GENUS-2 LEFSCHETZ FIBRATIONS

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AMS Special Session on Invariants in Geometric Topology: Low Dimensions and Beyond

OUTLINE:

Remark:

feel free to interrupt with questions!

A. BACKGROUND

i. Lefschetz fibrations

ii. relationship to mapping class group

B. CURRENT PROGRESS

i. π_1 of a Lefschetz fibration

ii. results on new b_1 bounds

C. FUTURE DIRECTIONS

i. future questions

ii. future techniques

A. BACKGROUND: Lefschetz fibrations

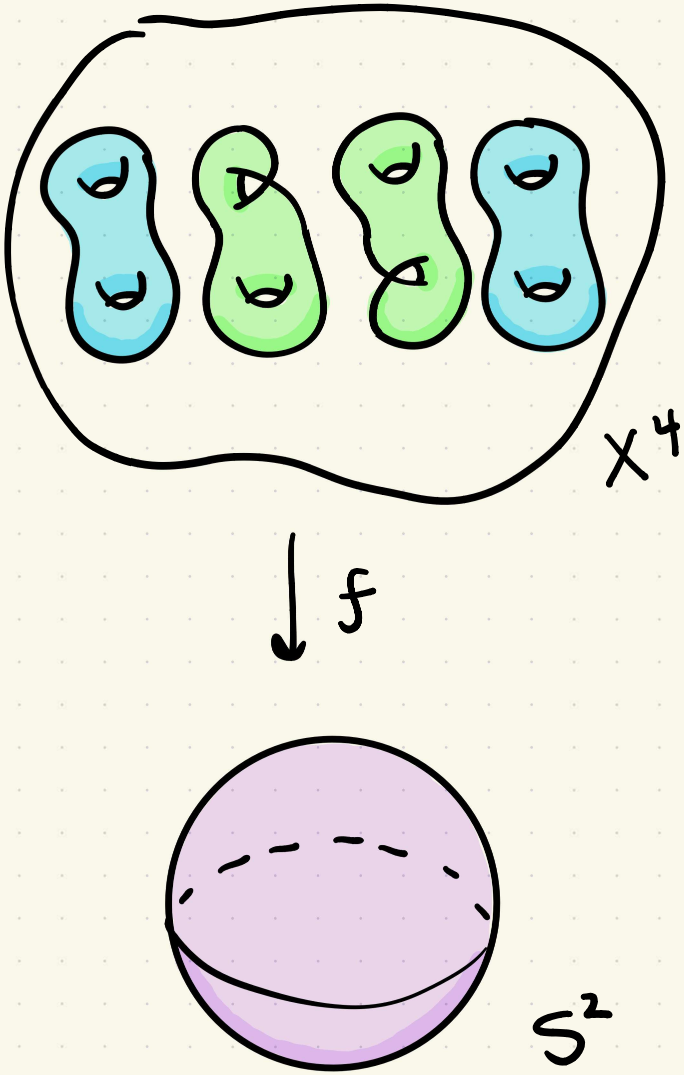
symplectic manifolds
have the structure of a
Lefschetz pencil

Lefschetz fibrations in symplectic geometry, 1998

Lefschetz pencils
have symplectic structures

The topology of symplectic manifolds, 2001

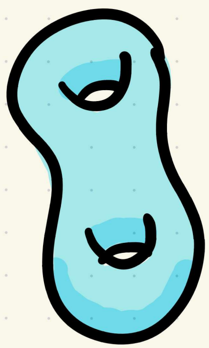
A. BACKGROUND: Lefschetz fibrations



1. $f : X^4 \longrightarrow S^2$ is a smooth surjection
2. finitely many critical values $q_1, \ldots, q_n$
3. each $f^{-1}(q_i) \in X$ has local coord. chart in which $$f(z,w) = zw$$
4. $f^{-1}(b)$ = regular fiber (is a genus-g surface)

