BETTI NUMBERS AND INDECOMPOSABILITY OF GENUS-G LEFSCHETZ FIBRATIONS

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OUTLINE:

Remark: feel free to interrupt with questions!

A. BACKGROUND

i. Lefschetz fibrations

ii. relationship to mapping class group

B. CURRENT PROGRESS

i. π_1 of a Lefschetz fibration

ii. results on new b_1 bounds

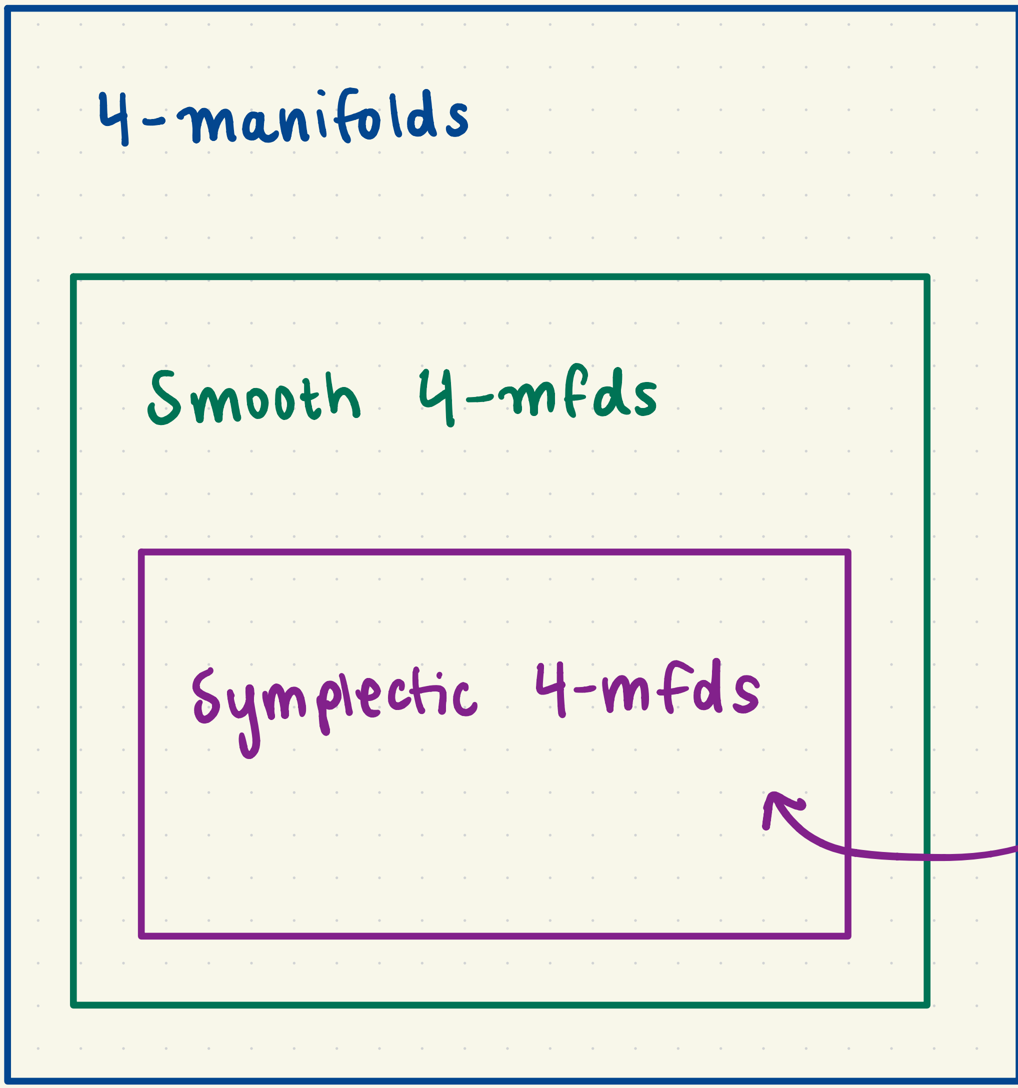
iii. indecomposability of a genus-2 family

C. FUTURE DIRECTIONS

i. future questions + techniques

ii. (if time) another proof of Matsumoto's relations

A. BACKGROUND: Lefschetz fibrations



A symplectic 4-mfd is a smooth 4-mfd X equipped with ω, a closed, nondegenerate 2-form.

$$\omega_{std} = \sum_{i=1}^{n} dx_i \wedge dy_i$$

(on $\mathbb{R}^n$)

A. BACKGROUND: Lefschetz fibrations

Why physicists care about symplectic $2n$-manifolds:

- many classical mechanics systems evolve on phase space which naturally carries a sympl. str.
- equations of motion come from a Hamiltonian function and are encoded by sympl. 2-forms
- electromagnetic fields fit into this framework, like one of maxwell's equations:

Faraday's law of induction: $dF = 0$

F behaves like a symplectic structure on spacetime coming from electric and magnetic fields

A. BACKGROUND: Lefschetz fibrations

Symplectic Structures:

Physics — gives all natural constructions of 4-manifolds

$\updownarrow$

Topology — seeks to classify and "understand" all possible ones

So what do the topologists know? ...

A. BACKGROUND: Lefschetz fibrations

symplectic manifolds
have the structure of a
Lefschetz pencil

Lefschetz fibrations in symplectic geometry, 1998

Lefschetz pencils
have symplectic structures

The topology of symplectic manifolds, 2001

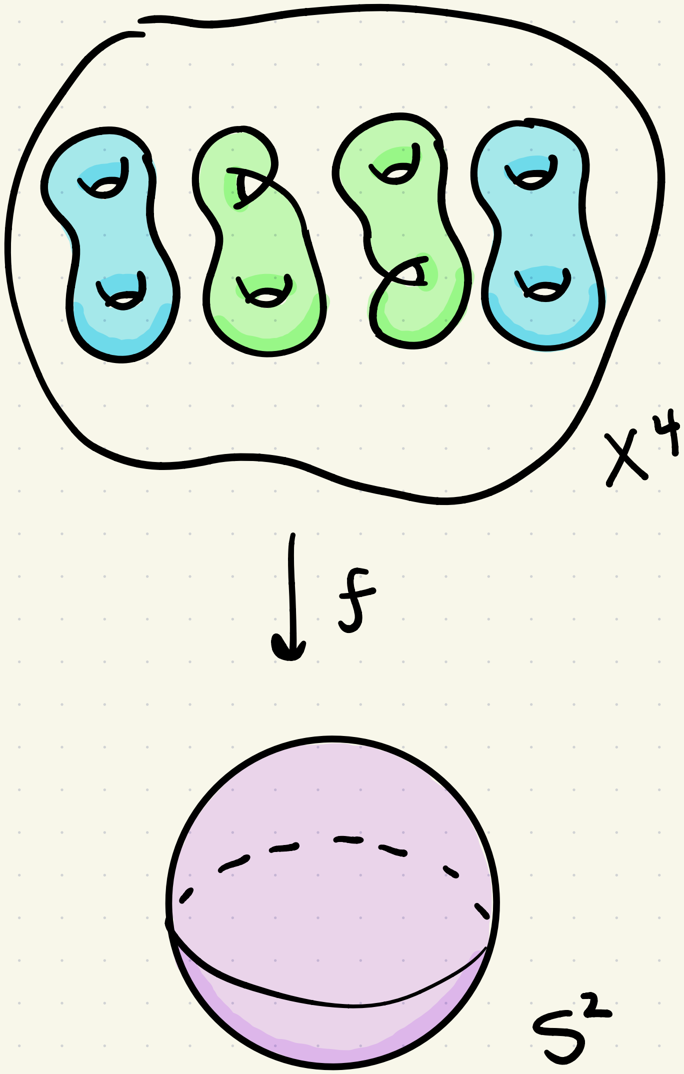
A. BACKGROUND: Lefschetz fibrations

$$\left\{ \begin{array}{c} \text{symplectic} \\ \text{4-mfds} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{genus-}g \\ \text{Lefschetz fibrations} \\ \text{over } S^2 \end{array} \right\}$$

Upshot: nearly surface bundles, combinatorial data, straightforward invariant computations

Q: what is a Lefschetz fibration?

