse formulation

$$\operatorname*{Min}_{\substack{A_1x_1=b_1\\ x_1\geq 0}} c_1^{\mathsf{T}}x_1 + \mathbb{E}\left[\min_{\substack{B_2x_1+A_2x_2=b_2\\ x_2\geq 0}} F_2(x_2) + \mathbb{E}\Big[\cdots + \mathbb{E}\left[\min_{\substack{B_Tx_{T-1}+A_Tx_T=b_T\\ x_T\geq 0}} F_T(x_T)\right]\Big]\right], \tag{5.32}$$

where

$$F_t(x_t) := c_t^{\mathsf{T}}x_t + \kappa_t[c_t^{\mathsf{T}}x_t - \nu_t]_+, \;\; t = 2, ..., T,$$

with ν_t and $\kappa_t \geq 0$ being chosen constants. The additional terms $\kappa_t[c_t^{\mathsf{T}}x_t - \nu_t]_+$ represent the penalty for exceeding the upper limits ν_t. An immediate question is how to choose constants ν_t and κ_t. One possible approach is to take ν_t to be the $(1-\alpha)$-quantile (say, the 95% quantile) of the distribution of the cost $c_t^{\mathsf{T}}\bar{x}_t$ of the *optimal* policy of the risk neutral problem. These quantiles can be estimated by, first, solving the risk neutral problem and hence computing its optimal policy $\bar{x}_t = \bar{x}_t(\xi_{[t]})$, $t = 2, ..., T$. Then at each stage the $(1-\alpha)$-quantile of the distribution of the cost $c_t^{\mathsf{T}}\bar{x}_t$ is estimated by randomly generating M realizations of the random process and computing respective costs in the forward step procedure. For the constants κ_t one can use the same value κ for all stages, with this value being gradually increased in experiments.

The SDDP algorithm with simple modifications can be applied to the problem (5.32) in a rather straightforward way. In the backward step of the algorithm at the last stage the corresponding problem (5.7), of the risk neutral formulation, should be replaced by

$$\operatorname*{Min}_{x_T} c_{Tj}^{\mathsf{T}}x_T + \kappa_T[c_{Tj}^{\mathsf{T}}x_T - \nu_T]_+ \;\text{ s.t. } B_{Tj}x_{T-1} + A_{Tj}x_T = b_{Tj},\; x_T \geq 0, \tag{5.33}$$

which can be written as the following linear programming problem

$$\begin{array}{ll} \operatorname*{Min}\limits_{x_T, w_T} & c_{Tj}^{\mathsf{T}}x_T + \kappa_T w_T \\ \text{s.t.} & B_{Tj}x_{T-1} + A_{Tj}x_T = b_{Tj},\; x_T \geq 0, \\ & w_T \geq c_{Tj}^{\mathsf{T}}x_T - \nu_T,\; w_T \geq 0. \end{array} \tag{5.34}$$

At stage $T-1$ the corresponding problem (5.13) of risk neutral formulation should be replaced by

$$\begin{array}{ll} \operatorname*{Min}\limits_{x_{T-1}, w_{T-1}, \theta} & c_{T-1,j}^{\mathsf{T}}x_{T-1} + \kappa_{T-1}w_{T-1} + \theta \\ \text{s.t.} & B_{T-1,j}\bar{x}_{T-2} + A_{T-1,j}x_{T-1} = b_{T-1,j},\; x_{T-1} \geq 0, \\ & \theta \geq \alpha_{Tk} + \beta_{Tk}^{\mathsf{T}}x_{T-1},\; k \in \mathcal{I}_T, \\ & w_{T-1} \geq c_{T-1,j}^{\mathsf{T}}x_{T-1} - \nu_{T-1},\; w_{T-1} \geq 0, \end{array} \tag{5.35}$$

and so on going backward in time. The forward step of the algorithm basically is the same as in the risk neutral case with costs $c_t^{\mathsf{T}} x_t$ replaced by $c_t^{\mathsf{T}} x_t + \kappa_t [c_t^{\mathsf{T}} x_t - \nu_t]_+$.