A. BACKGROUND: Lefschetz fibrations

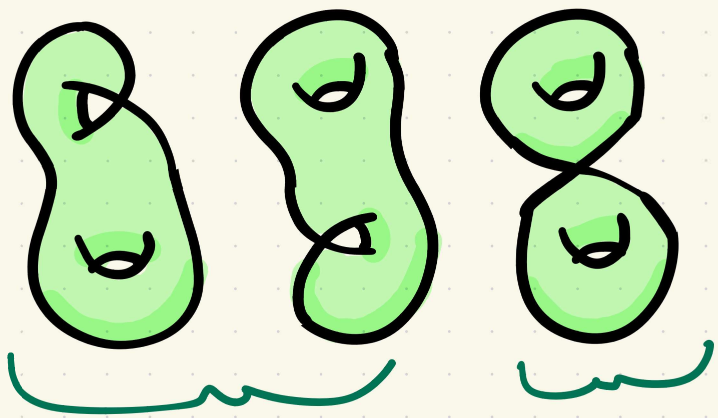
Regular fiber



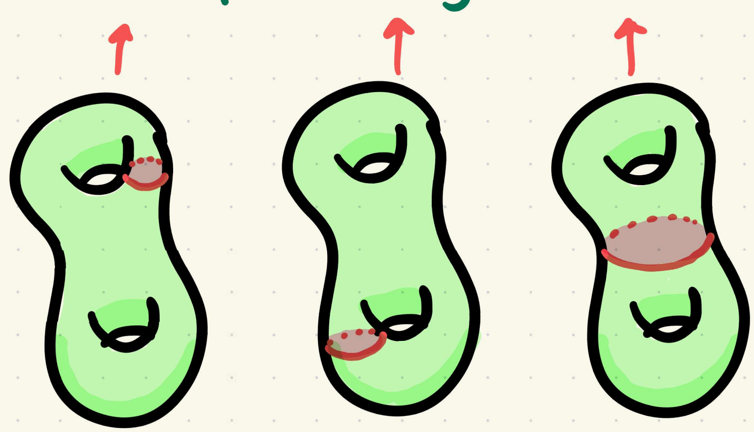
genus 2

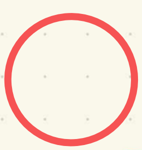
Σ_2

Singular fiber



nonseparating　　separating



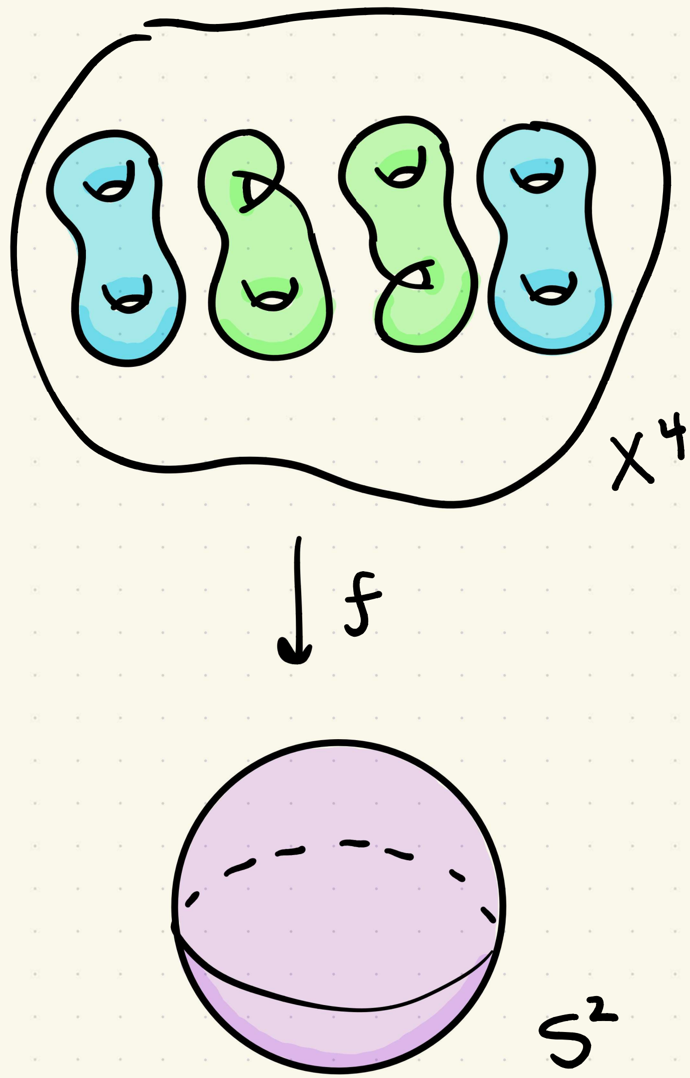


S^1 = vanishing cycle :

A. BACKGROUND: mapping class group

Remark 1:

genus of regular fiber
$\parallel$
genus of the L.F.



Remark 2:

the monodromy
determines the LF

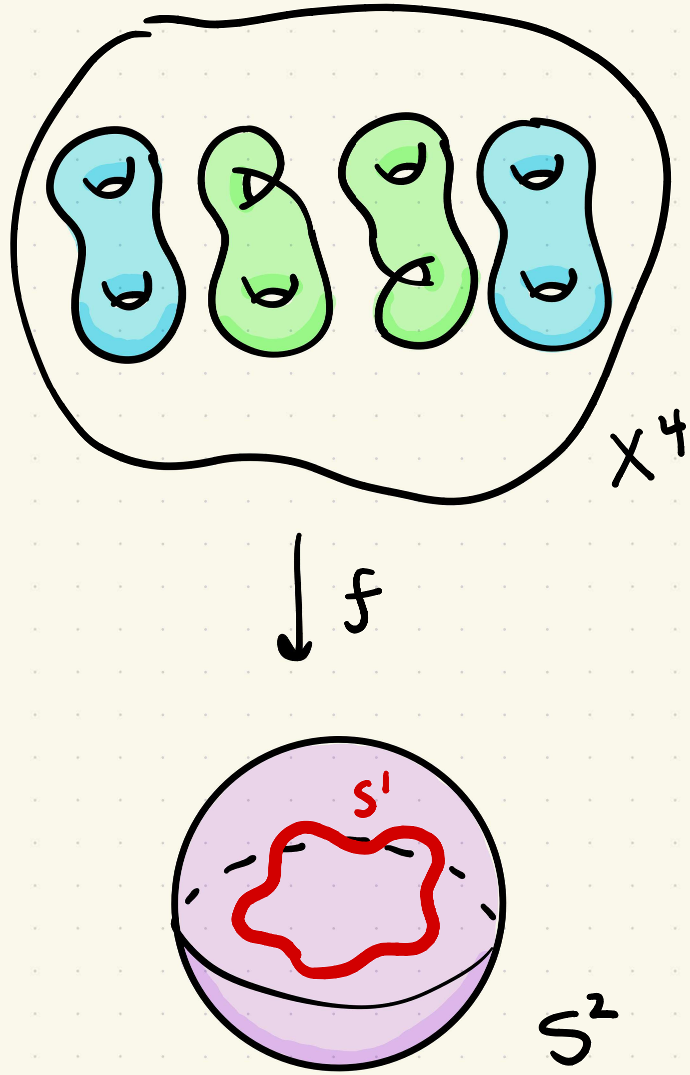
A. BACKGROUND: mapping class group

monodromy of a LF:

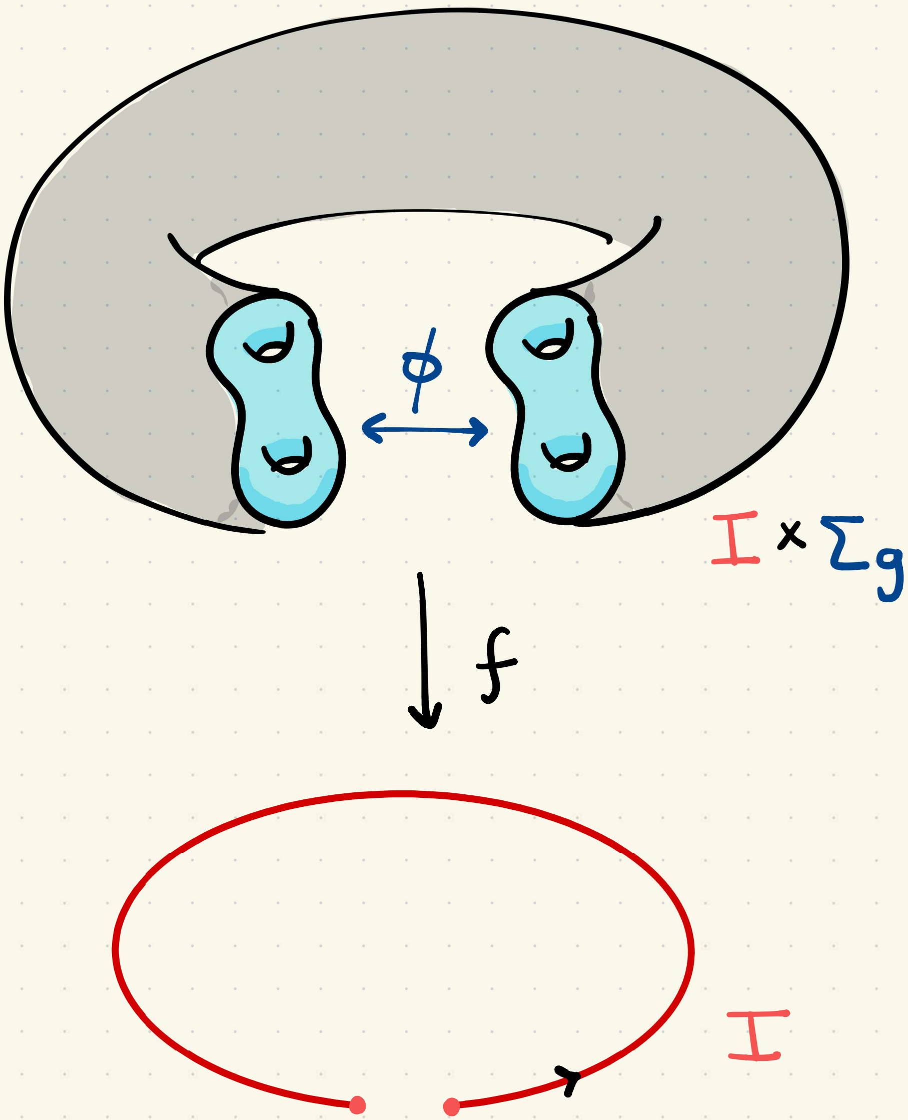
1. embedded S^1 in base space S^2

2. pre-image
$$= I \times \Sigma_g \big/ \left((\{0\} \times \Sigma_g) \sim (\{1\} \times \Sigma_g) \right)$$

3. **monodromy** ϕ = the self-diffeo of a regular fiber Σ_g to itself



A. BACKGROUND: mapping class group



Remarks:

- ϕ is self-diffeo of Σ_g to itself
- $\phi \in \mathrm{Mod}(\Sigma_g)$

$$\phi = \tau_{\gamma_n} \circ \ldots \circ \tau_{\gamma_2} \circ \tau_{\gamma_1}$$

