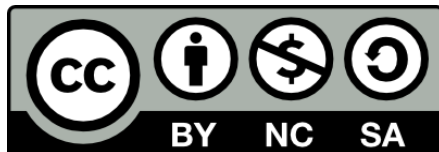


Plate Theory

with some exact analytical solutions



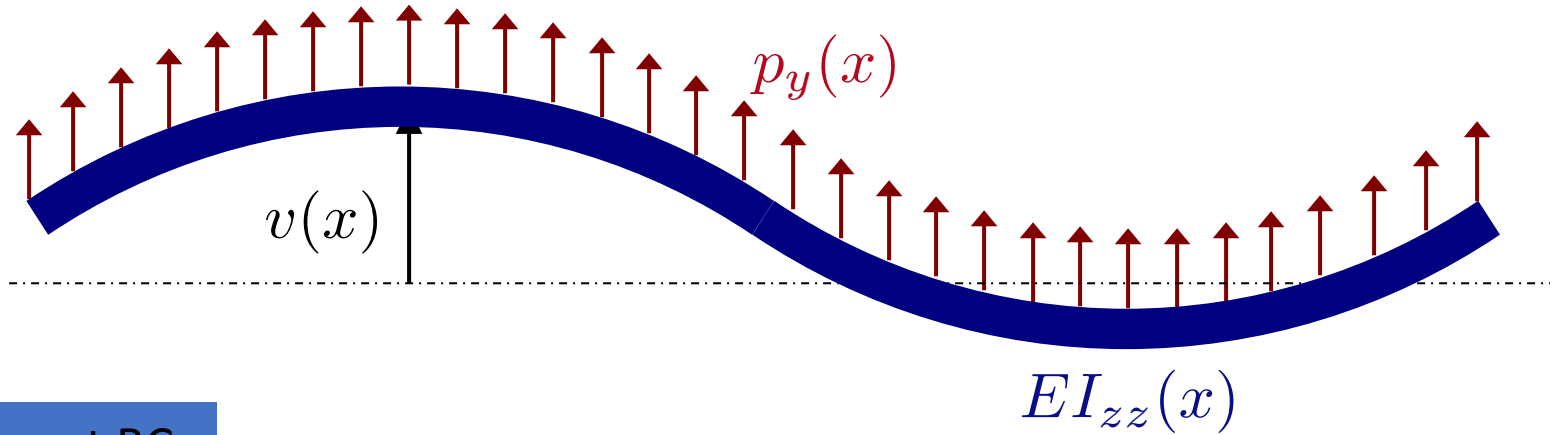
Structural Analysis: Plate Theory

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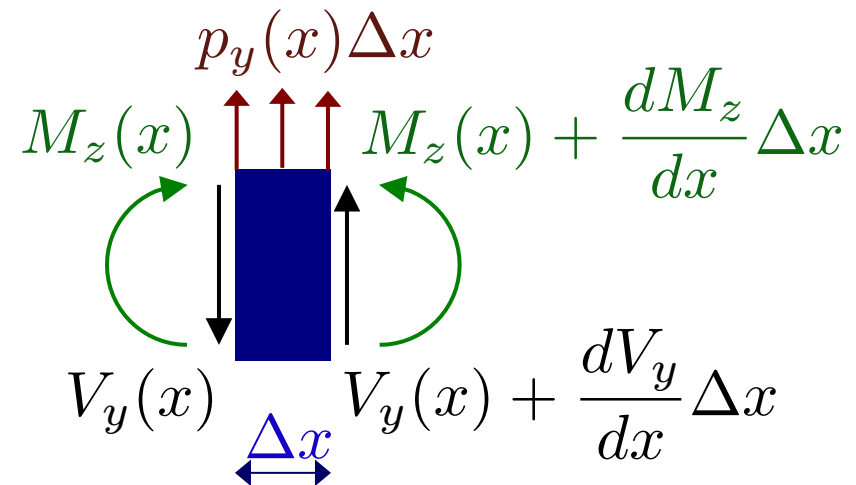
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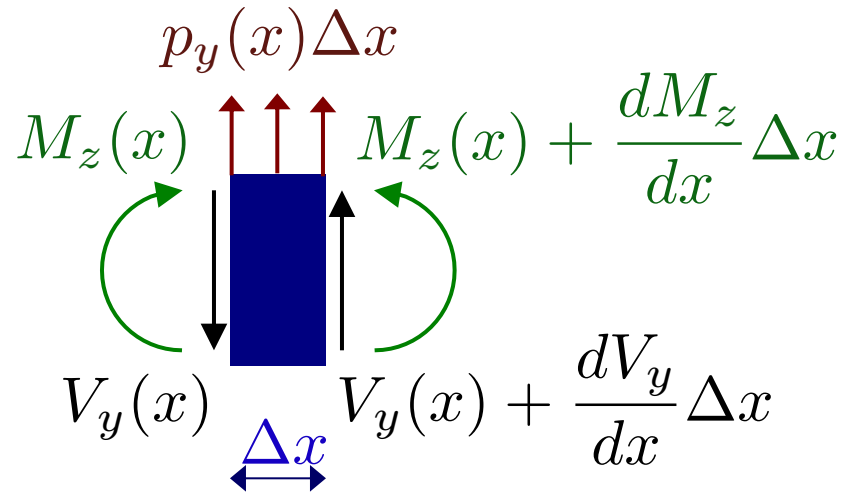
Beam Theory Derivation