It could be noted that in the above approach the upper limits ν_t are fixed and their calculations are based on solving the risk neutral problem which involves all possible realizations of the data process. In other words in formulation (5.32) the upper limits are not adapted to a current realization of the random process. Let us observe that optimal solutions of problem (5.32) will be not changed if the penalty term at t-th stage is changed to $\nu_t + \kappa_t [c_t^{\mathsf{T}} x_t - \nu_t]_+$ by adding the constant ν_t. Now if we adapt the upper limits ν_t to a realization of the data process by taking these upper limits to be $(1-\alpha_t)$-quantiles of $c_t^{\mathsf{T}} x_t$ *conditional* on observed history $\xi_{[t-1]}$, we end up with penalty terms given by $\mathsf{AV@R}_{\alpha_t|\xi_{[t-1]}}$ with $\alpha_t = 1/\kappa_t$. This leads to the nested risk averse formulation (3.107)–(3.108).

It is also possible to give the following interpretation of the risk averse formulation (3.107)–(3.108). Recall that $\mathsf{AV@R}_\alpha[Z] \geq \mathsf{V@R}_\alpha(Z)$. Therefore $\rho_{t|\xi_{[t-1]}}[Z] \geq \vartheta_{t|\xi_{[t-1]}}[Z]$, where

$$\vartheta_{t|\xi_{[t-1]}}[Z] := (1-\lambda_t)\mathbb{E}\left[Z|\xi_{[t-1]}\right] + \lambda_t \mathsf{V@R}_{\alpha_t}\left[Z|\xi_{[t-1]}\right]. \tag{5.36}$$

If we replace $\rho_{t|\xi_{[t-1]}}[Z]$ in the risk averse formulation (3.108) by $\vartheta_{t|\xi_{[t-1]}}[Z]$, we will be minimizing the weighted average of means and $(1-\alpha)$-quantiles, which will be a natural way of dealing with the involved risk. Unfortunately such formulation will lead to a nonconvex and computationally intractable problem. This is one of the main reasons of using $\mathsf{AV@R}_\alpha$ instead of $\mathsf{V@R}_\alpha$ in the corresponding risk averse formulation.

Dynamic programming equations for the risk averse problem (3.107)–(3.108) are given in (3.118)–(3.119). For the SAA problem (recall that we use the identical conditional sampling approach so that the stagewise independence property is preserved in the SAA problem) these equations for $t = T, ..., 2$, take the form

$$Q_{tj}(x_{t-1}) = \inf_{x_t \in \mathbb{R}^{n_t}, u_t \in \mathbb{R}} \left\{ c_{tj}^{\mathsf{T}} x_t + \lambda_{t+1} u_t + \mathcal{V}_{t+1}(x_t, u_t) : B_{tj} x_{t-1} + A_{tj} x_t = b_{tj},\ x_t \geq 0 \right\}, \tag{5.37}$$

$j = 1, ..., N_{t-1}$, where

$$\mathcal{V}_{t+1}(x_t, u_t) = \frac{1}{N_t} \sum_{j=1}^{N_t} \left\{ (1-\lambda_{t+1}) Q_{t+1,j}(x_t) + \lambda_{t+1} \alpha_{t+1}^{-1} \left[Q_{t+1,j}(x_t) - u_t \right]_+ \right\} \tag{5.38}$$

with $\mathcal{V}_{T+1}(\cdot) \equiv 0$ and $\lambda_{T+1} := 0$. At the first stage problem

$$\operatorname*{Min}_{x_1 \in \mathbb{R}^{n_1}, u_1 \in \mathbb{R}} c_1^{\mathsf{T}} x_1 + \lambda_2 u_1 + \mathcal{V}_2(x_1, u_1) \ \text{ s.t. } A_1 x_1 = b_1,\ x_1 \geq 0, \tag{5.39}$$

should be solved. Note again that in this formulation decision variables at t-th stage are $x_t \in \mathbb{R}^{n_t}$ and $u_t \in \mathbb{R}$.