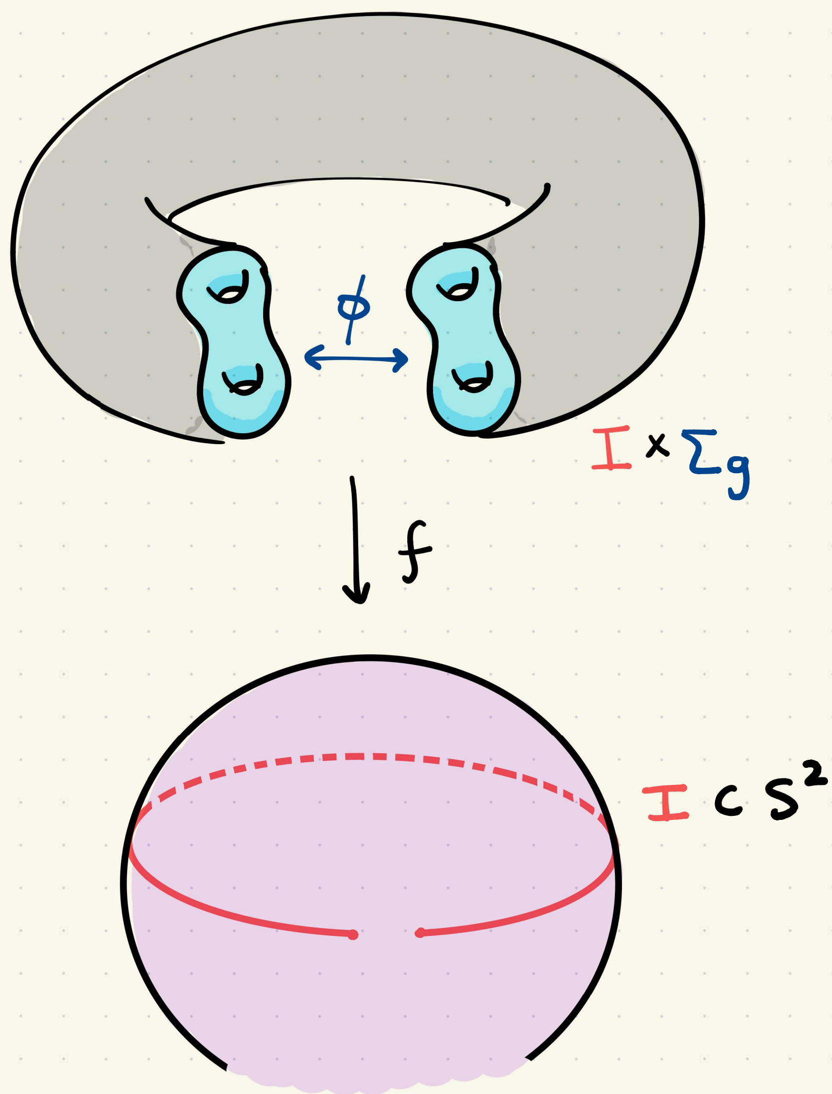
where τ are positive Dehn twists of order 1

A. BACKGROUND: mapping class group

motivation for $\phi = \mathrm{id}_{\mathrm{Mod}(\Sigma_g)}$

1. S^1 loop around only regular fibers, then

$$\phi = \mathrm{Id}_{\mathrm{mod}(\Sigma_g)}$$



A. BACKGROUND: mapping class group

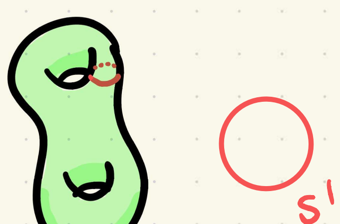
motivation for $\phi = \mathrm{id}_{\mathrm{Mod}(\Sigma_g)}$

2. S^1 loop around a critical value q, then

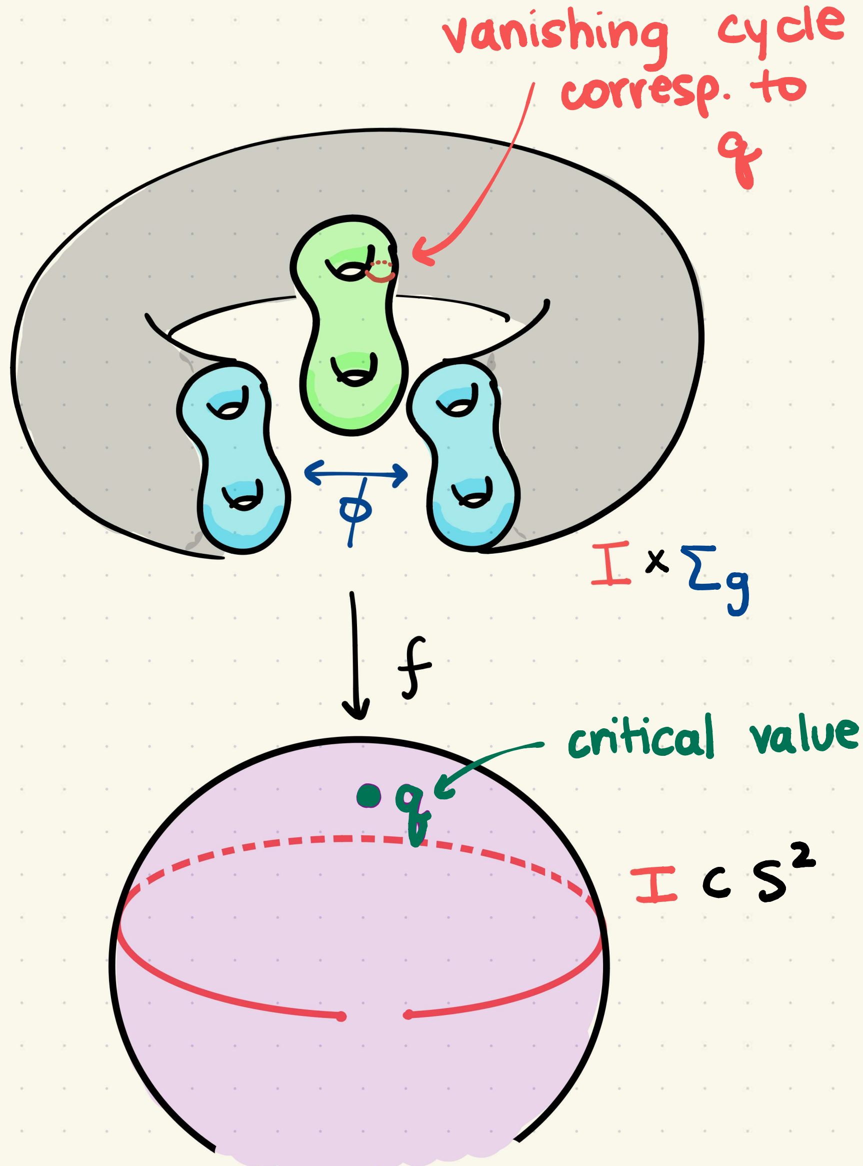
$$\phi = \tau_\gamma$$

where γ = vanishing cycle corresponding to singular fiber $f^{-1}(q)$

Recall:

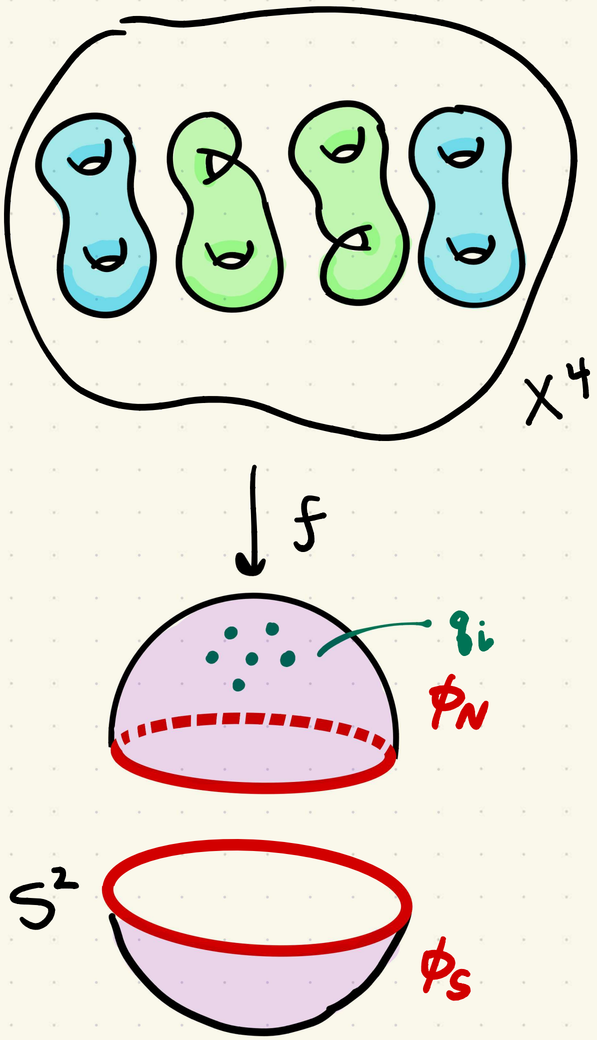


= vanishing cycle



A. BACKGROUND: mapping class group

Lefschetz fibrations over S^2:



If $\Sigma_g \longrightarrow X^4$ is a Lefschetz
$$\begin{array}{c} X^4 \\ \downarrow f \\ S^2 \end{array}$$
fibration, then $\phi_N = \phi_S$.

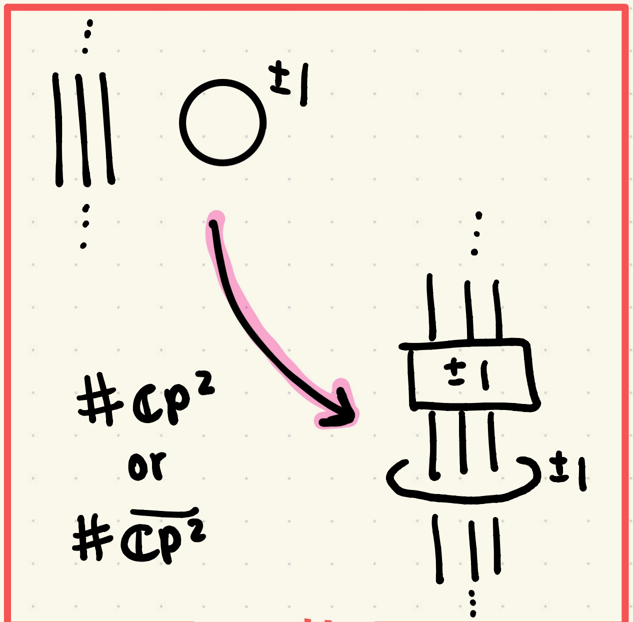
$$\phi_N = \tau_{\gamma_n} \circ \cdots \circ \tau_{\gamma_2} \circ \tau_{\gamma_1}$$

$$\phi_S = id_{\mathrm{mod}(\Sigma_g)}$$

$$\Rightarrow \tau_{\gamma_n} \circ \cdots \circ \tau_{\gamma_1} = id_{\mathrm{mod}(\Sigma_g)}$$