A. BACKGROUND: Lefschetz fibrations



1. $f: X^4 \longrightarrow S^2$ is a smooth surjection

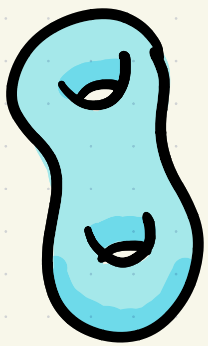
2. finitely many critical values $q_1, q_2, \ldots, q_n$

3. each $f^{-1}(q_i) \in X$ has local coordinate charts in which $f(z,w) = zw \qquad z, w \in \mathbb{C}$

4. $f^{-1}(b)$ for b a regular value is a regular fiber (a closed genus-g surface)

A. BACKGROUND: Lefschetz fibrations

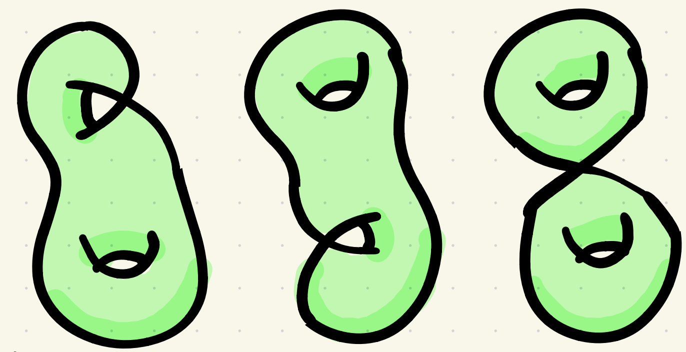
Regular fiber



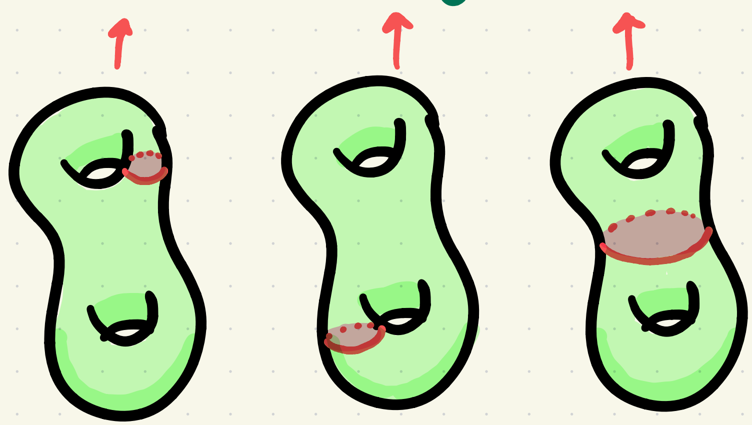
genus 2

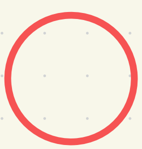
Σ_2

Singular fiber



nonseparating   separating



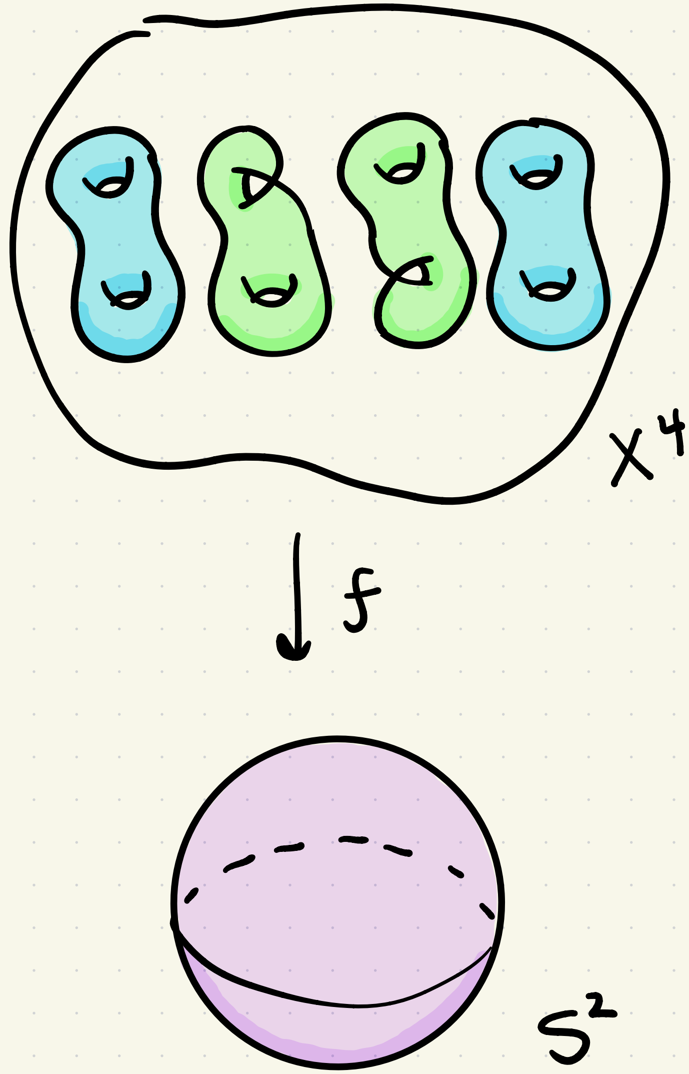


S^1 = vanishing cycle :

A. BACKGROUND: mapping class group

Remark 1:

genus of regular fiber
=
genus of the L.F.



Remark 2:

the monodromy
determines the LF

A. BACKGROUND: mapping class group

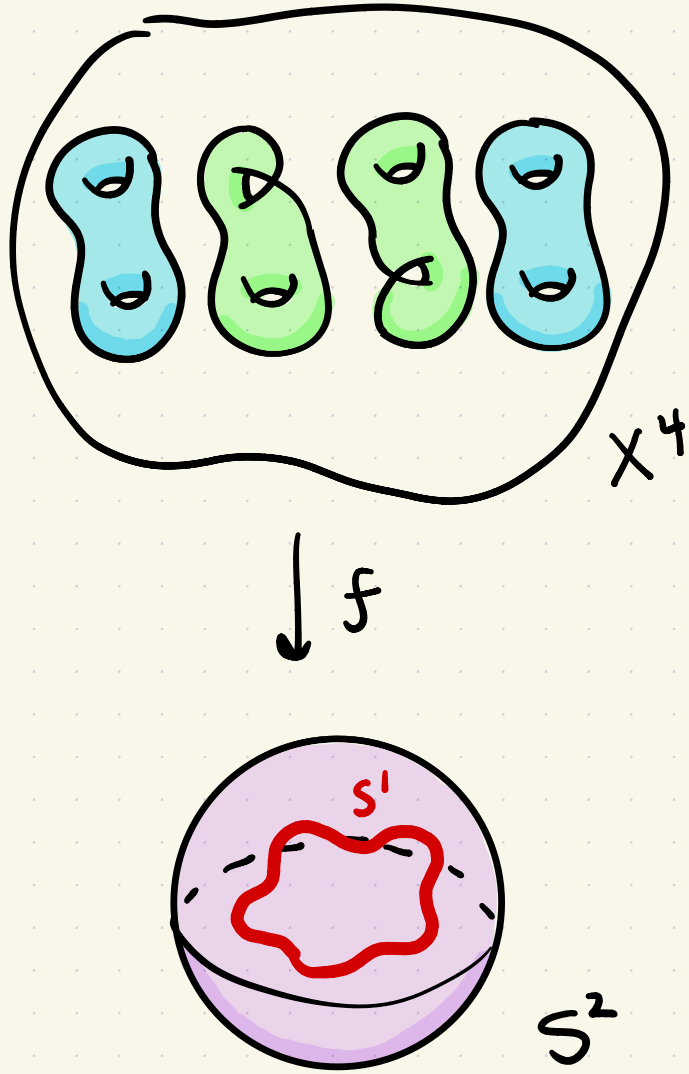
monodromy of a LF:

1. embedded S^1 in base space S^2

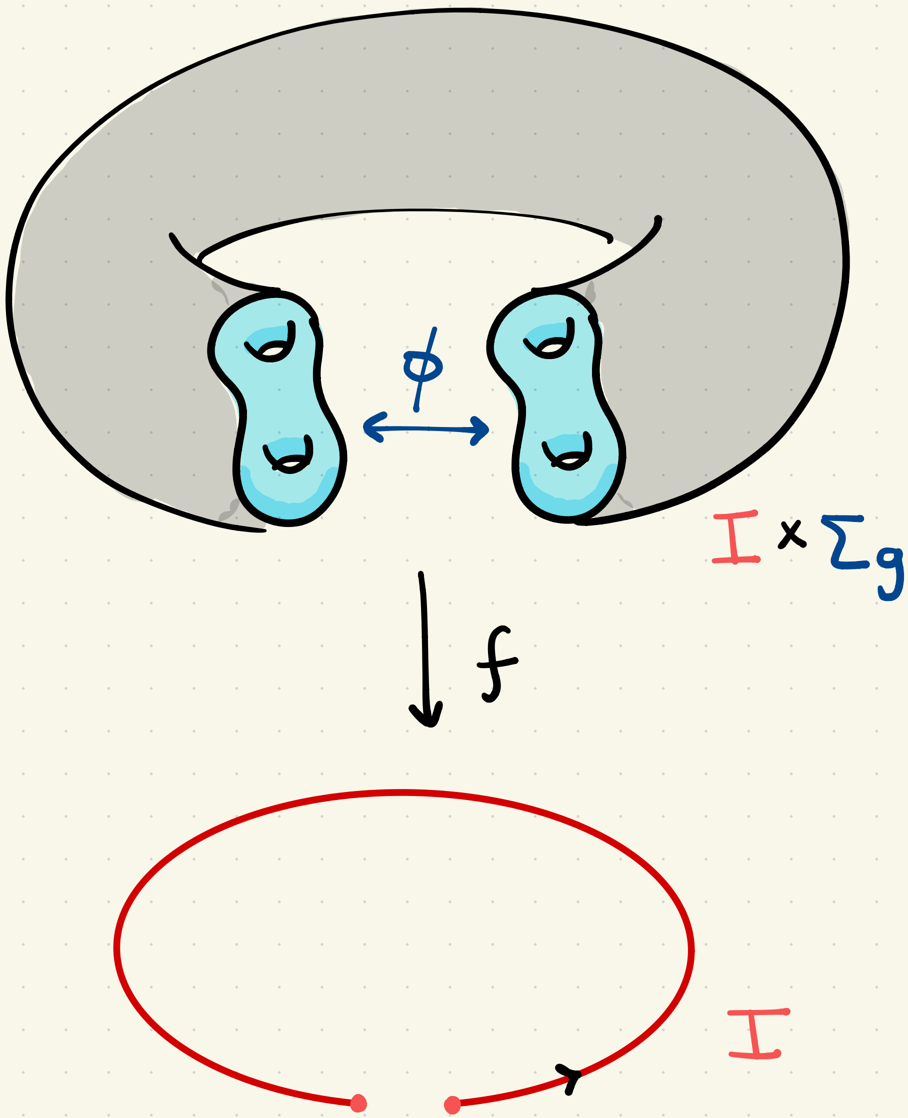
2. pre-image

$$= I \times \Sigma_g \Big/ \big((\{0\} \times \Sigma_g) \sim (\{1\} \times \Sigma_g) \big)$$

3. monodromy ϕ = the self-diffeo of a regular fiber Σ_g to itself



A. BACKGROUND: mapping class group



Remarks:

- ϕ is self-diffeo of Σ_g to itself
- $\phi \in \mathrm{Mod}(\Sigma_g)$

$$\phi = \tau_{\gamma_n} \circ \dots \circ \tau_{\gamma_2} \circ \tau_{\gamma_1}$$

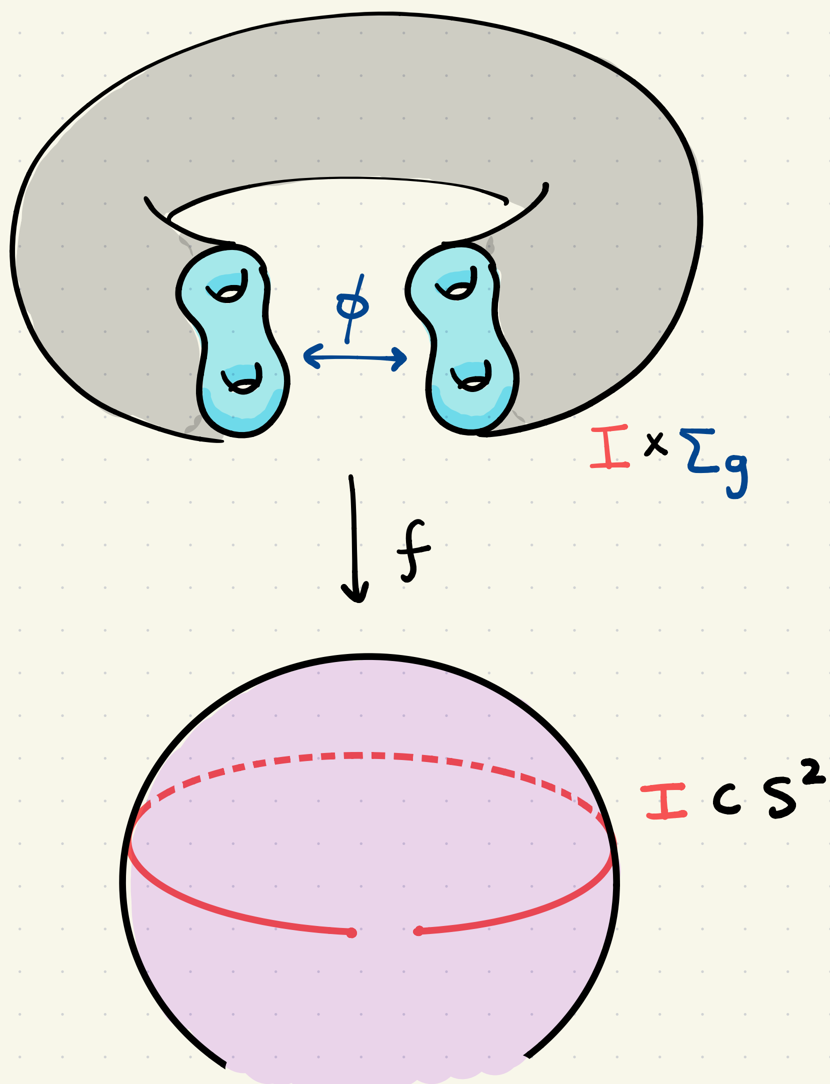
where τ are positive Dehn twists of order 1

A. BACKGROUND: mapping class group

motivation for $\phi = \mathrm{id}_{\mathrm{Mod}(\Sigma_g)}$

1. S^1 loop around only regular fibers, then

$$\phi = \mathrm{Id}_{\mathrm{mod}(\Sigma_g)}$$



A. BACKGROUND: mapping class group

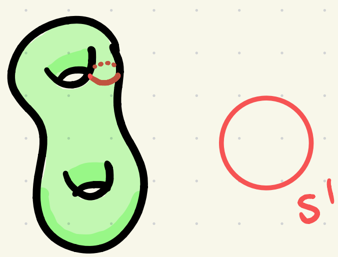
motivation for $\phi = \mathrm{id}_{\mathrm{Mod}(\Sigma_g)}$

2. S^1 loop around a critical value q, then

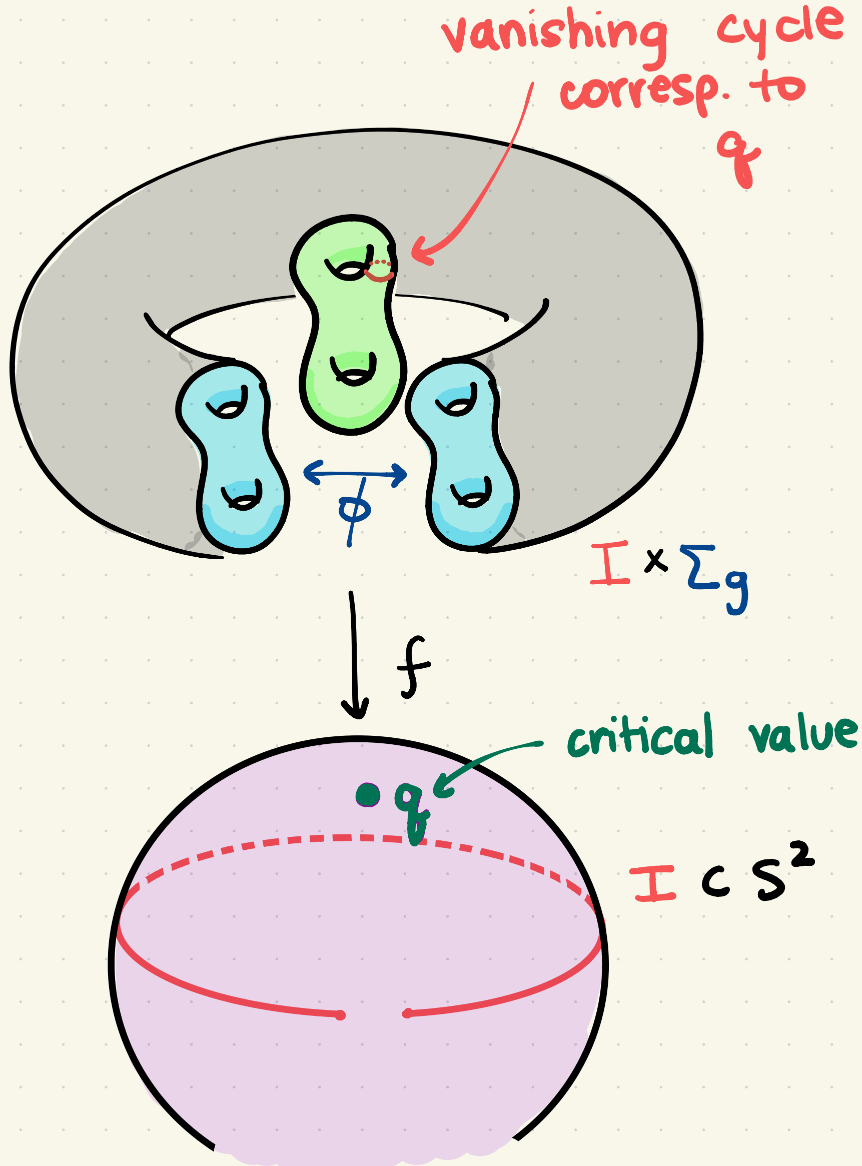
$$\phi = \tau_\gamma$$

where γ = vanishing cycle corresponding to singular fiber $f^{-1}(q)$

Recall:

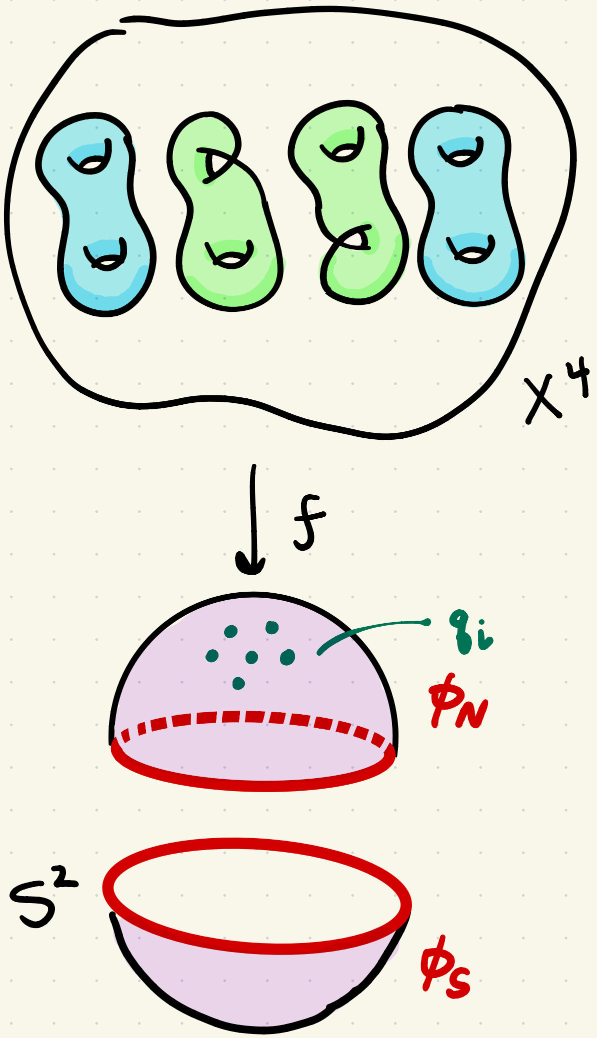


= vanishing cycle



A. BACKGROUND: mapping class group

Lefschetz fibrations over S^2:



If $\Sigma_g \longrightarrow X^4 \xrightarrow{f} S^2$ is a Lefschetz fibration, then $\phi_N = \phi_S$.

$$\phi_N = \tau_{\gamma_n} \circ \cdots \circ \tau_{\gamma_2} \circ \tau_{\gamma_1}$$

$$\phi_S = id_{\mathrm{mod}(\Sigma_g)}$$

$$\Longrightarrow \tau_{\gamma_n} \circ \cdots \circ \tau_{\gamma_1} = id_{\mathrm{mod}(\Sigma_g)}$$

B. CURRENT PROGRESS

Research questions surrounding $f: X \to S^2$, a Lefschetz fibration

1. What are all possible monodromies for X with a fixed fiber genus?

2. How to find new monodromies? (Q about id. fact. of mcg)

3. What is fundamental group of X?
... or signature, Euler char, 1^{st} Betti no, ...

4. Which X are indecomposable?

B. CURRENT PROGRESS

Research questions surrounding $f : X \to S^2$, a Lefschetz fibration

1. What are all possible monodromies for X with a fixed fiber genus? → answered for $g = 0, 1$; somewhat for $g = 2$; no hope? $g \geq 3$

2. How to find new monodromies? (Q about id. fact. of mcg) → current methods not exhaustive

3. What is fundamental group of X? ... or signature, Euler char, 1^{st} Betti no, ... → only when monodromy is (somewhat) known

4. Which X are indecomposable? → mostly unknown $g \geq 2$

B. CURRENT PROGRESS

Results for any genus:

Proposition 1 (K 202*)

Let $f: X^4 \to S^2$ be a genus-g nontrivial Lef. fib.
then $0 \leq b_1(X) \leq 2g-2$, this is sharp for $g=2$.

Proposition 2 (K 202*)

Let $f: X^4 \to S^2$ be a genus-g Lef. fib. with monodromy $\phi = \mathrm{id} \in \mathrm{Mod}(\Sigma_g)$.
If ϕ is transitive w.r.t. $\mathrm{Mod}(\Sigma_g)$, then $\pi_1(X) = 0$.

B. CURRENT PROGRESS

Lemma 3 (K 202*)

Let $f: X^4 \to S^2$ be a genus-2 nontrivial Lef. fib. with n nonseparating and s separating vc's. Then $2n - 5 \geq s$. This bound is sharp.

Lemma 4 (K 202*)

Let $f: X^4 \to S^2$ be a genus-2 nontrivial Lef. fib. Then X of type $(2k, 4k-5)$ have $b_1 = 2$ and are indecomposable Lef. fibrations (if they exist).

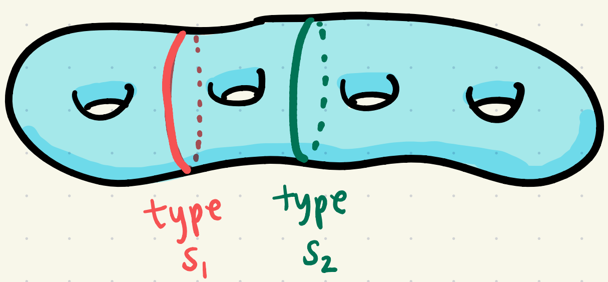
B. CURRENT PROGRESS

Let's begin with the following result:

Lemma 3 (K 202*)

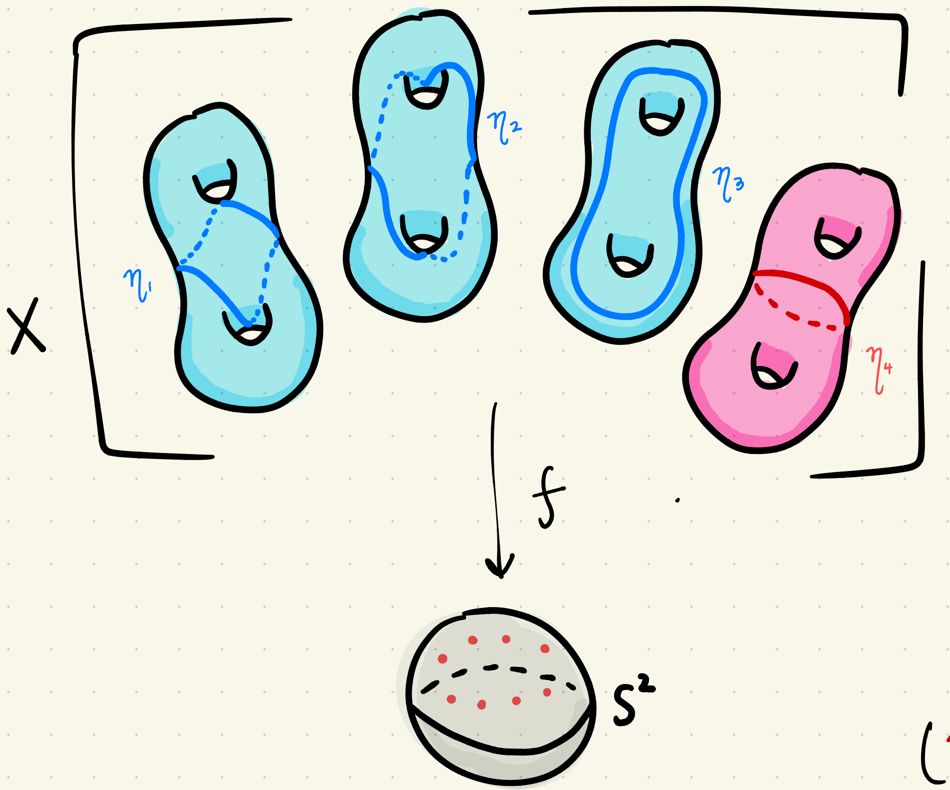
Let $f: X^4 \to S^2$ be a genus-2 nontrivial Lef. fib. with n nonseparating and s separating vc's. Then $2n - 5 \geq s$. This bound is sharp.