Forget BCs
for now



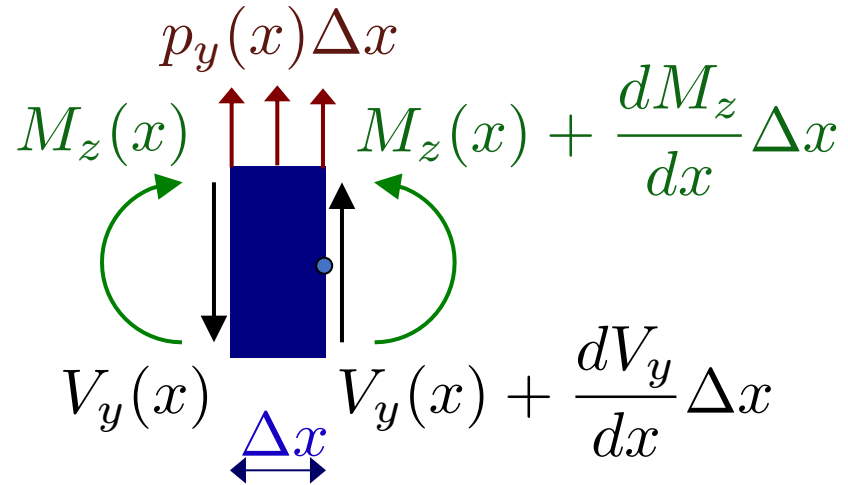
Force Equilibrium



$$\left[V_y(x) + \frac{dV_y}{dx} \Delta x \right] - V_y(x) + p_y(x) \Delta x = 0$$

$$-\frac{dV_y}{dx} = p_y(x)$$

Moment Equation



$$\left[M_z(x) + \frac{dM_z}{dx} \Delta x \right] - M_z(x) + V_y(x) \Delta x - p_y(x) \Delta x \frac{\Delta x}{2} = 0$$

$$-\frac{dM_z}{dx} = V_y(x)$$

Kinematics

- Curvature $\kappa = \frac{d^2 v}{dx^2}$
- Strain $\epsilon_{xx} = -y \frac{d^2 v}{dx^2}$
- Stress $\sigma_{xx} = -E y \frac{d^2 v}{dx^2}$
- Bending Moment $M_z = - \int_A \sigma_{xx} y dA$

