Lefschetz fibrations: genus two and beyond!

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symplectic manifolds have the structure of a Lefschetz pencil

Lefschetz fibrations in symplectic geometry, 1998

Lefschetz pencils have symplectic structures

The topology of symplectic manifolds, 2001

Lefschetz pencils:

- 4-mfds w/ "almost" a $\mathbb{C}P^1$'s worth of Σ_g

example: $f: \mathbb{C}P^2/B \longrightarrow S^2$

Every algebraic surface [Lefschetz]

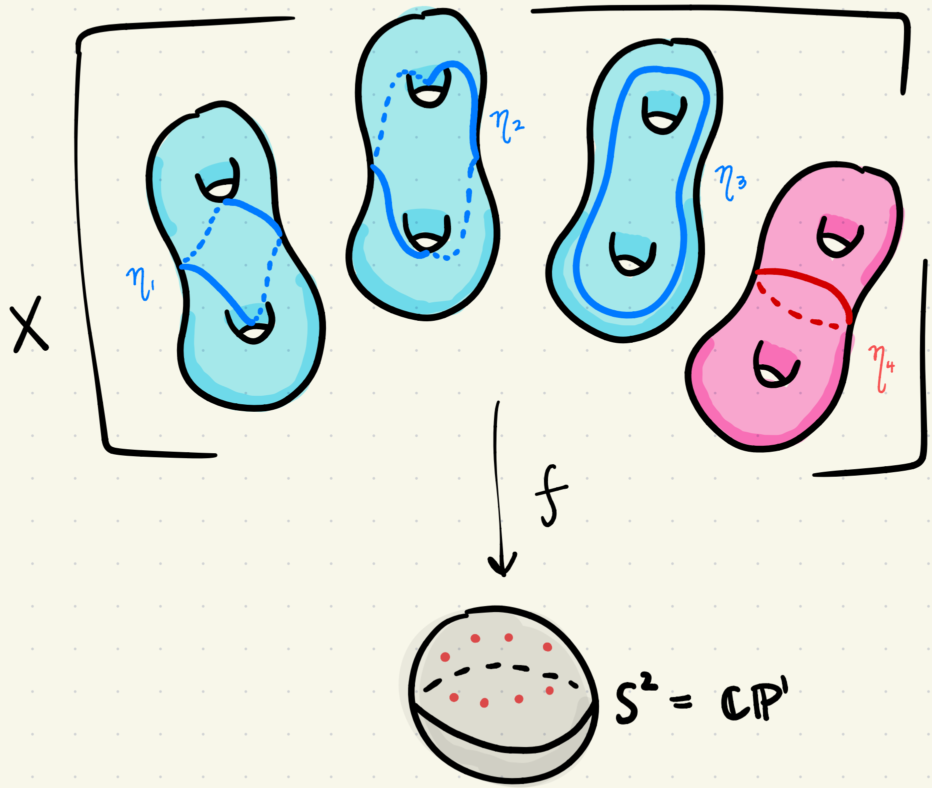
blow up $\Downarrow$ $\#_n \overline{\mathbb{C}P^2}$

Lefschetz fibration

example: $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, $\mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2}$, $3\mathbb{C}P^2 \# 13\overline{\mathbb{C}P^2}$

$S^2 \times \Sigma_g$, $(S^2 \times T^2) \# 4\overline{\mathbb{C}P^2}$, $K3 \# 2\overline{\mathbb{C}P^2}$

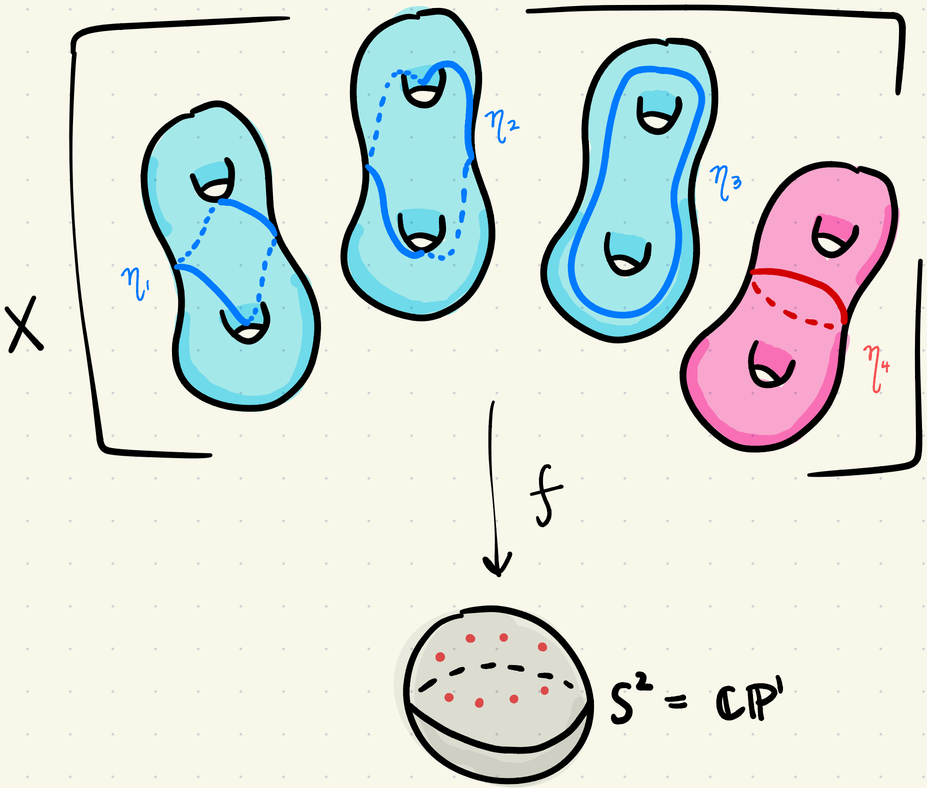
Example of a genus-2 Lefschetz fibration:



- This is the genus-2 member of Matsumoto's family of Lefschetz fibrations
- 8 vanishing cycles and ∴ 8 singular fibers
- with monodromy

$$\phi = (\tau_{\eta_4} \circ \tau_{\eta_3} \circ \tau_{\eta_2} \circ \tau_{\eta_1})^2$$

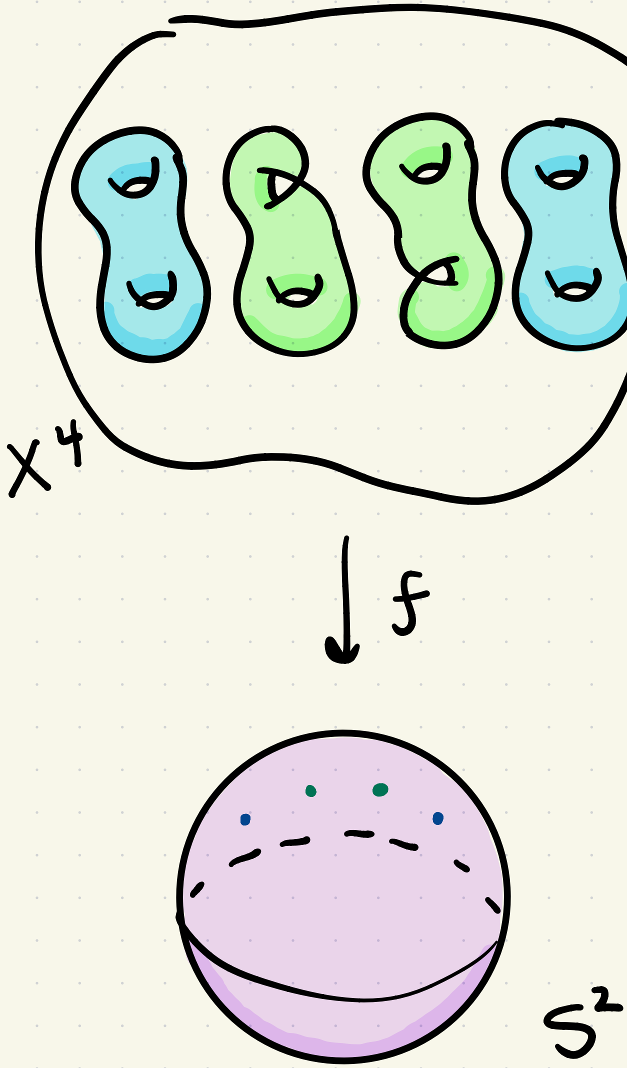
Example of a genus-2 Lefschetz fibration:



Why is it an important example?

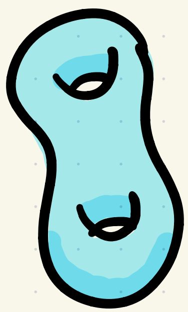
- [Korkmaz] constructs $f: X^4 \to S^2$ w/ $\pi_1 = G$
- small, highly not simply connected
- used to construct new & exotic LFs

Definition: Lefschetz fibration



1. $f: X^4 \to S^2$ is a smooth surjection
2. finitely many critical values
   $q_1, q_2, \dots, q_n$
3. each $f^{-1}(q_i) \in X$ has local coord. charts in which $f(z,w) = zw$
   $z, w \in \mathbb{C}$
4. $f^{-1}(b)$ for b a regular value is a regular fiber (a closed genus-g surface)

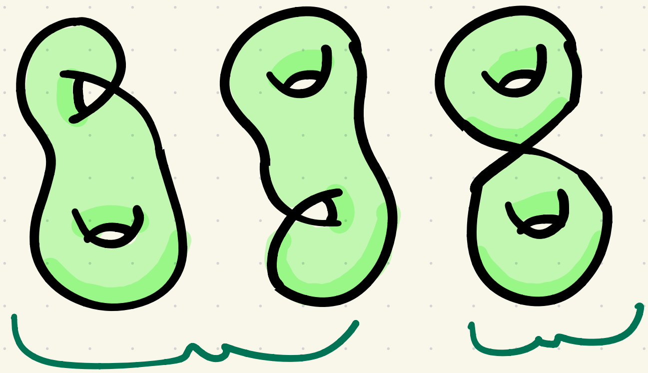
Regular fiber



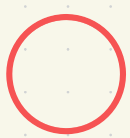
genus 2

Σ_2

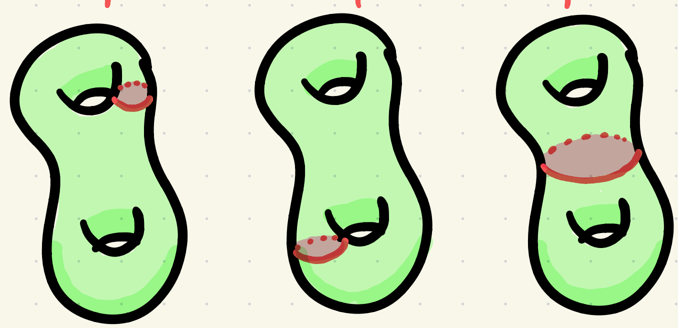
Singular fiber



nonseparating   separating



S^1 = vanishing cycle :