Defn: A Lefschetz fibration with genus-g fibers is of type $(n, s_1, s_2, \ldots, s_k)$ for $k = \lfloor \frac{g}{2} \rfloor$ if it has n nonseparating and s_i separating vanishing cycles.



B. CURRENT PROGRESS

Example:



This is Matsumoto's genus-2 LF and it is of type (6,2)

- genus-2 Lefschetz fib with 8 vanishing cycles

$$\phi = (\tau_{\eta_4} \circ \tau_{\eta_3} \circ \tau_{\eta_2} \circ \tau_{\eta_1})^2$$

By Lemma, $2n - 5 \geq s$

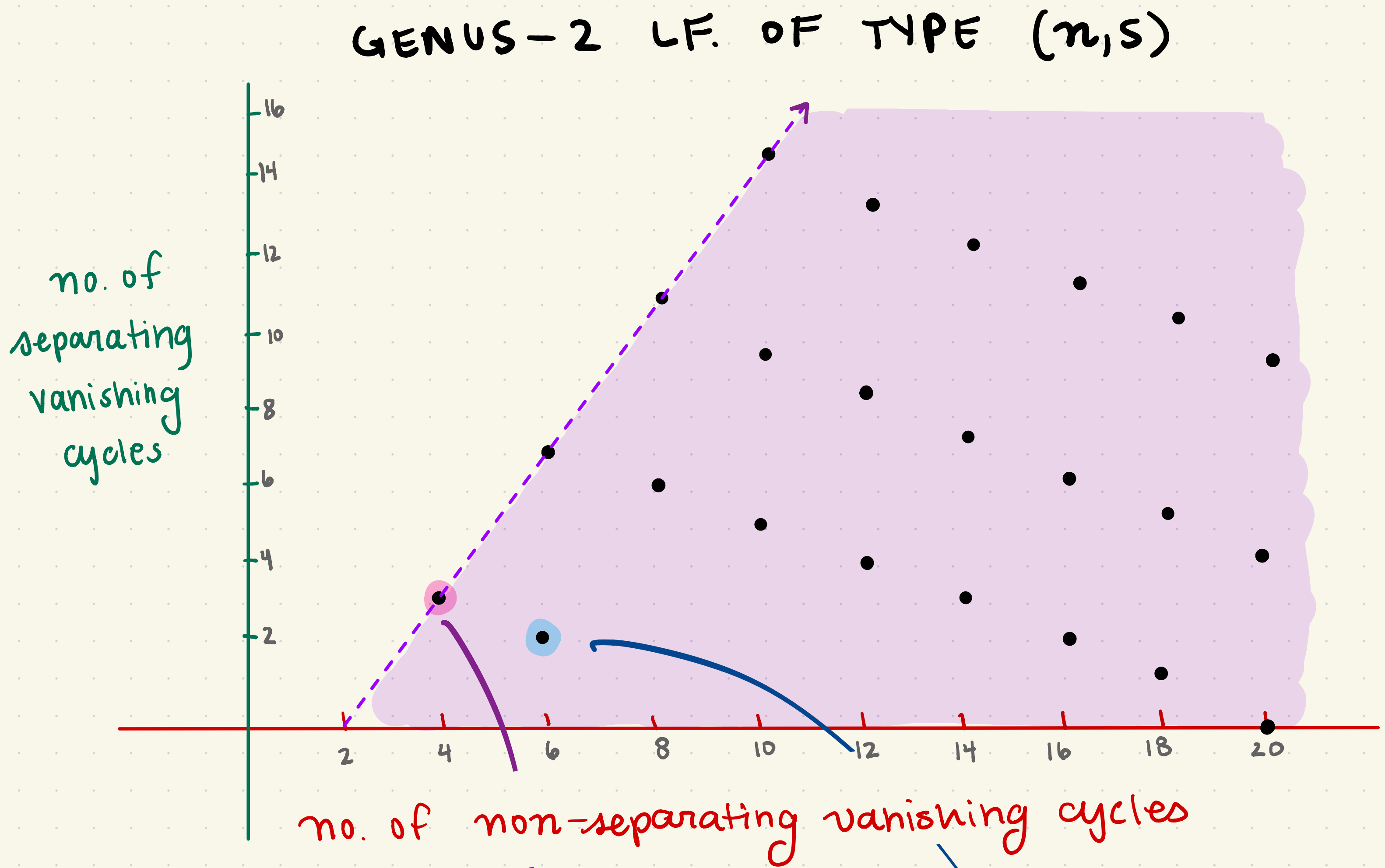
$(n, s) = (6, 2) \rightsquigarrow 2(6) - 5 \geq 2$

$7 \geq 2$

$(n, s) = (4, 3) \rightsquigarrow 2(4) - 5 \geq 3$

$3 \geq 3$

B. CURRENT PROGRESS



(4,3)

Smallest LF

“Small Lefschetz fibrations and exotic 4-manifolds” (Baykur-Korkmaz, '17)

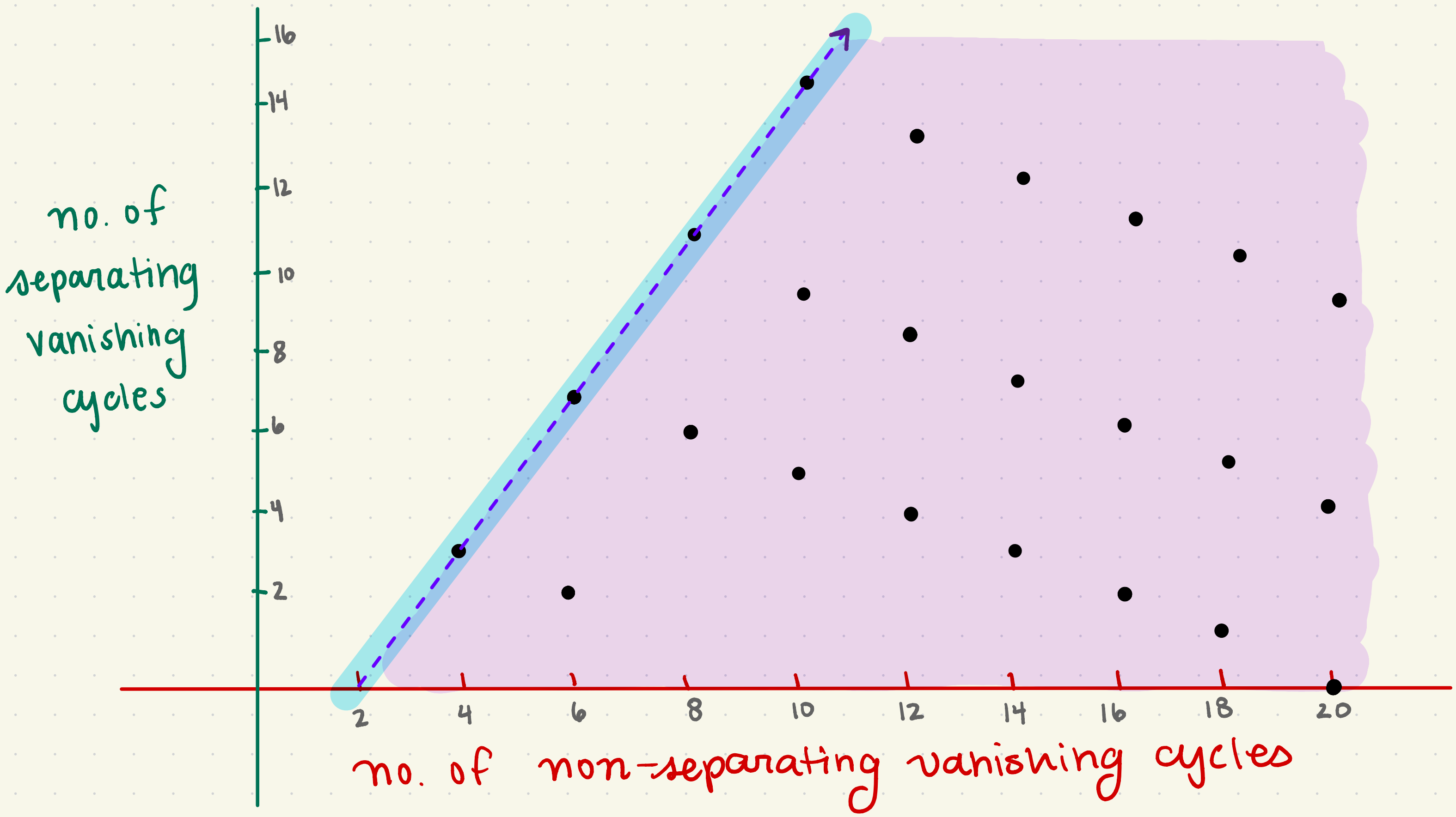
(6,2)

Matsumoto's LF

(Matsumoto, '96) “Lefschetz fibrations of genus two – a topological approach”

B. CURRENT PROGRESS

GENUS-2 LF. OF TYPE (n,s)



Lemma 4 (K 202*)

Let $f: X^4 \to S^2$ be a genus-2 nontrivial Lef. fib. Then X of type $(2k, 4k-5)$ have $b_1 = 2$ and are indecomposable Lef. fibrations (if they exist).