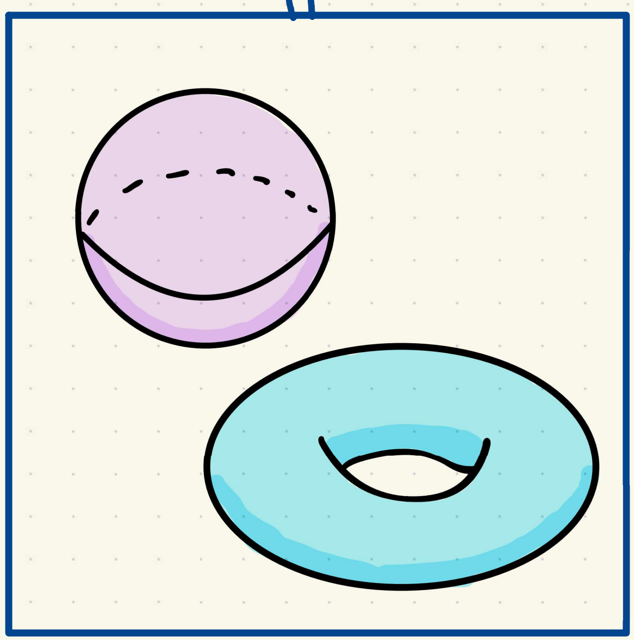
B. CURRENT PROGRESS

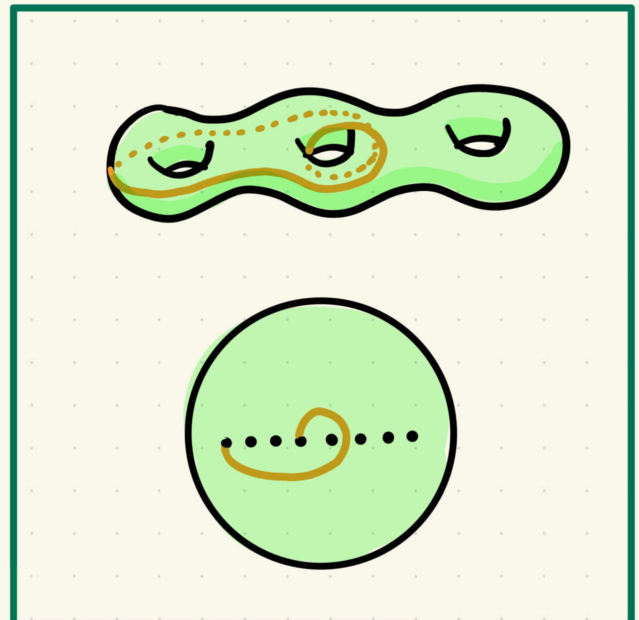
Goal: understand all possible groups that can be π_1 of a nontrivial genus-2 Lefschetz fib.



Blow-ups do not change π_1 of the 4 manifold

genera 0 and 1 already classified





Q's about curves on surfaces (or braids on n-strands)

B. CURRENT PROGRESS

[Gompf] every finitely presented group is π_1 of some closed symplectic 4-manifold

A new construction of symplectic manifolds, 1995

Question:

What are all possible fundamental groups of a nontrivial genus 2 Lefschetz fibration?

Current known fundamental groups:

[Ozbagci–Stipsicz '98, Korkmaz '07, Cai–Chafee–Lytle–Vorontsova '24]:

$$0,\ \mathbb{Z},\ \mathbb{Z}/n\mathbb{Z},\ \mathbb{Z}\oplus\mathbb{Z},\ \mathbb{Z}/n\oplus\mathbb{Z}/m,\ \mathbb{Z}\oplus\mathbb{Z}/n$$

B. CURRENT PROGRESS

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Question:

What are all possible fundamental groups of a nontrivial genus 2 Lefschetz fibration?

Conjecture: Abelian and ≤ 2 generators

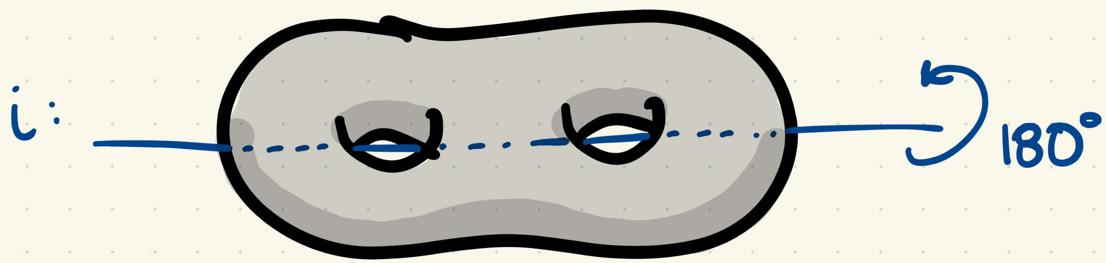
B. CURRENT PROGRESS

- rely on hyperellipticity of scc's on Σ_2

[Margalit-Winarski]

Every element of $\mathrm{Mod}(\Sigma_2)$ has a representative that commutes with i