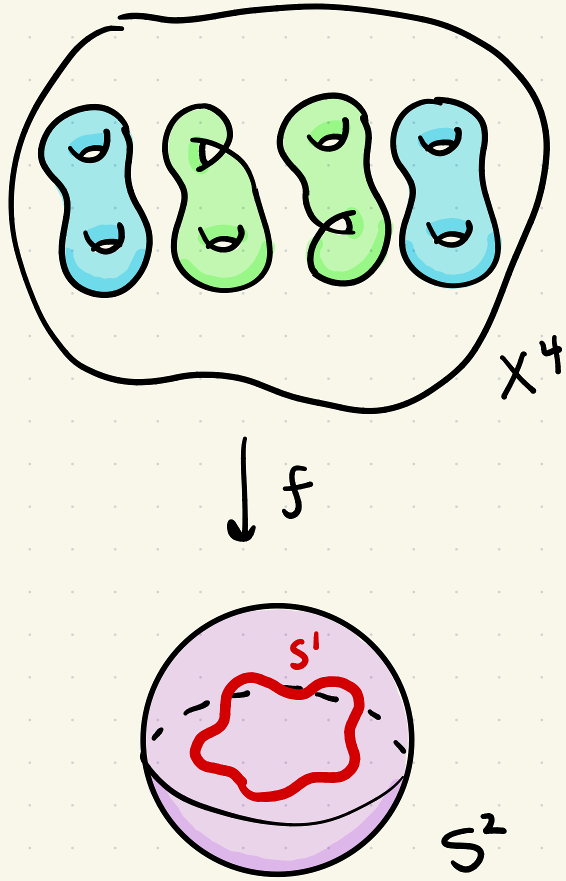
The vanishing cycles determine the mondromy:

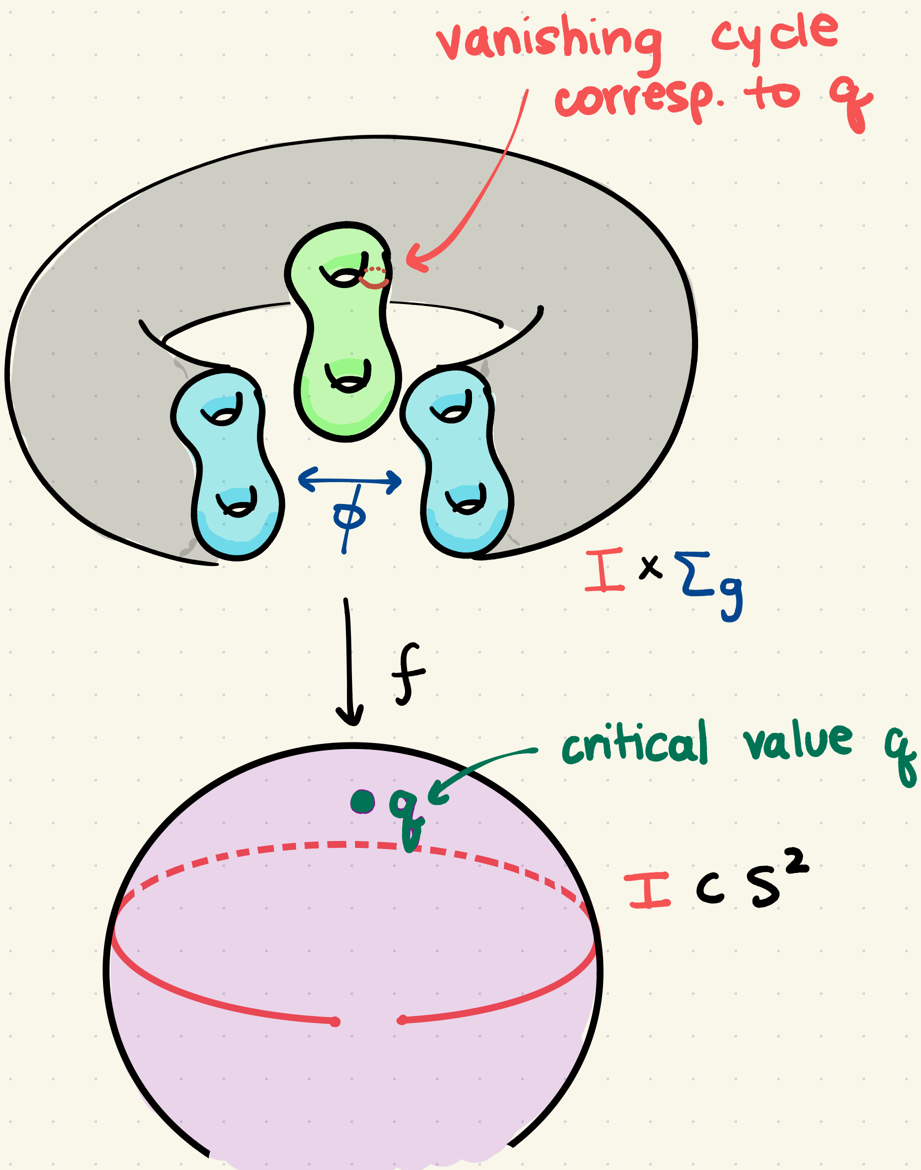
monodromy of a LF:

1. embedded S^1 in base S^2

2. pre-image $= I \times \Sigma_g \big/ \left((\{0\} \times \Sigma_g) \sim (\{1\} \times \Sigma_g) \right)$

3. monodromy ϕ = the self-diffeo of a regular fiber Σ_g to itself





monodromy:

$$\phi = \tau_\eta^1$$

$\shortparallel$

right-handed order 1

Dehn twist about

η = van. cycle

Fact:

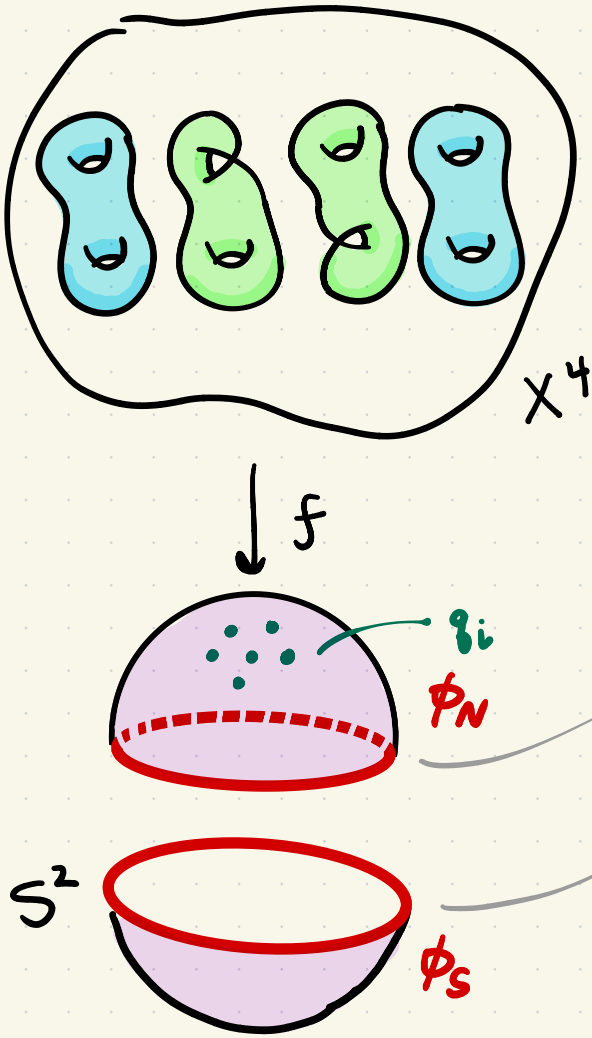
$\phi = \mathrm{id} \in \mathrm{Mod}(\Sigma_g)$

when $I \in S^2$ only

encircles regular values

For a Lefschetz fibration $f: X^4 \longrightarrow S^2$

$$\phi_N = \phi_S$$



$$\phi_N = \tau_{\gamma_n} \circ \dots \circ \tau_{\gamma_2} \circ \tau_{\gamma_1}$$

$$\phi_S = id_{mod(\Sigma_g)}$$

$$\Longrightarrow \tau_{\gamma_n} \circ \dots \circ \tau_{\gamma_1} = id_{mod(\Sigma_g)}$$

So what do we want to know about these objects?

1. What are all possible monodromies for X with a fixed fiber genus?

2. How to find new monodromies? (Q about id. fact. of mcg)

3. What is fundamental group of X? ... or signature, Euler char, 1^{st} Betti no, ...

4. Which X are indecomposable?

So what do we want to know about these objects?

1. What are all possible monodromies for X with a fixed fiber genus? → answered for $g=0,1$; somewhat for $g=2$; no hope? $g \geq 3$

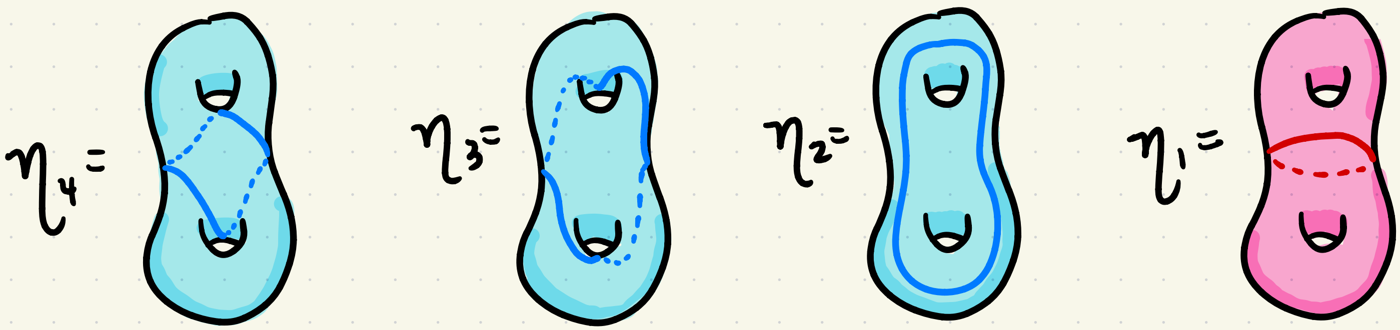
2. How to find new monodromies? (Q about id. fact. of mcg) → current methods not exhaustive

3. What is fundamental group of X? ... or signature, Euler char, 1st Betti no, ... → only when monodromy is (somewhat) known

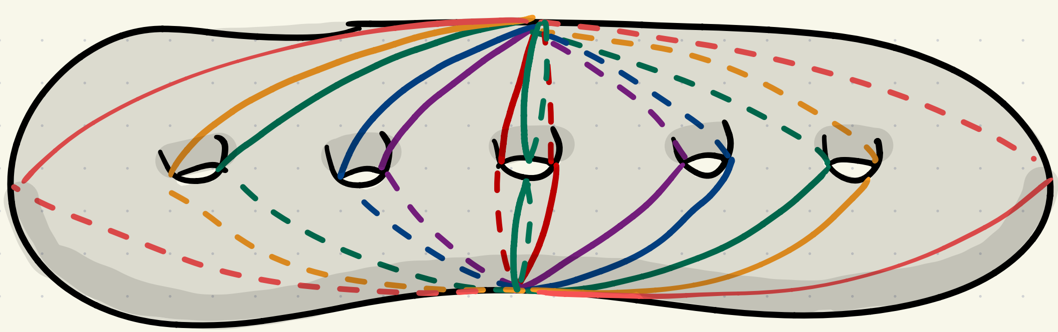
4. Which X are indecomposable? → mostly unknown $g \geq 2$