PROOF:

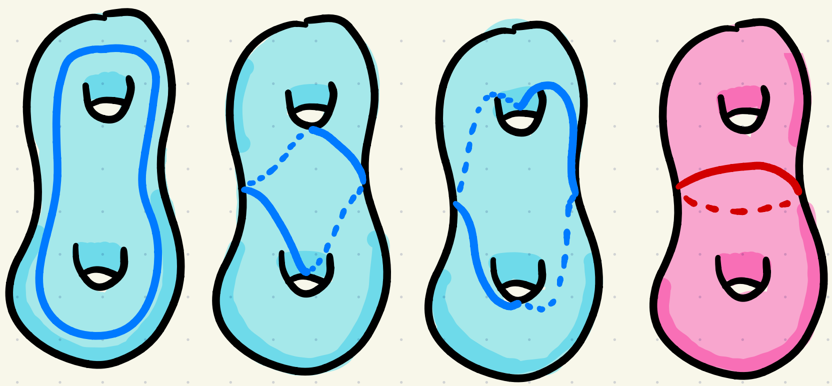
indecomposability : vector argument

$b_1(X) = 2$: $b_1 = \frac{1}{2}(\chi + \sigma) + b_2^+ - 1$ and $b_2^- \geq s+1$ and $b_1 \leq 2$

B. CURRENT PROGRESS

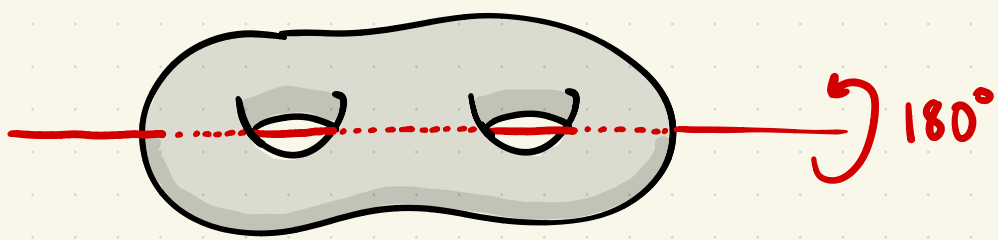
How do we know how to compute

$$b_2^+(X),\ b_2^-(X),\ b_1(X) \text{ or } \chi(X)?$$

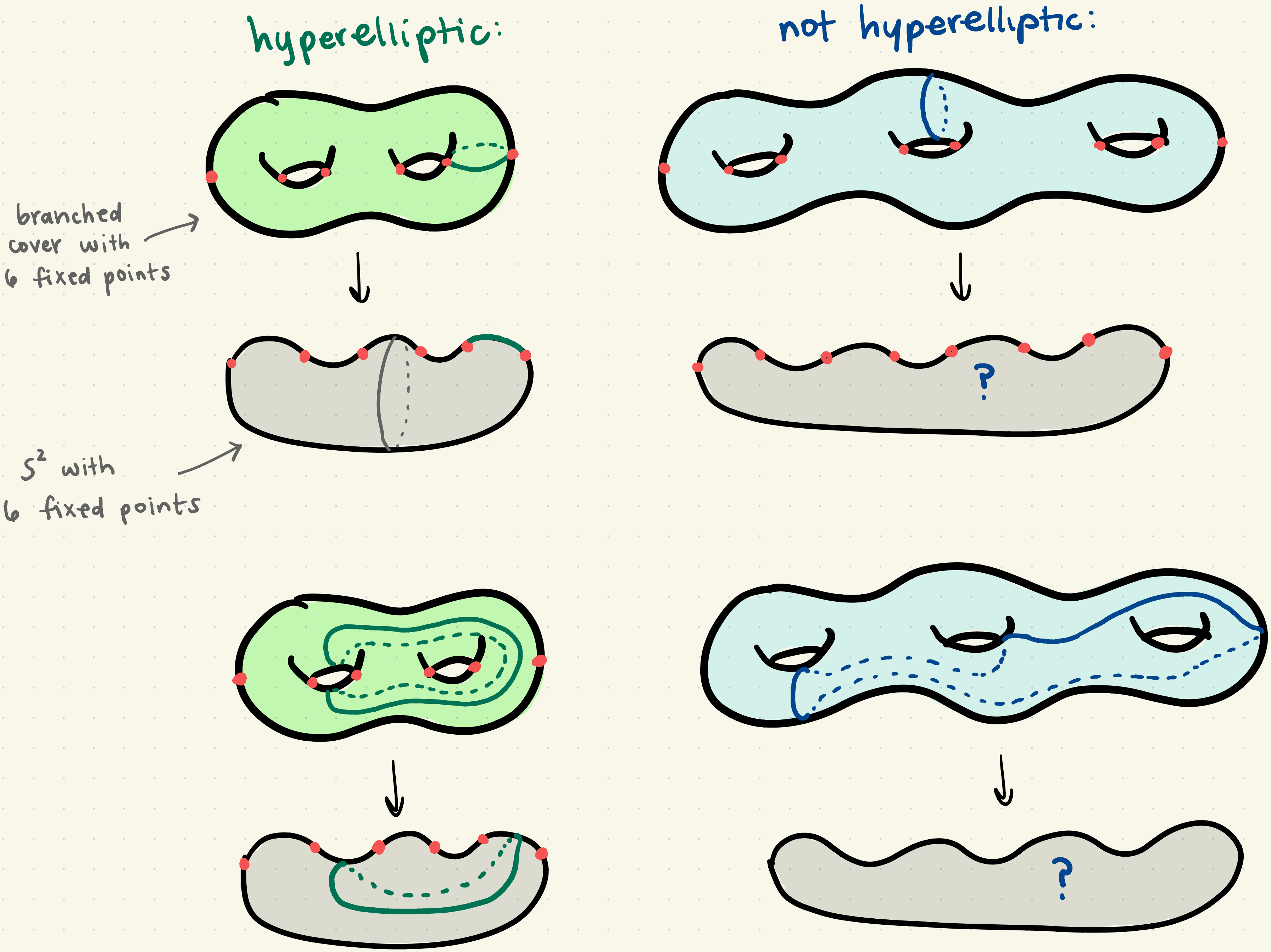


<u>Fact</u>: simple closed curves on Σ_2 are hyperelliptic.

[Margalit–Winarski]

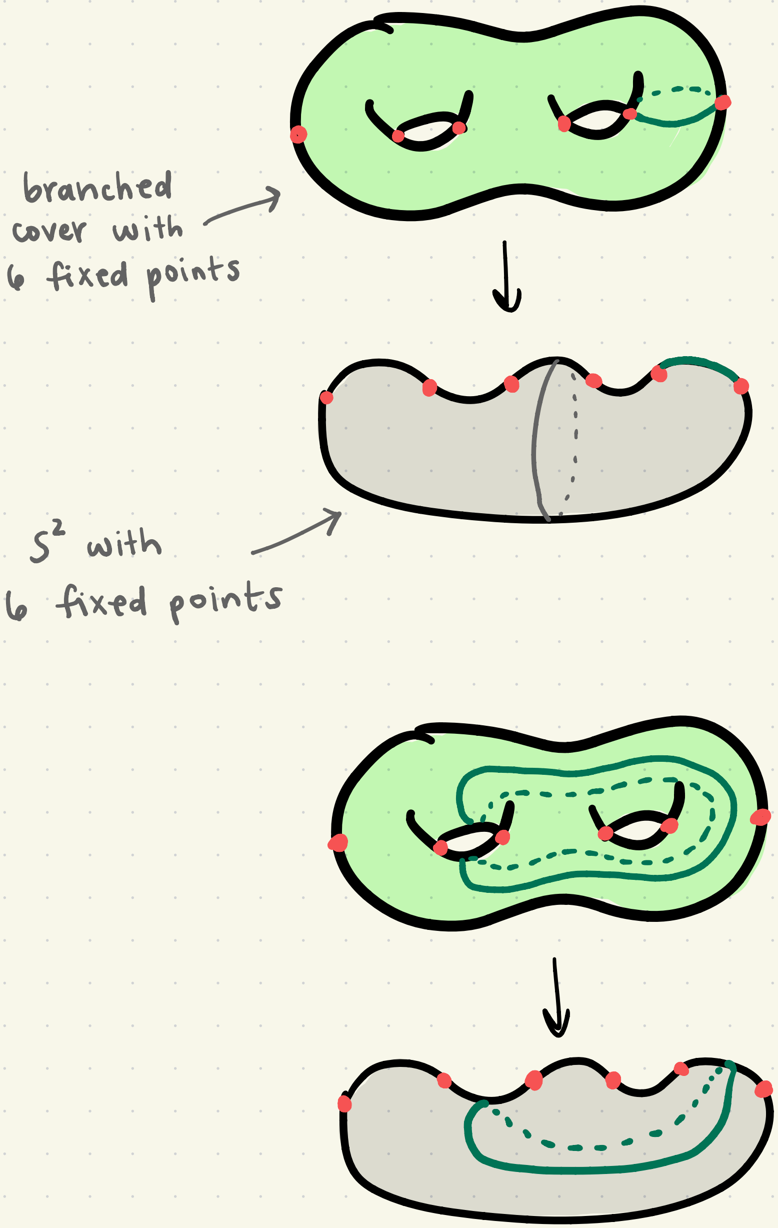


B. CURRENT PROGRESS



B. CURRENT PROGRESS

hyperelliptic:



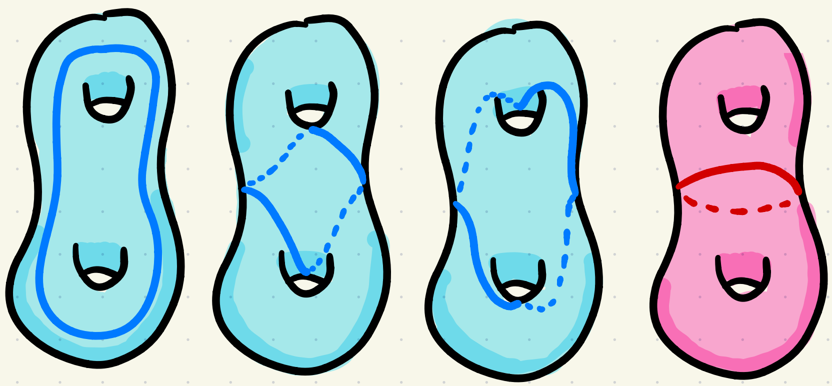
Consequences:

- separating vanishing cycles are lifts of two copies of full twists around fixed points
- nonsep. v.c.s are lifts of half twists between fixed points of S^2
- can explicitly see how each s and n contribute to χ and signature, σ

B. CURRENT PROGRESS

How do we know how to compute

$b_2^+(X)$, $b_2^-(X)$, $b_1(X)$ or $\chi(X)$?



Answer: hyperelliptic LFs are a well-structured subset

How do we know that $b_1(X) \leq 2$? (Proposition 1)

B. CURRENT PROGRESS

How do we know that $b_1(X) \leq 2$? (Proposition 1)

Proposition 1 (K 202*)

Let $f: X^4 \to S^2$ be a nontrivial genus-2 Lef. fib. then $0 \leq b_1(X) \leq 2$

Proof:

Fact: For a, b isotopy classes of scc's in Σ_2, for any $k \geq 0$,

$$\Psi(\tau_b^k)[a] = [a] + k \cdot \hat{i}(a,b)[b]$$

$$\Psi : \mathrm{Mod}(\Sigma_g) \longrightarrow Sp(2g;\mathbb{Z})$$

symplectic representation of $\mathrm{Mod}(\Sigma_g)$

B. CURRENT PROGRESS

How do we know that $b_1(X) \le 2$? (Proposition 1)

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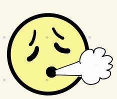
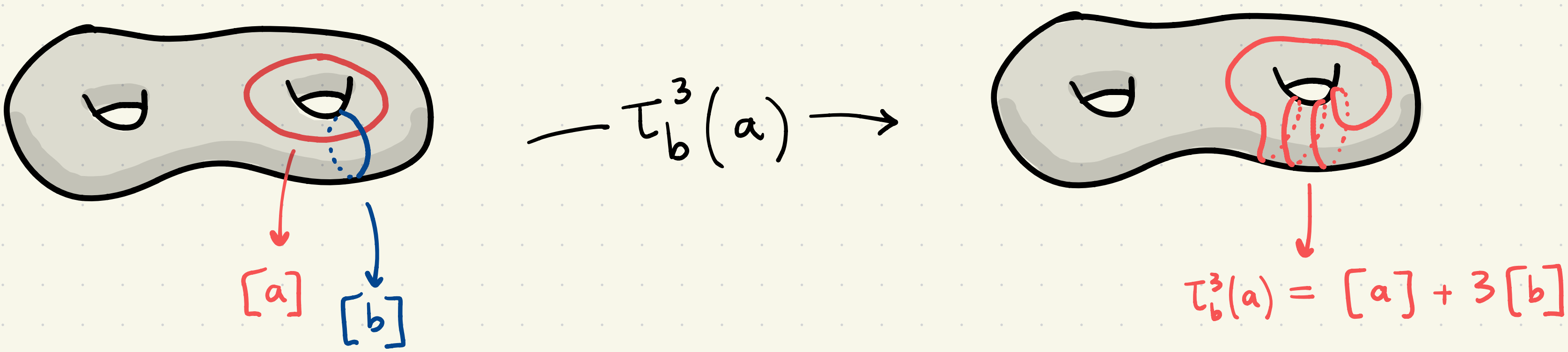
Example: 



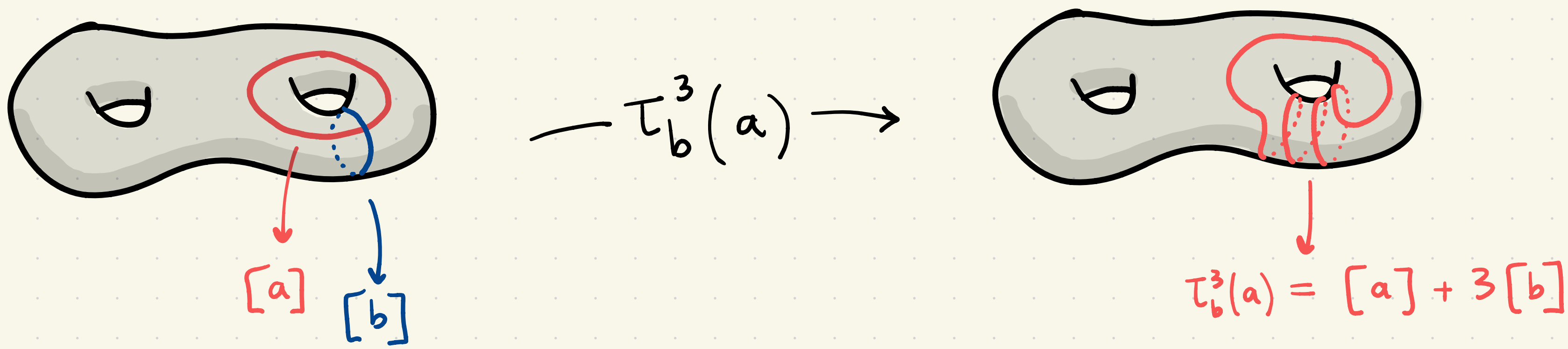
B. CURRENT PROGRESS

Proof:

Fact: For a,b isotopy classes of scc's in Σ_2, for any $k \geq 0$,

$$\Psi(\tau_b^k)[a] = [a] + k \cdot \hat{i}(a,b)[b]$$

Example:

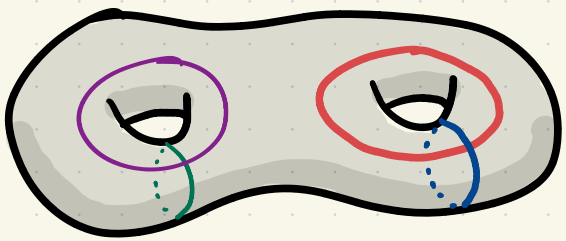


And we know that sep. curve do not change homology.

So if all nonsep η_i are homologous,

$\Psi(\tau_{\eta_n} \circ \dots \circ \tau_{\eta_1})([\delta])$ can't possibly be equal to $\Psi(Id)([\delta])$

B. CURRENT PROGRESS



Let α, β, γ, & δ be generators of homology.

Then each nonsep vanishing cycle looks like

$$\eta_i = a_i[\alpha] + b_i[\beta] + c_i[\gamma] + d_i[\delta]$$

At least two vc's, say η_ℓ and η_k, are distinct.