$$M_z = EI_{zz} \frac{d^2 v}{dx^2}$$

$$I_{zz} = \int_A y^2 dA$$

Beam Equations

Force Equilibrium

In y -direction

$$-\frac{dV_y}{dx} = p_y(x)$$

Moment Equilibrium

In z -direction

$$-\frac{dM_z}{dx} = V_y(x)$$

Kinematics +

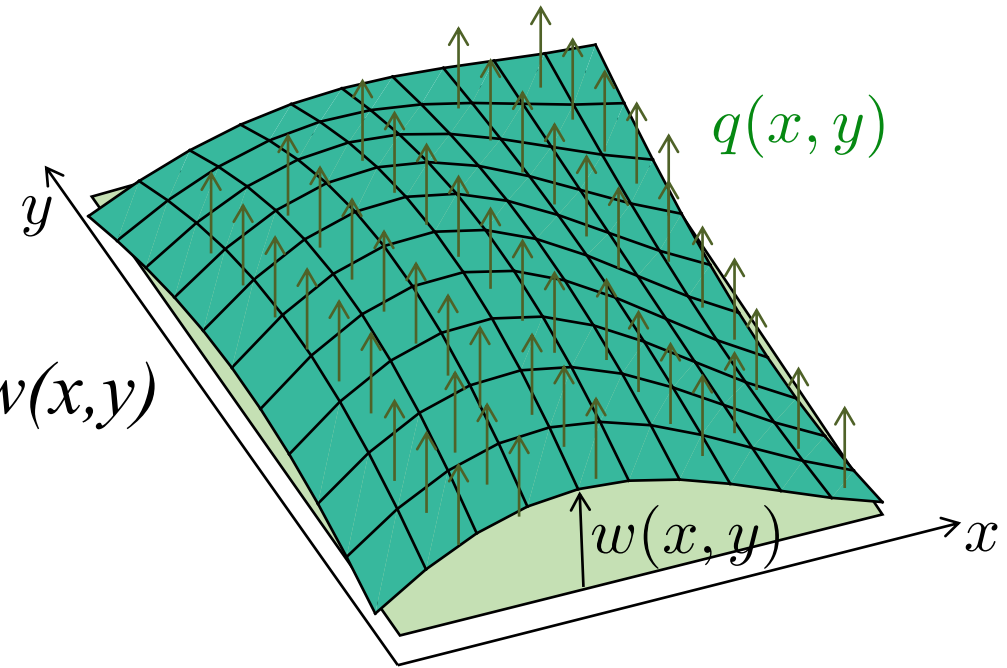
Material (Constitutive Law)

$$M_z = EI_{zz} \frac{d^2 v}{dx^2}$$

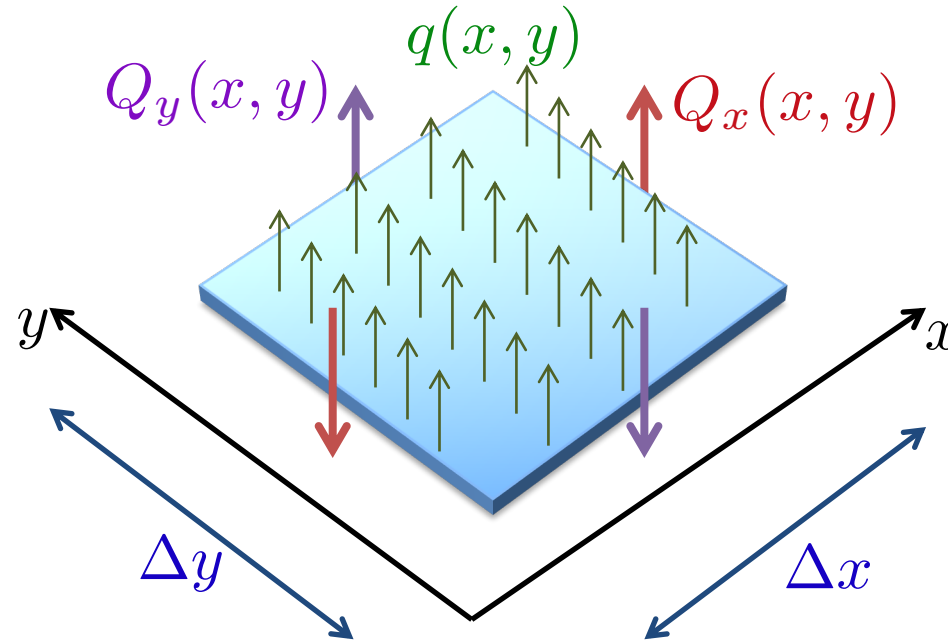
$$\frac{d^2}{dx^2} \left[EI_{zz}(x) \frac{d^2 v(x)}{dx^2} \right] = p_y(x)$$

Plate Theory Derivation

- Plate in the x - y plane
- Transverse load in the z -direction: $q(x,y)$
- Calculate deformation in the z -direction: $w(x,y)$