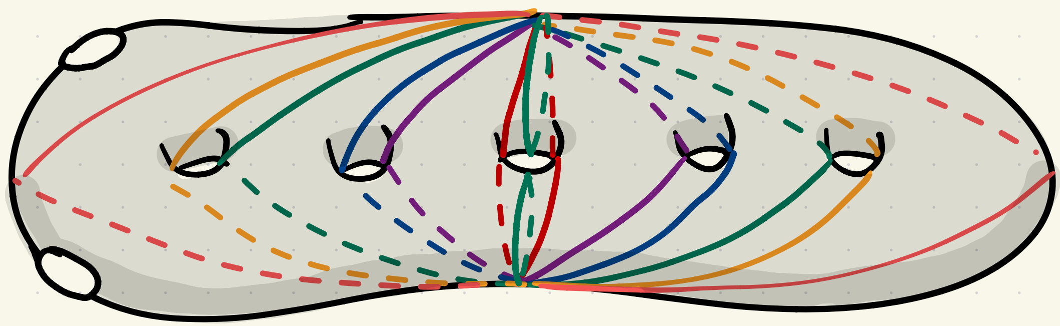
An interesting family

Yukio Matsumoto found a genus-2 Lefschetz fibration

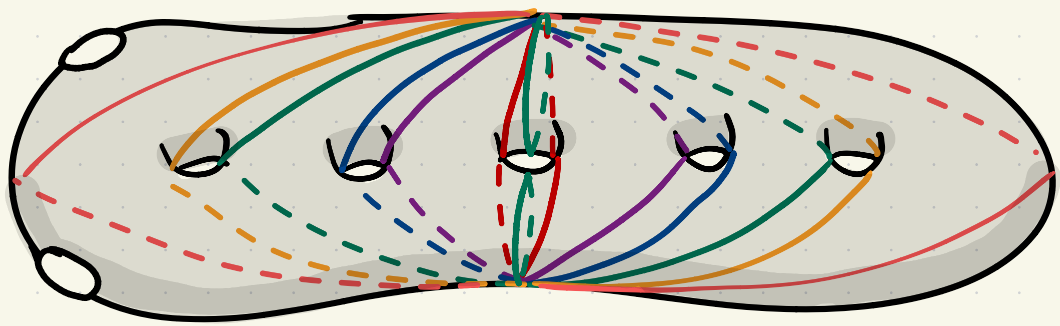


Several others extended this example to higher genera





- has at least 2 (symplectic) sections
- first non-simply connected family
- non-hyperelliptic for odd genera
- built from atomic singular fibers,
  configurations of sing. fibs,
  and Mathematica code



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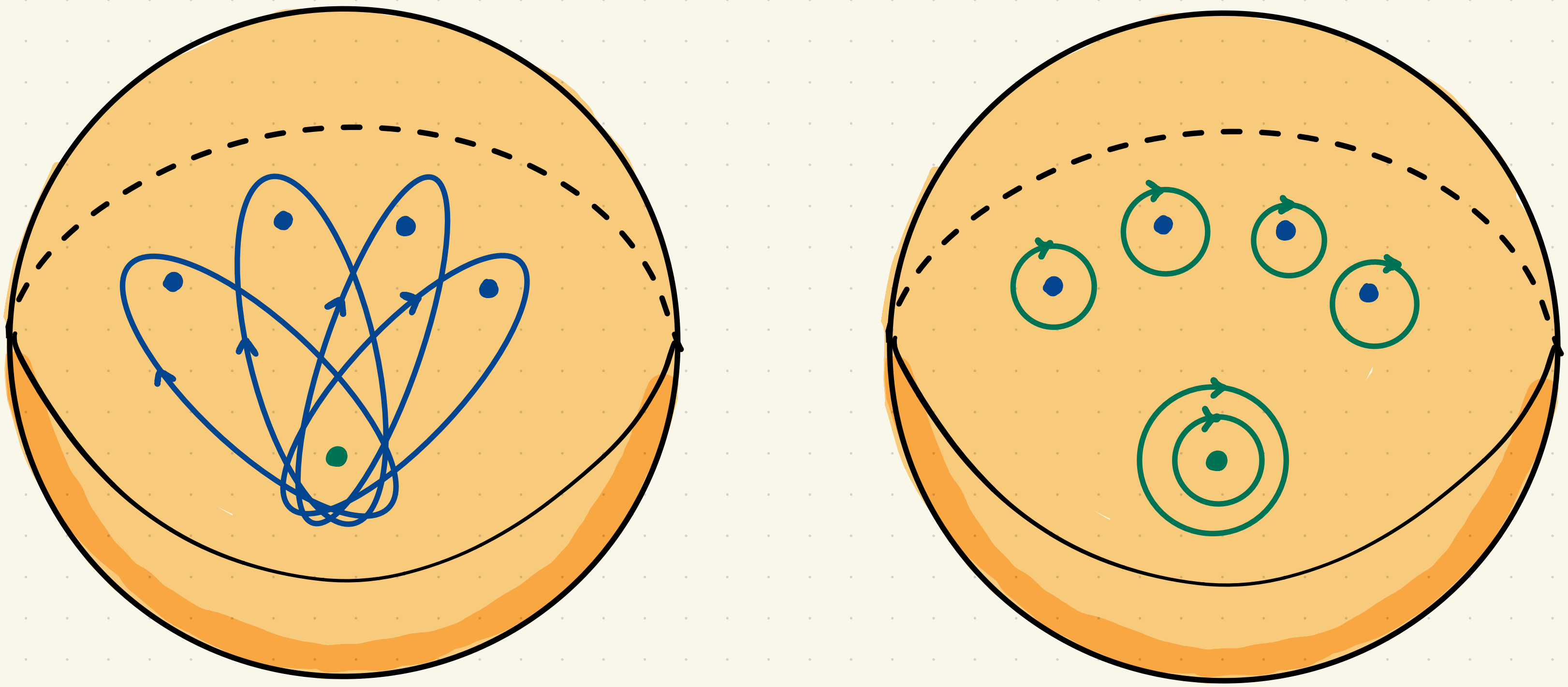
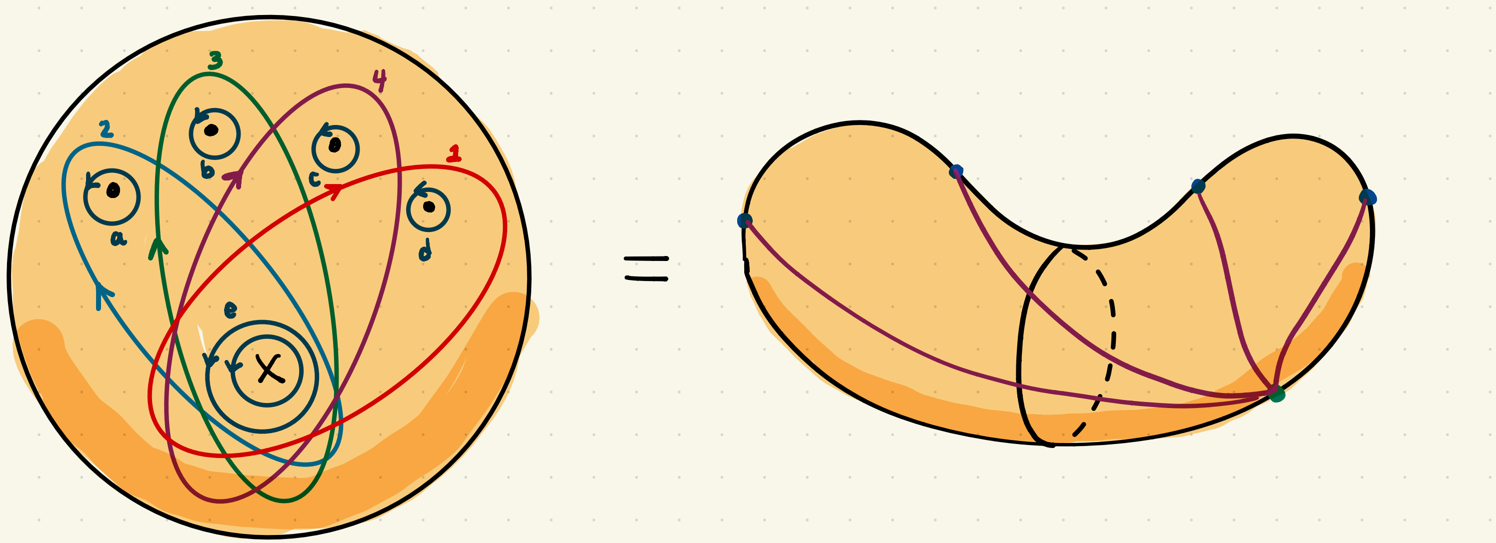
Why no top. proof?