So $H_1(X)$ is some subgroup of $\dfrac{\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}}{\mathbb{Z}\oplus\mathbb{Z}_p}$

from η_ℓ from η_k

$\therefore \quad 0 \leq b_1(X) \leq 2$

B. CURRENT PROGRESS

Theorem (Ivan Smith)

A nontrivial genus-g LF over S^2 must have at least one nonseparating vanishing cycle

IMPROVEMENT

Theorem 5

A nontrivial genus-g LF over S^2 must have at least two non-homologous nonseparating VCs

$\frac{4g+2}{5}$ improvement on total number

B. CURRENT PROGRESS

CONJECTURES: (for all genus-2 LFs)

- $[\eta_l]$ & $[\eta_k]$ are always a part of an integral basis for $\mathbb{Z}^4$
- At least one class $[\eta_i]$ intersects $[\eta_l]$ or $[\eta_k]$ homologically once

MOTIVATION: $\pi_1(X, x_0)$ has at most 2 generators

i.e. a longstanding conjecture

B. CURRENT PROGRESS

Proposition 2 (K 202*)

Let $f : X^4 \to S^2$ be a genus-g Lef. fib. with monodromy $\phi = \mathrm{id} \in \mathrm{Mod}(\Sigma_g)$.

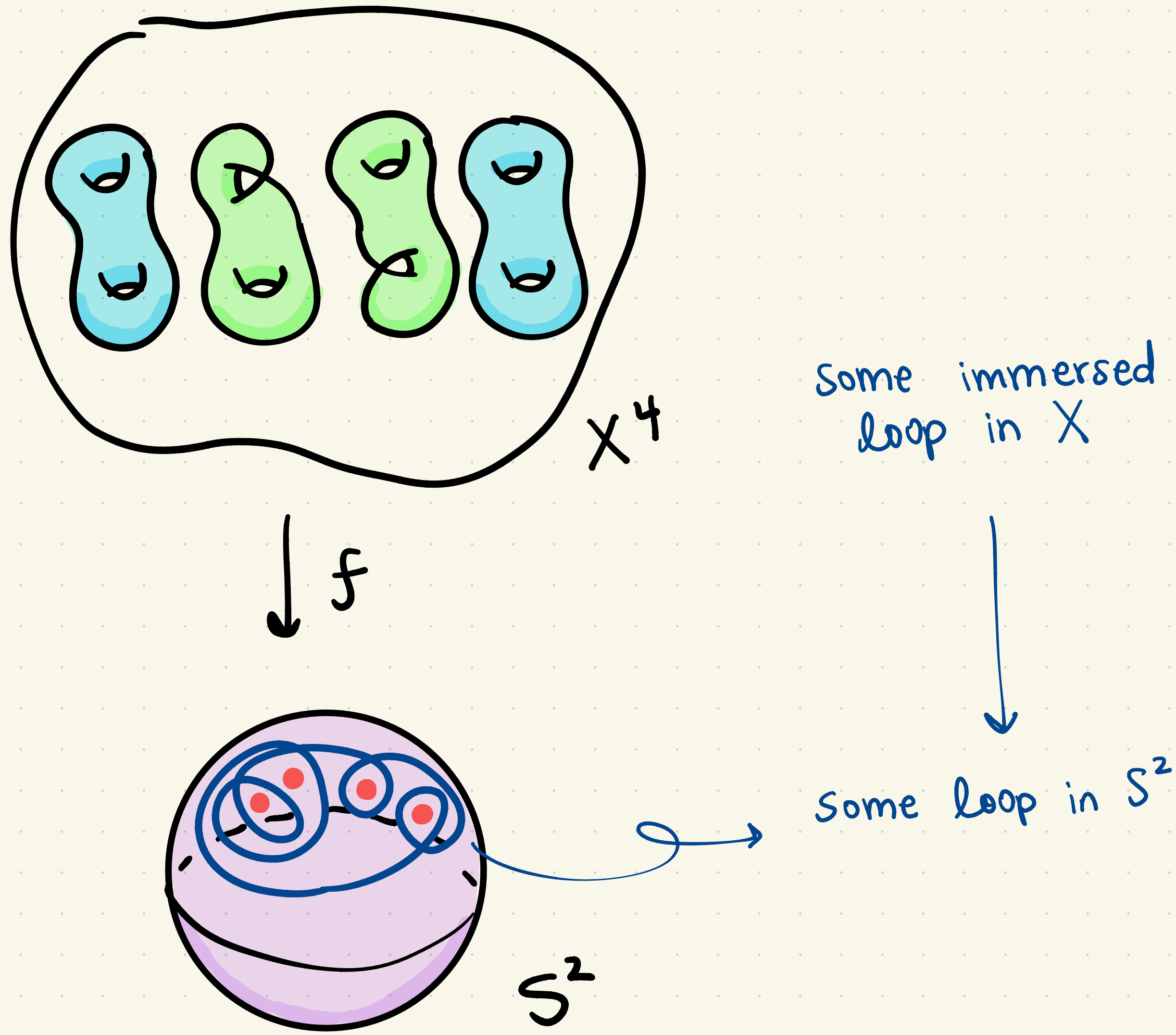
If ϕ is transitive w.r.t. $\mathrm{Mod}(\Sigma_g)$, then $\pi_1(X) = 0$.

[Seibert-Tian, '99] Nontrivial holomorphic genus $g \geq 2$ Lefschetz fibrations with only non-separating vanishing cycles are simply connected

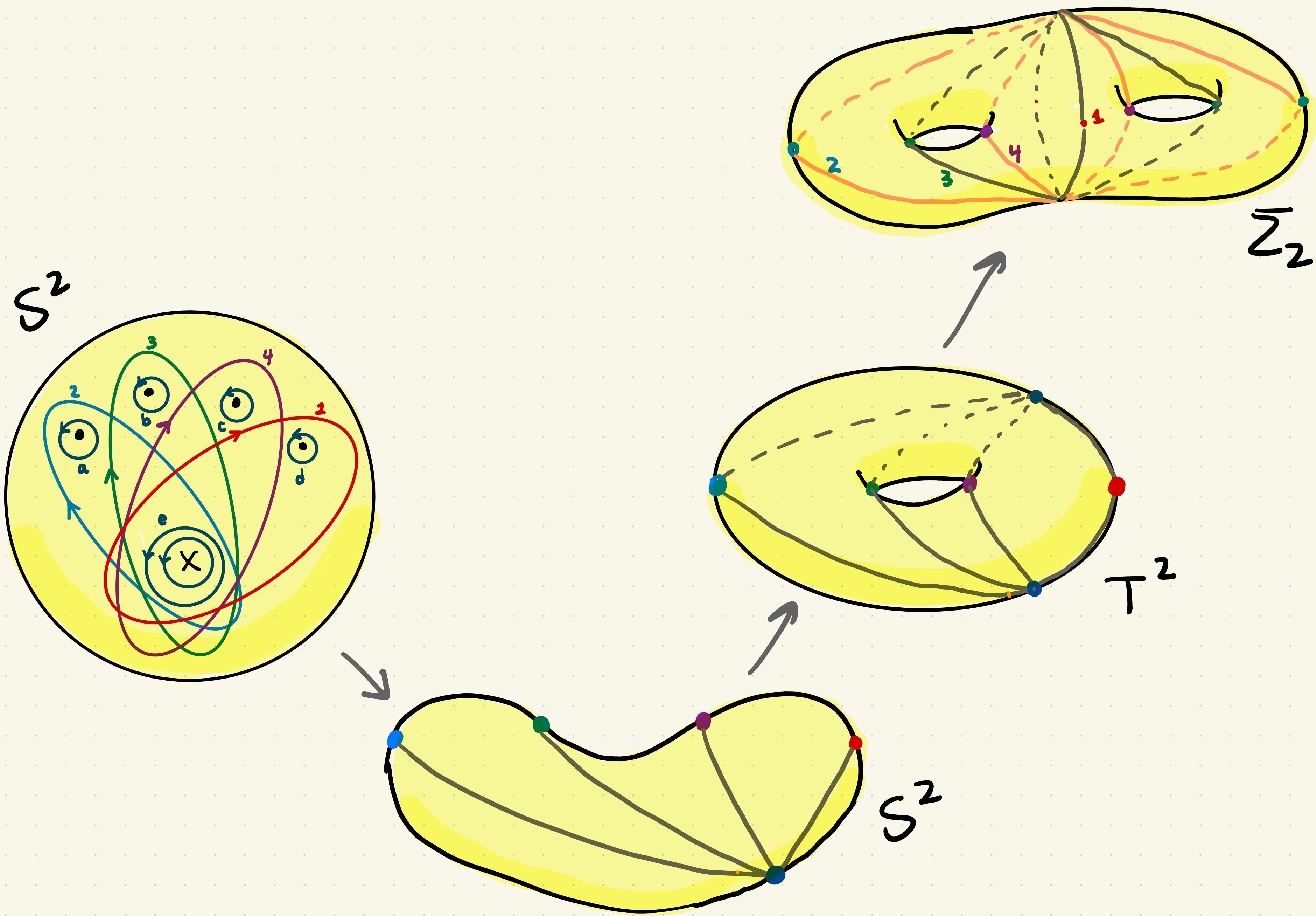
"On the holomorphicity of genus 2 Lefschetz fibrations" 2003

B. CURRENT PROGRESS

Picture proof:



C. FUTURE DIRECTIONS



THANKS FOR LISTENING!