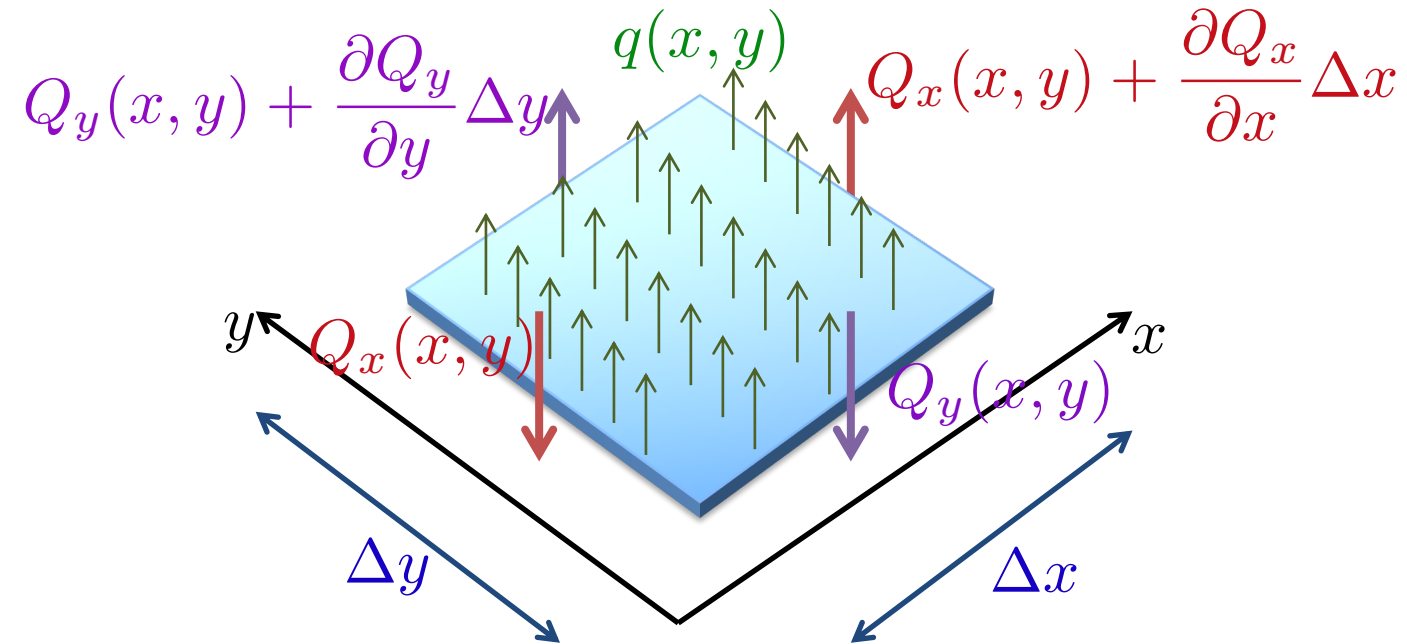


Shear Forces



- Shear Force per unit length in z -direction
 - On x -edge: $Q_x(x, y)$
 - On y -edge: $Q_y(x, y)$
 - Positive on positive edge in positive direction
- Applied distributed load per unit area in z -direction: $q(x, y)$

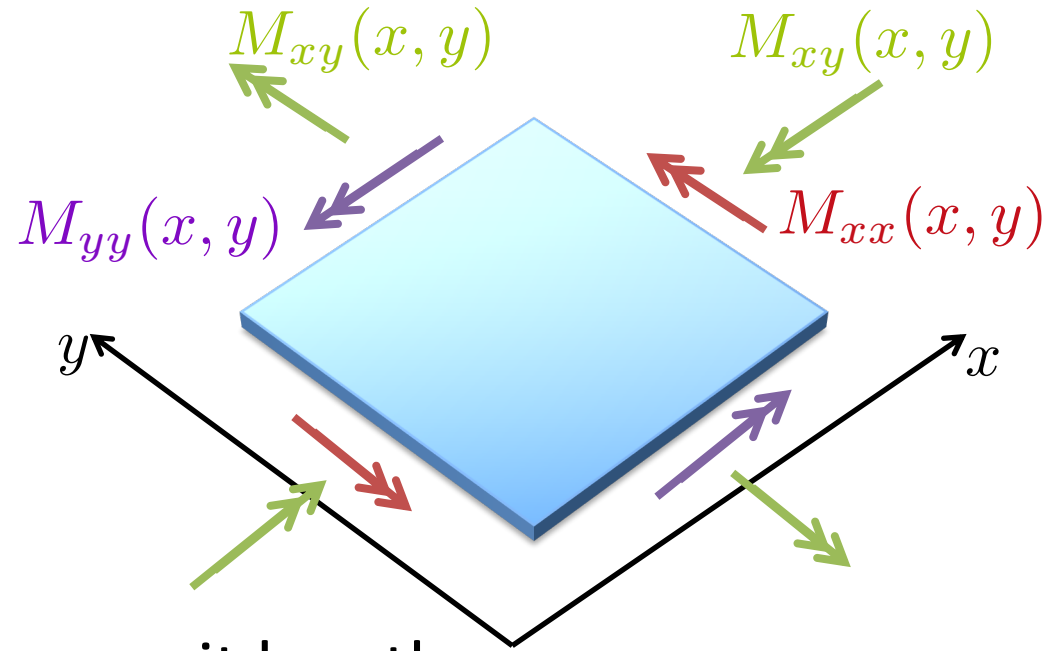
Force Equilibrium



$$\begin{aligned}
 & \left[Q_x(x, y) + \frac{\partial Q_x}{\partial x} \Delta x \right] \Delta y - Q_x(x, y) \Delta y \\
 & + \left[Q_y(x, y) + \frac{\partial Q_y}{\partial y} \Delta y \right] \Delta x - Q_y(x, y) \Delta x + q(x, y) \Delta x \Delta y = 0
 \end{aligned}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q(x, y) = 0$$