Let's build it from daisy relations!

aka generalized lantern relations



$$\tau_4 \tau_3 \tau_2 \tau_1 = \tau_E^2 \, \tau_D \tau_C \tau_B \tau_A$$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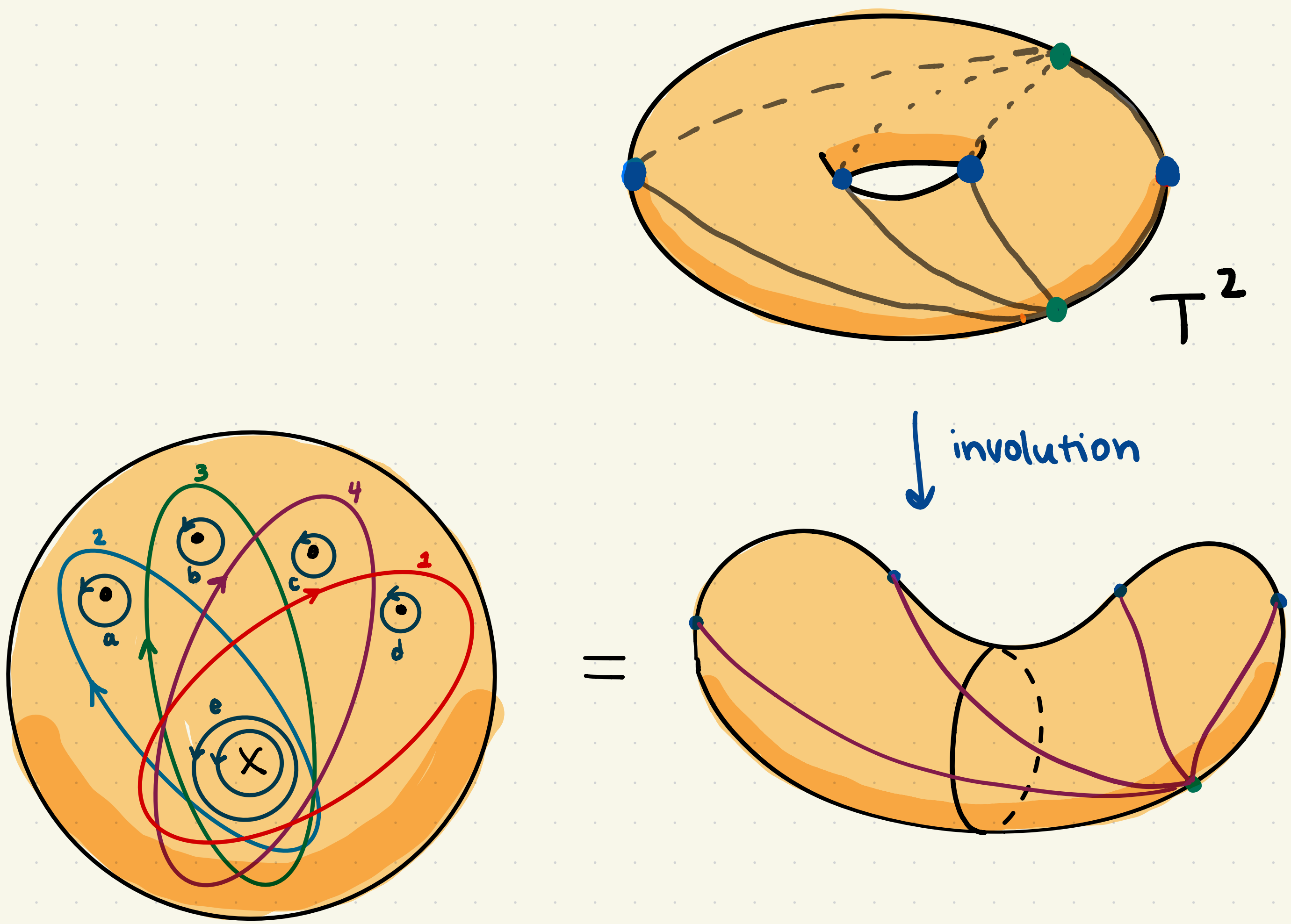