By the chain rule for subdifferentials we have that the subdifferential of the (convex) function $\phi(x_{t-1}, u_{t-1}) := [Q_{tj}(x_{t-1}) - u_{t-1}]_+$ at a point $(x_{t-1}, u_{t-1}) = (\bar{x}_{t-1}, \bar{u}_{t-1})$ can be written as

$$\partial\phi(\bar{x}_{t-1}, \bar{u}_{t-1}) = \begin{cases} [0,0] & \text{if} \quad Q_{tj}(\bar{x}_{t-1}) < \bar{u}_{t-1}, \\ \bigcup_{g \in \partial Q_{tj}(\bar{x}_{t-1})} [g, -1] & \text{if} \quad Q_{tj}(\bar{x}_{t-1}) > \bar{u}_{t-1}, \\ \bigcup_{\substack{g \in \partial Q_{tj}(\bar{x}_{t-1}) \\ t \in [0,1]}} [tg, -t] & \text{if} \quad Q_{tj}(\bar{x}_{t-1}) = \bar{u}_{t-1}. \end{cases} \tag{5.40}$$

Consequently, if $g_{tj} \in \partial Q_{tj}(\bar{x}_{t-1})$, $j = 1, ..., N_{t-1}$, then a subgradient of $\mathcal{V}_t(x_{t-1}, u_{t-1})$ at $(\bar{x}_{t-1}, \bar{u}_{t-1})$ is given by[23]

$$\frac{1}{N_{t-1}} \left[(1 - \lambda_t) \sum_{j=1}^{N_{t-1}} g_{tj} + \lambda_t \alpha_t^{-1} \sum_{j \in \mathcal{J}_t} g_{tj},\ -\lambda_t \alpha_t^{-1} |\mathcal{J}_t| \right], \tag{5.41}$$

where

$$\mathcal{J}_t := \{ j : Q_{tj}(\bar{x}_{t-1}) > \bar{u}_{t-1},\ j = 1, ..., N_{t-1} \}.$$

One can proceed now in backward steps of the SDDP algorithm in a way similar to the risk neutral case by adding cutting planes of the cost-to-go functions $\mathcal{V}_t(x_{t-1}, u_{t-1})$.

Let us consider now construction of the corresponding forward step procedure. Given a feasible first stage solution $(\bar{x}_1, \bar{u}_1)$ and a current set of piecewise linear lower approximations $\mathfrak{V}_t(x_{t-1}, u_{t-1})$ of cost-to-go functions $\mathcal{V}_t(x_{t-1}, u_{t-1})$, $t = 2, ..., T$, we can proceed iteratively forward by solving problems

$$\operatorname*{Min}_{x_t, u_t} c_t^{\mathsf{T}} x_t + \lambda_{t+1} u_t + \mathfrak{V}_{t+1}(x_t, u_t) \ \text{ s.t. } A_t x_t = b_t - B_t x_{t-1},\ x_t \geq 0, \tag{5.42}$$

for a (randomly) generated scenario $\xi_2, ..., \xi_T$. Let $(\bar{x}_t, \bar{u}_t)$, $t = 1, ..., T$, be respective optimal solutions. These solutions can be used in constructions of cutting planes in the backward step procedure. We have that $\bar{x}_t = \bar{x}_t(\xi_{[t]})$ and $\bar{u}_t = \bar{u}_t(\xi_{[t]})$, $t = 1, ..., T$, are functions of the data process, and $\bar{x}_t(\xi_{[t]})$ gives a feasible and implementable policy. Unfortunately, here the forward step of the SDDP algorithm cannot be easily adapted for estimating value of this policy.

