Summer School on Linear Programs: Robust IPM Exercises

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Hints: Hints are on the last page. It is recommended to think about the problem without hints for a while, and then look at the hints when stuck.

Problem 1: Self-Concordance

In the two lectures today, we did all the IPM analysis using the logarithmic barrier function. Are there other barrier functions which can be used to design IPMs? In this exercise we explore this notion a little bit, using the concept of *self-concordance*.

We will state the definition of ν-self-concordance here. It may seem unnatural at first, but hopefully through the exercises you will see why it would be natural to arrive at such a definition from first principles.

Definition (ν-self-concordance)**.** For a convex set $K \subseteq \mathbb{R}^n$, the function $\Phi : K \to \mathbb{R} \cup \{+\infty\}$ is ν-self-concordant if:

1. For all $x \in K$, it holds that $\nabla\Phi(x)^\top \nabla^2\Phi(x)^{-1}\nabla\Phi(x) \leq \nu$, and
2. For all $x \in K$ and $u, v, w \in \mathbb{R}^n$ it holds that

$$\left|\nabla^3\Phi(x)[u, v, w]\right| \leq 2\|u\|_{\nabla^2\Phi(x)}\|v\|_{\nabla^2\Phi(x)}\|w\|_{\nabla^2\Phi(x)}.$$

Here, $\|x\|_M := \sqrt{x^\top Mx}$ is the matrix norm, and $\nabla^3\Phi(x)[u, v, w]$ is the third order directional derivative in directions u, v, w.

(a) Prove that the function $\Phi(x) = -\log x$ on $K\{x : x > 0\}$ is 1-self-concordant.

(b) Let $\Phi_1, \Phi_2, \ldots, \Phi_k$ be $\nu_1, \ldots, \nu_k$-self-concordant functions on a domain K. Prove that $\Phi_1 + \cdots + \Phi_k$ is $\nu_1 + \cdots + \nu_k$-self-concordant.

(c) Prove that if Φ is a self-concordant function, then for x and $\|\Delta\|_{\nabla^2\Phi(x)} \leq \varepsilon < 1$ that

$$(1-\varepsilon)\nabla^2\Phi(x) \preceq \nabla^2\Phi(x+\Delta) \preceq \frac{1}{1-\varepsilon}\nabla^2\Phi(x).$$

Problem 2: IPM Using Self-Concordance

In this section we set up an IPM using self-concordant barriers. We consider the dual version of linear programs:

$$\min_{x\in K} c^\top x, \tag{0.1}$$

for a convex set K admitting a ν-self-concordant barrier function.

Central path and optimality. This motivates setting up the central path as follows:

$$x_\mu := \operatorname{argmin}_x c^\top x + \mu\Phi(x).$$

Then KKT conditions say that the centrality conditions are $c + \mu\nabla\Phi(x_\mu) = 0$. This brings us to the definition of ε-centered point in this setting.

Definition. We say that a point x is ε-centered for path parameter μ if $\|c+\mu\nabla\Phi(x_\mu)\|_{\nabla^2\Phi(x)^{-1}} \leq \varepsilon\mu$.

Let us analyze this method.

(a) (Advancing) Let $\delta \leq 1/2$ and $\nu \geq 1$. Prove that if x is ε-centered at path parameter μ, then for $\mu' = \left(1 - \frac{\delta}{\sqrt{\nu}}\right)\mu$, x is $2(\varepsilon + \delta)$-centered at path parameter μ'.

(b) (Solving for the step) Let $g = -(c + \mu\nabla\Phi(x))$. Check that the natural first order step to correct this gradient error is given by $x \to x + \delta_x$ for

$$\delta_x = \frac{1}{\mu}\nabla^2\Phi(x)^{-1}g.$$

(c) (Recentering) Let x be ε-centered for $\varepsilon \leq 1/100$. Prove that for this choice of δ that $x + \delta_x$ is at $10\varepsilon^2$-centered at path parameter μ.

Combining these steps gives an algorithm which takes $\tilde{O}(\sqrt{\nu})$ steps to find a solution to (0.1) to high-accuracy.

Hints

Problem 1(b): The hardest part is proving the following inequality for vectors $v_1, \dots, v_k$ and PSD matrices $M_1, \dots, M_k$

$$v^\top M^{-1} v \le \sum_{i=1}^k v_i^\top M_i^{-1} v_i \quad \text{where } v = \sum_{i=1}^k v_i, M = \sum_{i=1}^k M_i.$$

To prove this, use that $v_i^\top M_i^{-1} v_i \le \nu_i$ is equivalent to $\frac{1}{\nu_i} v_i v_i^\top \preceq M_i$ in the PSD ordering. Summing this over all i, it suffices then to prove that

$$\frac{1}{\sum_{i=1}^k \nu_i} v v^\top \preceq \sum_{i=1}^k \frac{1}{\nu_i} v_i v_i^\top,$$

which is much easier.

Problem 1(c): We want to prove that $v^\top \nabla^2 \Phi(x + \Delta) v \le \frac{1}{1-\varepsilon} v^\top \nabla^2 \Phi(x) v$. Define

$$f(t) = v^\top \nabla^2 \Phi(x + t\Delta) v$$

and use that $f'(t) = \nabla^3 \Phi(x + t\Delta)[\Delta, v, v]$ and use property 2 of self-concordance.

Problem 2(a): Use the triangle inequality and property 1 of self-concordance.

Problem 2(b): Just as discussed in the lecture, plug in $x \to x + \delta_x$ and note that $\nabla \Phi(x + \delta_x) \approx \nabla \Phi(x) + \nabla^2 \Phi(x) \delta_x$ is the first-order Taylor expansion.

Problem 2(c): Use the fundamental theorem of calculus to bound the higher-order error of the expansion $\nabla \Phi(x + \delta_x) \approx \nabla \Phi(x) + \nabla^2 \Phi(x) \delta_x$. You will need to use Problem 1(c).

References