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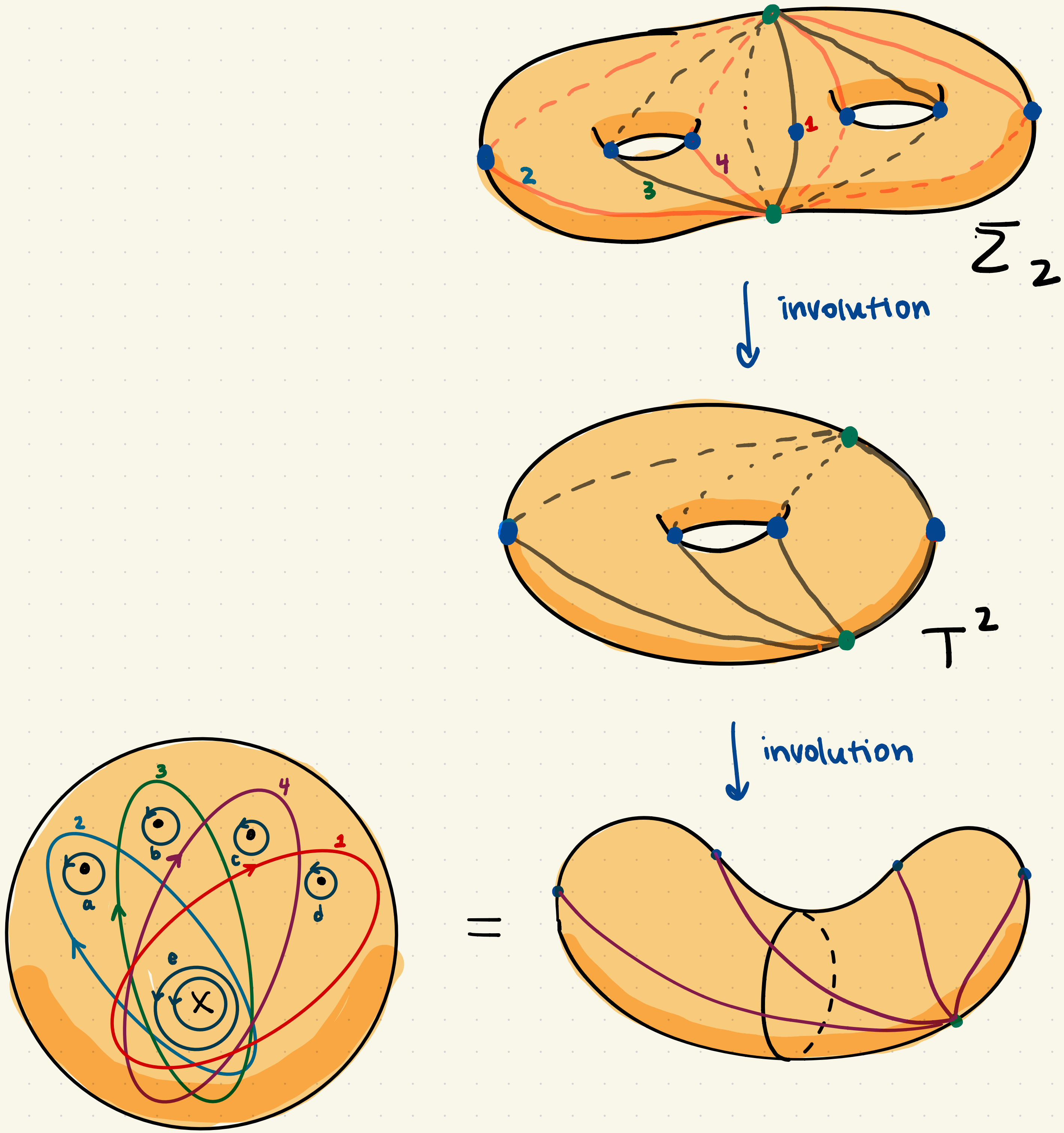

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What else do we know about nontrivial genus-g Lefschetz fibrations?

Proposition (K 202*)

Let $f: X^4 \to S^2$ be a genus-g nontrivial Lef. fib.
then $0 \leq b_1(X) \leq 2g-2$, this is sharp for $g=2$.

Motivation:

Conjecture (Korkmaz, Stipicz, Baykur, ...)

$\exists$ no genus Lefschetz fibration with 3 generators of π_1

What else do we know about nontrivial genus-g Lefschetz fibrations?