5.3 Reduction to Static Problems

Another possible approach to multistage stochastic programming is to reduce dynamic setting to a static case. Consider the multistage problem (3.5). Suppose that we can

[23] Recall that $|\mathcal{J}_t|$ denotes the cardinality, i.e., the number of elements, of the set $\mathcal{J}_t$.

identify a parametric family of policies $\bar{x}_t(\xi_{[t]}, \theta)$, $t = 1, ..., T$, depending on a finite number of parameters $\theta \in \mathbb{R}^q$ (compare with section 2.2 where this was discussed for two stage problems). Suppose further that we can construct a set $\Theta \subset \mathbb{R}^q$ such that these policies are feasible for all $\theta \in \Theta$. That is, for all $\theta \in \Theta$ it holds that $\bar{x}_1(\theta) \in \mathcal{X}_1$ and $\bar{x}_t(\xi_{[t]}, \theta) \in \mathcal{X}_t\big(\bar{x}_{t-1}(\xi_{[t-1]}, \theta), \xi_t\big)$, $t = 2, ..., T$, w.p.1. Consider the following stochastic program

$$\operatorname*{Min}_{\theta \in \Theta} F_1\big(\bar{x}_1(\theta)\big) + \mathbb{E}\left[\sum_{t=2}^{T} F_t\big(\bar{x}_t(\xi_{[t]}, \theta), \xi_t\big)\right]. \tag{5.43}$$

Since by construction the considered policies are feasible, the optimal value of problem (5.43) gives an upper bound for the optimal value of the original multistage problem (3.5). Of course, quality of a solution $\bar{x}_t(\xi_{[t]}, \theta^*)$, $t = 1, ..., T$, of (5.43), viewed as a solution of the original multistage problem (3.5), depends on a successful choice of the parametric family.

The above problem (5.43) is a (static) stochastic problem and could be solved, say by the SAA method, provided that the set Θ is defined in a computationally accessible way. That is, a random sample of N scenarios $\xi_2^j, ..., \xi_T^j$, $j = 1, ..., N$, of the data process is generated and problem (5.43) is approximated by the corresponding SAA problem

$$\operatorname*{Min}_{\theta \in \Theta} F_1\big(\bar{x}_1(\theta)\big) + \frac{1}{N} \sum_{j=1}^{N} \sum_{t=2}^{T} F_t\big(\bar{x}_t(\xi_{[t]}^j, \theta), \xi_t^j\big). \tag{5.44}$$

Recall that the number N of generated scenarios required to solve static problem (5.43) with a given accuracy $\varepsilon > 0$, by employing the SAA problem (5.44), is of order $O(\varepsilon^{-2})$ provided some standard regularity conditions hold (see [30, section 5.3]).

Example 11 Suppose that we have a finite family of feasible policies

$$\big\{x_t^k(\xi_{[t]}),\ t = 1, ..., T\big\},\ k = 1, ..., K.$$

Suppose, further, that the multifunctions $\mathcal{X}_t(\cdot, \xi_t)$ are *convex*, i.e., the set $\mathcal{X}_1$ is convex and for a.e. ξ_t and all x_{t-1}, x'_{t-1} and $\tau \in [0, 1]$ it holds that

$$\tau \mathcal{X}_t(x_{t-1}, \xi_t) + (1 - \tau)\mathcal{X}_t(x'_{t-1}, \xi_t) \subset \mathcal{X}_t\big(\tau x_{t-1} + (1 - \tau)x'_{t-1}, \xi_t\big),\ t = 2, ..., T. \tag{5.45}$$

