Summer School on LP Solvers
Problem Set on Strongly Polynomial Algorithms

Throughout this exercise sheet we consider LP in primal-dual standard form:

$$\begin{aligned} &\min c^\top x & &\max y^\top b \\ &\text{s.t. } Ax = b & &A^\top y + s = c \\ &x \geq 0 & &s \geq 0 \end{aligned} \tag{1}$$

for $A \in \mathbb{R}^{d \times n}$, $b \in \mathbb{R}^d$ and $c \in \mathbb{R}^n$.

The central path solution for parameter $\mu \geq 0$ is defined to be the pair of primal-dual feasible solutions $(x^{cp}(\mu), s^{cp}(\mu))$ such that for all $i \in [n]$ we have that $x^{cp}(\mu)_i s^{cp}(\mu)_i = \mu$. Let furthermore (x^*, s^*) be a primal-dual pair of optimal solutions.

1. Recall the definition of the max-central path as

$$x_i^m(\mu) \coloneqq \max\{x_i : Ax = b, x \geq 0, c^\top x \leq c^\top x^* + \mu\} \tag{2}$$

   Prove that the max-central path approximates the central path up to a factor $2n$. More precisely, show that $(2n)^{-1} x_i^m(n\mu) \leq x_i^{cp}(\mu) \leq x_i^m(n\mu)$. *Hint: Make use of the fact that for any primal-dual feasible solution* (x, s) *we have that* $0 = (x - x^*)^\top (s - s^*)$ *and so* $x^\top s = x^\top s^* + s^\top x^* - (x^*)^\top s^* = x^\top s^* + s^\top x^*$.

2. Recall that a circuit $C \subset [n]$ of A is a minimal subset of dependent columns of A. That is: A_C does not have full column rank, however for all $\emptyset \subsetneq D \subsetneq C$ the matrix A_D does have full column rank. An elementary vector $x^C \in \ker(A)$ corresponding to circuit C is a vector s.t. $\operatorname{supp}(x^C) = C$, where $\operatorname{supp}(v) \coloneqq \{i \in [n] : v_i \neq 0\}$ denotes the support. The circuit imbalance $\kappa(A)$ of A is defined to be the maximal ratio of two entries in the support of an elementary vector. That is, $\kappa(A) \coloneqq \max_{x \in \mathcal{E}(A)} \max_{i,j \in \operatorname{supp}(x)} |x_j / x_i|$, where $\mathcal{E}(A)$ denotes the set of elementary vectors of A.
   Tomorrow we will see that Interior Point Methods can find an optimal solution to (1) in time $O\big(\operatorname{poly}(n) \log(\kappa(A))\big)$.
   Let A be the incidence matrix of an undirected graph, i.e., each column has exactly two non-zeros entries, both of which are 1.

   (a) Characterize the circuits of A.

(b) Show that $\kappa(A) \in \{1, 2\}$. For which graphs is $\kappa(A) = 1$?

3. So far, we have not discussed how to find an initial starting point (x, s) near the central path. As LP feasibility is generally as hard as LP optimization, we can not expect this to be an easy task. Therefore, in both theory and practice, IPMs are usually initialized on auxiliary LPs. One of the standard initialization techniques is the following: For some sufficiently large parameter $M > 0$, consider the primal dual program

$$\begin{aligned} &\min c^\top x + M\mathbf{1}^\top \hat{x} && \max y^\top b + 2M\mathbf{1}^\top z \\ &\text{s.t. } Ax - A\hat{x} = b && A^\top y + z + s = c \\ &x + \bar{x} = 2M\mathbf{1} && -A^\top y + \hat{s} = M\mathbf{1} \\ & && z + \bar{s} = 0 \\ &x, \hat{x}, \bar{x} \geq 0 && s, \hat{s}, \bar{s} \geq 0 \end{aligned} \tag{3}$$

(a) Show that if M is large enough, (in particular $M \gg \|b\|, \|c\|$) we can initialize this system near the central path (with some ε-error) for some

$$x \approx \hat{x} \approx \bar{x} \approx s \approx \hat{s} \approx \bar{s} \approx M\mathbf{1} \tag{4}$$

and a parameter $\mu \approx M^2$. Provide some intuition on why optimal solutions to (3) correspond to optimal solutions of (1).

(b) The constraint matrix of the modified system (3) is now

$$M = \begin{bmatrix} A & -A & 0 \\ I & 0 & I \end{bmatrix} \tag{5}$$

Show that $\kappa(M) = \kappa(A)$. In particular, this initialization technique preserves the circuit imbalance.

4. For a given partition $B \cup N = [n]$ of variables and a feasible vector $x > 0$ recall the trust region program for a particular type of update step in interior points methods:

$$\min_{\delta \in \ker(A)} \left\| \frac{x_N + \delta_N}{x_N} \right\|_2 \quad \text{s.t.} \quad \left\| \frac{\delta_B}{x_B} \right\|_2 \leq \varepsilon\,. \tag{6}$$

If we square constraint and objective and furthermore lagrangify the constraint with a fixed $\lambda \geq 0$ we obtain the unconstrained problem

$$\min_{\delta \in \ker(A)} \left\| \frac{x_N + \delta_N}{x_N} \right\|_2^2 + \lambda \left\| \frac{\delta_B}{x_B} \right\|_2^2 . \tag{7}$$

For fixed $\lambda > 0$, give a closed formula for the optimal solution of (7). *Note: This illustrates that the complexity in solving (6) stem from the fact that it is hard to guess a value of $\lambda \geq 0$ for which the optimal solution to (7) fulfills $\|\delta_B/x_B\|_2 \approx \varepsilon$.*