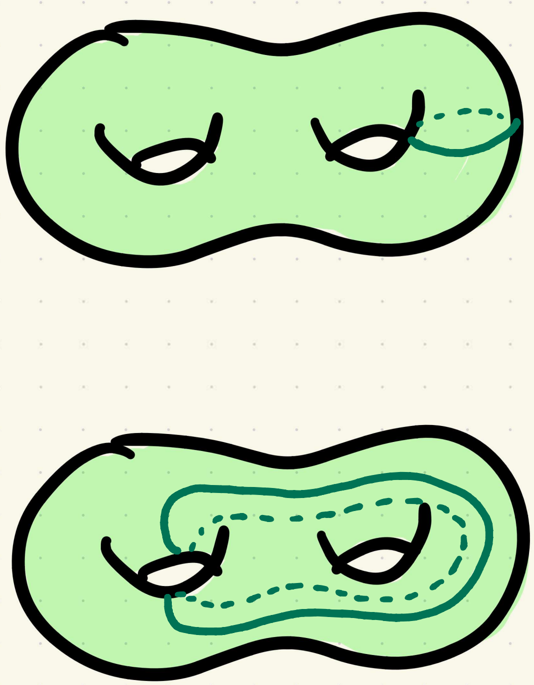
hyperelliptic involution



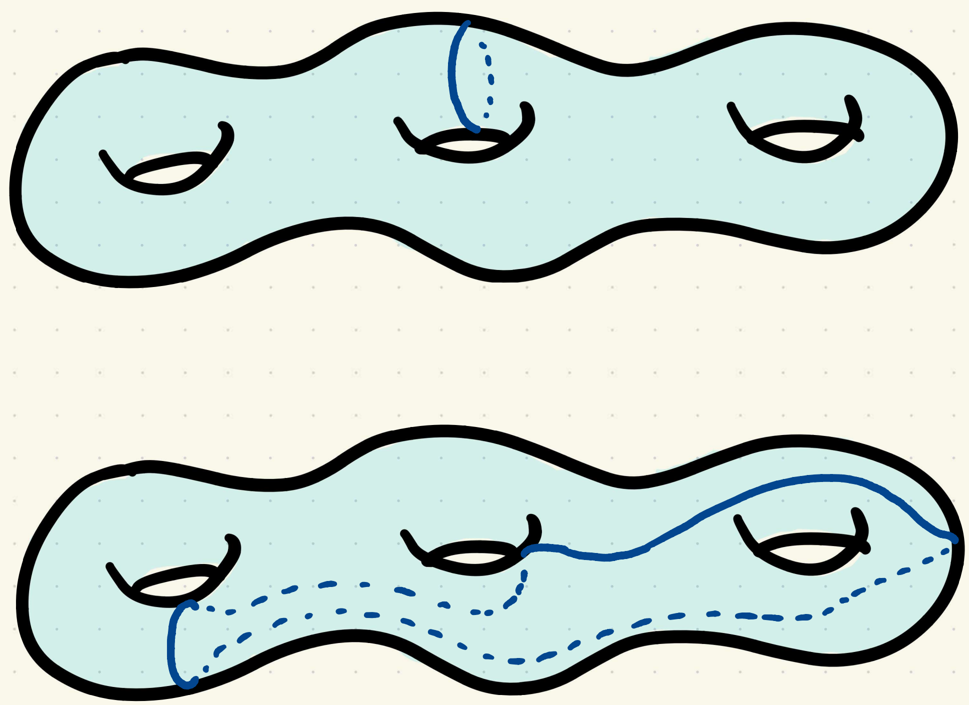
B. CURRENT PROGRESS

- rely on hyperellipticity of scc's on Σ_2

hyperelliptic:



not hyperelliptic:



B. CURRENT PROGRESS

- rely on hyperellipticity of scc's on Σ_2

CONSEQUENCES OF HYPERELLIPTICITY:

- can compute χ, σ, c_1^2, χ_h, ...

- n = no. of nonseparating vc's
  s = no. of separating vc's

$$H_1(\mathrm{mod}(\Sigma_2)) = \mathbb{Z}/10\mathbb{Z} \Rightarrow n + 12s \equiv 0 \pmod{10}$$

- we know a lot about a genus-2 L.F. of type (n,s) if it exists

B. CURRENT PROGRESS

KNOWN

$2n - s \geq 3$ and

$n + 7s \geq 20$

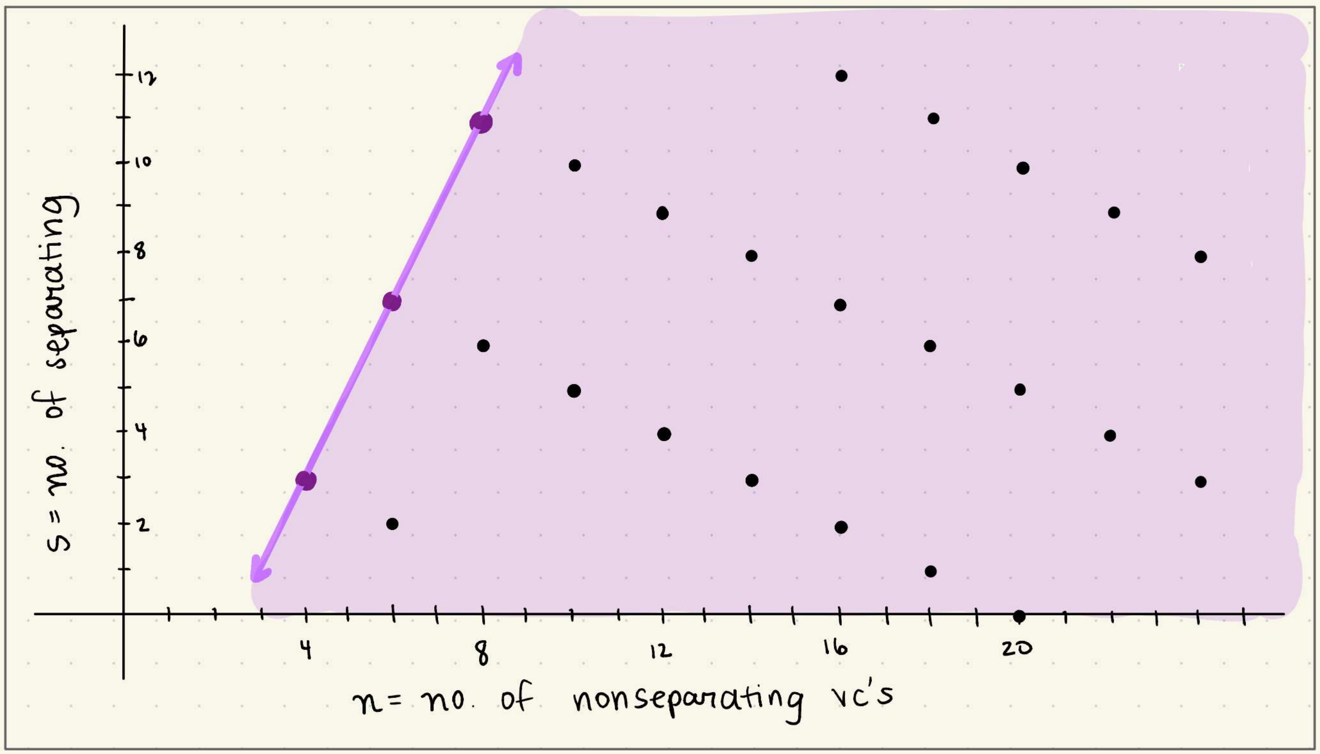
[Baykur – Korkmaz, 2016]