For example, the multifunctions $\mathcal{X}_t(\cdot, \xi_t)$ representing feasible sets of linear multistage programs, defined in (3.3), are convex. Consider convex combination

$$\bar{x}_t(\xi_{[t]}, \theta) := \sum_{k=1}^{K} \theta_k x_t^k(\xi_{[t]}),\ t = 1, ..., T, \tag{5.46}$$

of these policies. Here $\theta = (\theta_1, ..., \theta_K) \in \Delta_K$, with $\Delta_K := \left\{\theta \in \mathbb{R}^K_+ : \sum_{k=1}^K \theta_k = 1\right\}$ being the K-dimensional simplex. Feasibility of the considered policies means that for $k = 1, ..., K$,

$$x_t^k(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}^k(\xi_{[t-1]}), \xi_t),\ t = 2, \ldots, T,\ \text{w.p.1}, \tag{5.47}$$

and hence

$$\bar{x}_t(\xi_{[t]}, \theta) \in \sum_{k=1}^K \theta_k \mathcal{X}_t(x_{t-1}^k(\xi_{[t-1]}), \xi_t),\ t = 2, \ldots, T,\ \text{w.p.1}. \tag{5.48}$$

By convexity of the multifunctions $\mathcal{X}_t(\cdot, \xi_t)$ we have that the right hand side of (5.48) is included in $\mathcal{X}_t(\bar{x}_t(\xi_{[t]}, \theta), \xi_t)$, and hence it follows that $\bar{x}_t(\xi_{[t]}, \theta)$, $t = 1, ..., T$, is a feasible policy for any $\theta \in \Delta_K$. This approach with several examples is discussed in [12]. Note that for the linear multistage problem (3.13), with linear objective function, the corresponding problem (5.43) has an optimal solution at one of the extreme points of Δ_K, i.e., at a point of the form $\theta^* = (0, ..., 0, 1, 0, ..., 0)$. That is, in that case the optimal policy $\bar{x}_t(\xi_{[t]}, \theta^*)$ coincides with one of the considered policies. $\Diamond$

5.3.1 Affine Policies

Another way of constructing parametric policies is to consider policies which are affine functions of the data process (compare with section 2.2.1). As an example let us consider the inventory model (4.15) with a specified probability distribution of the demand process:

$$\begin{array}{ll} \underset{x_t \geq y_t}{\text{Min}} & \mathbb{E}\left\{\sum_{t=1}^T c_t(x_t - y_t) + b_t[D_t - x_t]_+ + h_t[x_t - D_t]_+\right\} \\ \text{s.t.} & y_{t+1} = x_t - D_t,\ t = 1, ..., T-1. \end{array} \tag{5.49}$$

Recall that the minimization in (5.49) is performed over (nonanticipative) policies of the form $x_1, x_2(D_{[1]}), ..., x_T(D_{[T-1]})$ satisfying the feasibility constraints of (5.49) for almost every realization of the demand process.

Consider now policies of the form

$$x_t(D_{[t-1]}) := \theta_{1t} + \sum_{\tau=2}^t \theta_{\tau t} D_{\tau-1},\ t = 2, ..., T, \tag{5.50}$$

depending on the parameter vector $\theta = (\theta_{12}, ..., \theta_{TT})$. Since these policies are given as affine functions of the data (demand) process, such policies are called *affine*. Sub-

stituting these policies into (5.49) we obtain the following optimization problem

$$\begin{array}{ll} \underset{x_1 \geq y_1,\, \theta}{\text{Min}} & \mathbb{E}\Big\{ c_1(x_1 - y_1) + b_1[D_1 - x_1]_+ + h_1[x_1 - D_1]_+ \\ & \quad + \sum_{t=2}^{T} c_t(\theta_{1t} + \sum_{\tau=2}^{t} \theta_{\tau t} D_{\tau-1} - \theta_{1,t-1} - \sum_{\tau=2}^{t-1} \theta_{\tau,t-1} D_{\tau-2} + D_{t-1}) \\ & \quad + b_t[D_t - \theta_{1t} - \sum_{\tau=2}^{t} \theta_{\tau t} D_{\tau-1}]_+ + h_t[\theta_{1t} + \sum_{\tau=2}^{t} \theta_{\tau t} D_{\tau-1} - D_t]_+ \Big\} \\ \text{s.t.} & \theta_{1t} + \sum_{\tau=2}^{t} \theta_{\tau t} D_{\tau-1} \geq \theta_{1,t-1} + \sum_{\tau=2}^{t-1} \theta_{\tau,t-1} D_{\tau-2} - D_{t-1}, \\ & t = 2, ..., T-1, \ (D_1, ..., D_T) \in \mathfrak{D}, \end{array} \tag{5.51}$$