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Of course $b_1(\Sigma_g \times S^2) = 2g$

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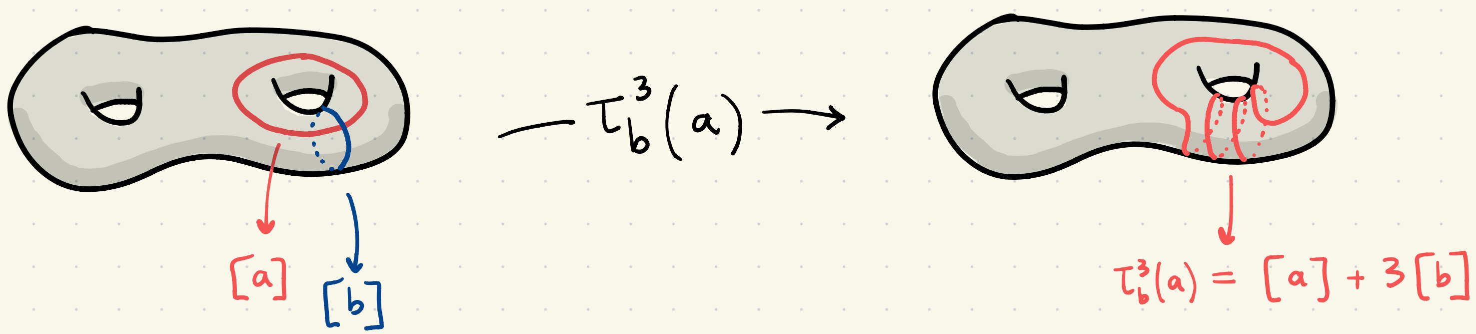
proof: Fact: For a, b isotopy classes of scc's in Σ_2, for any $k \geq 0$,

$$\Psi(\tau_b^k)[a] = [a] + k \cdot \hat{i}(a,b)[b]$$

Example:



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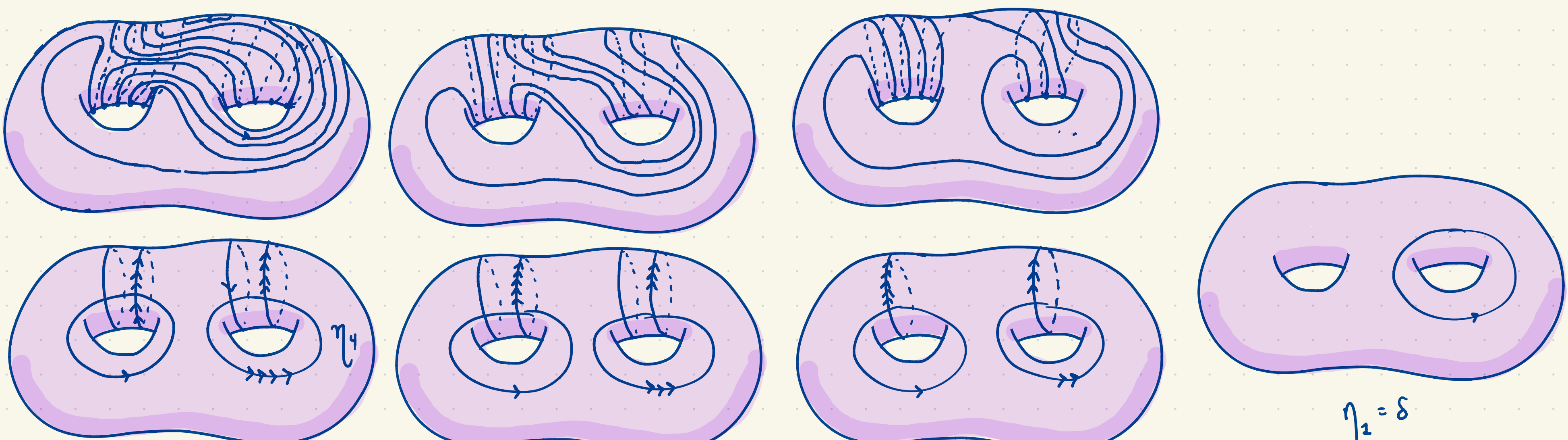
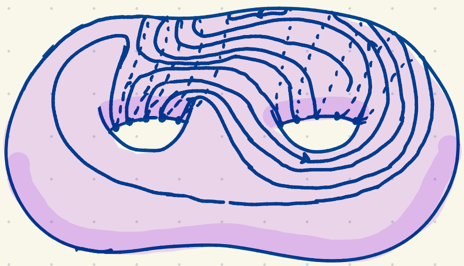


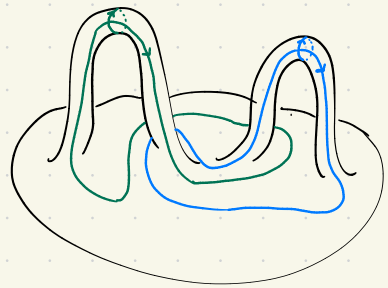
$= 5\alpha + \beta + 3\gamma + 4\delta$

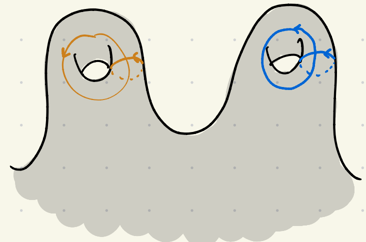
$\eta_3 = 5\alpha + \beta + 3\gamma + 3\delta$

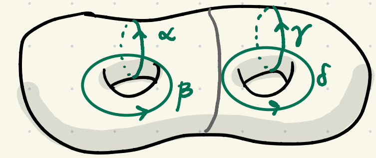
$\eta_2 = 5\alpha + \beta + 3\gamma + 2\delta$

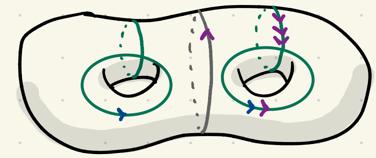
$\eta_1 = \delta$

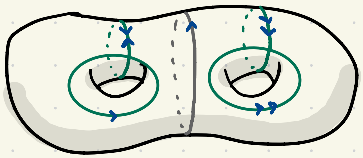












THANKS FOR LISTENING!

$$\bar\sigma a \bar\sigma b = b \bar\sigma a \sigma$$
$$a \bar\sigma b \bar\sigma = \sigma b \bar\sigma a$$
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