IMPROVEMENT

$2n - s \geq 5$

(tighter bound)

but more interestingly ...



the LF's on this line are indecomposable if they exist

Proof:

no way to add two vectors in parallelogram and output $(n, 2n-5)$

B. CURRENT PROGRESS

[Seibert-Tian, '99] Nontrivial hyperelliptic genus $g \geq 2$ Lefschetz fibrations with only nonseparating vanishing cycles are simply connected.

Let X be a genus-2 LF. of type (n, s):

- $(n, 0) \implies b_1(X) = 0$
- $(n, s)\ s > 0 \implies b_1(X) \geq -\frac{4}{5}n + \frac{2}{5}s + 4$

[Baykur-Korkmaz] Small LFs and exotic 4-manifolds, 2017

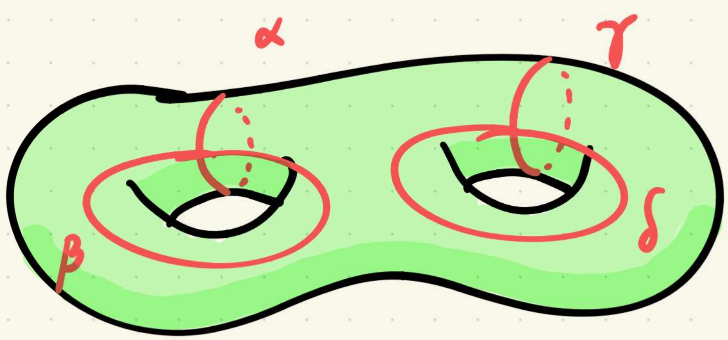
$\hookrightarrow \forall\, n, s$, this lower bound is either 0 or 2.

B. CURRENT PROGRESS

Let X be a genus-2 L.F. of type (n,s):

KNOWN → **IMPROVEMENT**

$0 \leq b_1(X) \leq 4$ → $0 \leq b_1(X) \leq 2$



$\{\alpha, \beta, \gamma, \delta\}$ a basis for $H_1(\Sigma_2)$

B. CURRENT PROGRESS

Let X be a genus-2 LF. of type (n,s):

KNOWN → IMPROVEMENT

$0 \leq b_1(X) \leq 3$

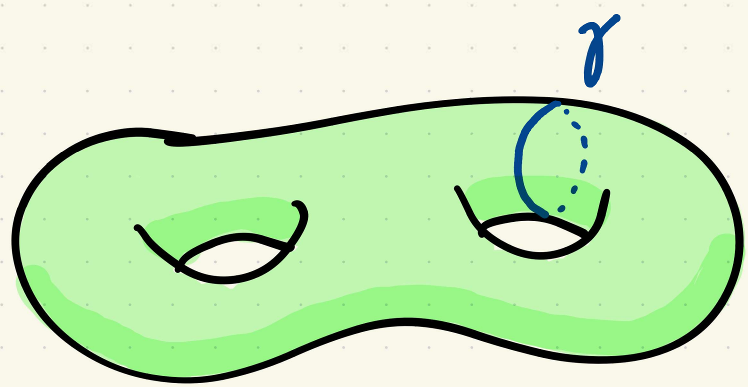
$\Uparrow$

$0 \leq b_1(X) \leq 2$

Improvement due to Smith:

[Smith] A nontrivial genus-g LF over S^2 must have at least 1 nonseparating v.c.

↳ improved to $\frac{4g+2}{5}$



can always find γ after some homeomorphism

B. CURRENT PROGRESS

Let X be a genus-2 LF. of type (n,s):

KNOWN → **IMPROVEMENT**

$0 \leq b_1(X) \leq 3$

$0 \leq b_1(X) \leq 2$

(sharp)

Idea of Proof that $b_1(X) \neq 3$:

Proposition

For a, b isotopy classes of scc's in Σ_g, for any $k \geq 0$,

$$\boxed{\Psi(\tau_b^k)[a] = [a] + k \cdot \hat{i}(a,b)[b]}$$

$\hat{i}$ = alg. intersection number

Suppose $b_1(X) = 3$

$\Downarrow$

all nonsep. vc's are in same homology class

$\Downarrow$

together with Prop, ϕ can't be identity

THANKS FOR LISTENING!