where $D_0 := 0$ and $\mathfrak{D} \subset \mathbb{R}^T_+$ is the support of the distribution of random vector $(D_1, ..., D_T)$. Since in problem (5.51) optimization is constrained to affine policies, its optimal value gives an upper bound for the optimal value of the original problem (5.49). Of course, if optimal values of problems (5.49) and (5.51) do coincide, then an optimal solution of problem (5.51) defines an affine policy which is an optimal solution of problem (5.49).

Problem (5.51) is a static stochastic problem with a finite number of decision variables, $\theta_{12}, ..., \theta_{TT}$ and x_1, and an infinite number of linear constraints. There are two basic difficulties in solving problem (5.51). One is that the expected value cannot be computed in a closed form and should be approximated. This can be approached by generating a random sample of the demand vector $(D_1, ..., D_T)$ and approximating the corresponding expectation by the sample average, that is by constructing the Sample Average Approximation (SAA) problem. Still the infinite number of linear constraints could be difficult to handle in a general case.

Note that the feasibility constraints of problem (5.51) can be written in the following equivalent form

$$\begin{array}{ll} \underset{(D_1,...,D_T) \in \mathfrak{D}}{\max} & \big\{ \theta_{1,t-1} + \sum_{\tau=2}^{t-1} \theta_{\tau,t-1} D_{\tau-2} - D_{t-1} - \theta_{1t} - \sum_{\tau=2}^{t} \theta_{\tau t} D_{\tau-1} \big\} \leq 0, \\ & t = 2, ..., T-1. \end{array} \tag{5.52}$$

In some cases the maximum in (5.52) can be written in a closed form. Suppose that random variables $D_1, ..., D_T$ are independent, i.e., the demand process is stagewise independent, with distribution of D_t being supported on interval $I_t = [l_t, u_t] \subset \mathbb{R}_+$, $t = 1, ..., T$. That is, $\mathfrak{D} = I_1 \times \cdots \times I_T$. In that case the maximization in (5.52) separates into maximization with respect to each $D_t \in I_t$ individually, and the maximum is attained either at $D_t = l_t$ or $D_t = u_t$, $t = 1, ..., T$. Consequently the feasibility constraints of problem (5.51) can be written explicitly as a finite number of linear constraints.

Suppose now that the demand process can be modeled as an autoregresive process $AR(p)$ of the form

$$D_t = \mu + \phi_1 D_{t-1} + ... + \phi_p D_{t-p} + \varepsilon_t, \tag{5.53}$$

with ε_t being a sequence of iid random variables, $t = 1, ..., T$, and $D_{1-p}, ..., D_0$ being given (observed) values. We can write this $AR(p)$ in the following matrix form $w_t = M + \Phi w_{t-1} + \Upsilon_t$, where $w_t := (D_{t-1}, ..., D_{t-p})^\mathsf{T}$,

$$\Phi := \begin{bmatrix} \phi_1 & & \cdots & \phi_p \\ 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & \cdots & 1 & 0 \end{bmatrix},$$

$M := (\mu, 0, ..., 0)^\mathsf{T}$ and $\Upsilon_t := (\varepsilon_{t-1}, 0, ..., 0)^\mathsf{T}$. Substituting this into (5.49) we obtain problem

$$\begin{array}{ll} \underset{x_t \geq y_t}{\text{Min}} & \mathbb{E}\Big\{ \sum_{t=1}^T c_t(x_t - y_t) + b_t[\mu + \phi^\mathsf{T} w_t - x_t + \varepsilon_t]_+ + h_t[x_t - \mu - \phi^\mathsf{T} w_t - \varepsilon_t]_+ \Big\} \\ \text{s.t.} & y_{t+1} = x_t - \mu - \phi^\mathsf{T} w_t - \varepsilon_t, \ t = 1, ..., T-1, \\ & w_{t+1} = M + \Phi w_t + \Upsilon_{t+1}, \ t = 1, ..., T-1, \end{array} \tag{5.54}$$

where $\phi := (\phi_1, ..., \phi_p)^\mathsf{T}$. Note that optimization in (5.54) is performed over decision variables x_t and w_t, and the random process is $\varepsilon_1, ..., \varepsilon_T$ (compare with Example 10 on page 93).

Formulation (5.54) allows to write dynamic programming equations, with cost-to-go functions $Q_t(y_t, w_t)$, and hence to apply, say, the SDDP algorithm. In order to formulate affine policies in terms of the random process $\varepsilon_1, ..., \varepsilon_T$ we can proceed as follows. We can write the demand process in the form $D_t = \mu_t + \sum_{\tau=1}^t \beta_{\tau t} \varepsilon_\tau$, where the coefficients μ_t and $\beta_{\tau t}$ can be computed recursively using equation (5.53). We can consider now affine policies of the form

$$x_t(\varepsilon_{[t-1]}) := \gamma_{1t} + \sum_{\tau=2}^t \gamma_{\tau t} \varepsilon_{\tau-1}, \ t = 2, ..., T. \tag{5.55}$$

Substituting this into (5.49) we obtain similar to (5.51) a static stochastic program in terms of decision variables x_1 and $\gamma_{\tau t}$ and random variables ε_t.

Such affine policies can be formulated for more complex stochastic programs. For example for linear stochastic programs of the form (3.13), with only right hand sides $b_2, ..., b_T$ being random, we can consider affine policies of the form

$$x_t(b_{[t]}) = \mu_t + \sum_{\tau=2}^t \Psi_{\tau t} b_\tau, \ \ t = 2, ..., T, \tag{5.56}$$

depending on parameters – vectors μ_t and matrices $\Psi_{\tau t}$. In this case the objective function is linear in b_t and hence its expectation can be written explicitly as $c_1^\intercal x_1 + \sum_{t=2}^T c_t^\intercal \left(\mu_t + \sum_{\tau=2}^t \Psi_{\tau t}\beta_\tau\right)$, where $\beta_t := \mathbb{E}[b_t]$. The feasibility constraints could be more difficult to handle and would require a careful formulation adjusted to a particular problem.

Remark 9 It is possible to show that a linear stochastic program of the form (3.13) has a continuous piecewise affine optimal policy, provided that: (i) its optimal value is finite, (ii) only its right hand sides $b_2, ..., b_T$ are random, (iii) the process $b_2, ..., b_T$ is stagewise independent, (iv) the number of scenarios is finite. The arguments are similar to those of Example 1 on page 6. That is, because of the stagewise independence the (expectation) cost-to-go functions $\mathcal{Q}_{t+1}(x_t)$ do not depend on the data process, are convex and piecewise linear since the number of scenarios is finite. Consequently the optimization problems in dynamic equations, given in the right hand side of (5.3), can be formulated as linear programming problems with the corresponding right hand sides being linear functions of b_{tj} and $B_{tj}x_{t-1}$. By a standard result of the theory of parametric linear programming we have then that an optimal solution of the right hand side of (5.3) is a continuous piecewise affine function of b_{tj} and x_{t-1}. The proof can be completed by induction going backward in time (cf., [2]). Note that for these arguments to be valid the conditions (i)–(iv) are essential. $\Diamond$

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