Bending Moments



- Bending moment per unit length

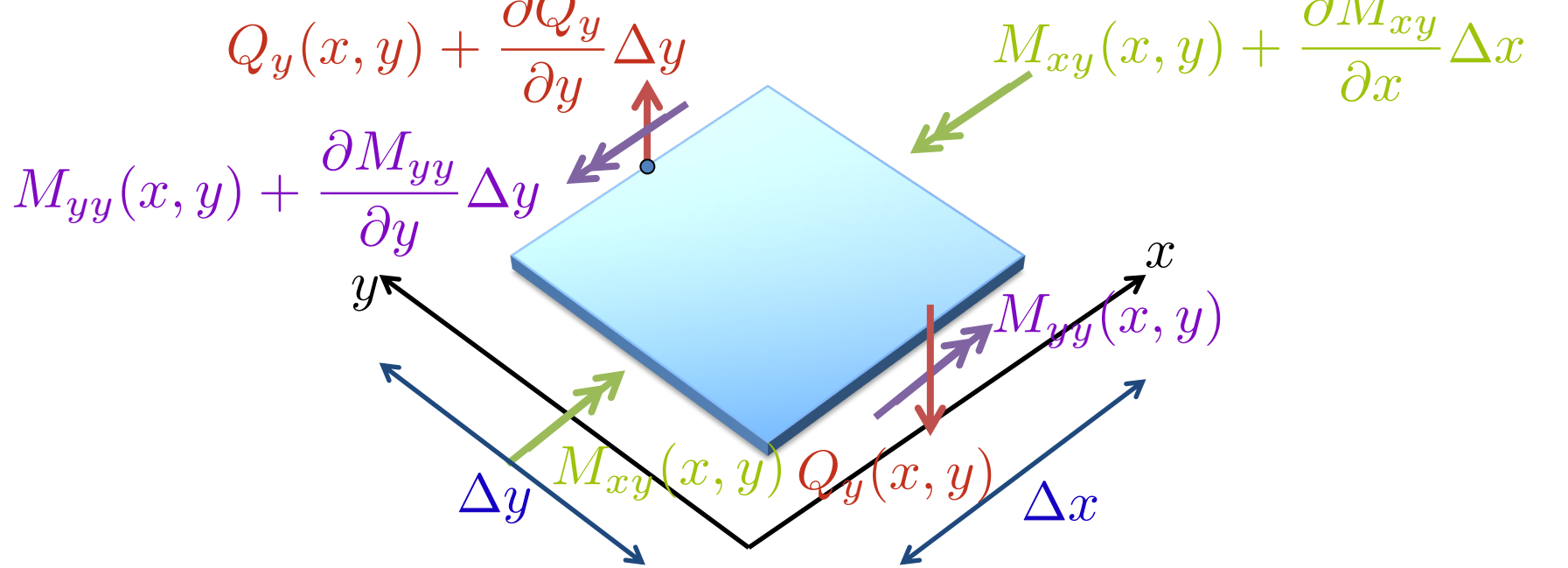
- On x -edge: $M_{xx}(x, y) = \int \sigma_{xx} z dz$

- On y -edge: $M_{yy}(x, y) = \int \sigma_{yy} z dz$

- On both edges: $M_{xy}(x, y) = \int \sigma_{xy} z dz$

- Positive based on the positive directions of the stresses

Moments Equilibrium in x -direction



$$\begin{aligned}
 & - \left[M_{xy}(x, y) + \frac{\partial M_{xy}}{\partial x} \Delta x \right] \Delta y + M_{xy}(x, y) \Delta y \\
 & - \left[M_{yy}(x, y) + \frac{\partial M_{yy}}{\partial y} \Delta y \right] \Delta x + M_{yy}(x, y) \Delta x + Q_y(x, y) \Delta x \Delta y = 0
 \end{aligned}$$

$$-\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_{yy}}{\partial y} + Q_y(x, y) = 0$$

Equilibrium Equations

Force Equilibrium
In z -direction

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q(x, y) = 0$$

Moment Equilibrium
In x -direction

$$-\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_{yy}}{\partial y} + Q_y(x, y) = 0$$

Moment Equilibrium
In y -direction

$$-\frac{\partial M_{xx}}{\partial x} - \frac{\partial M_{xy}}{\partial y} + Q_x(x, y) = 0$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + q(x, y) = 0$$

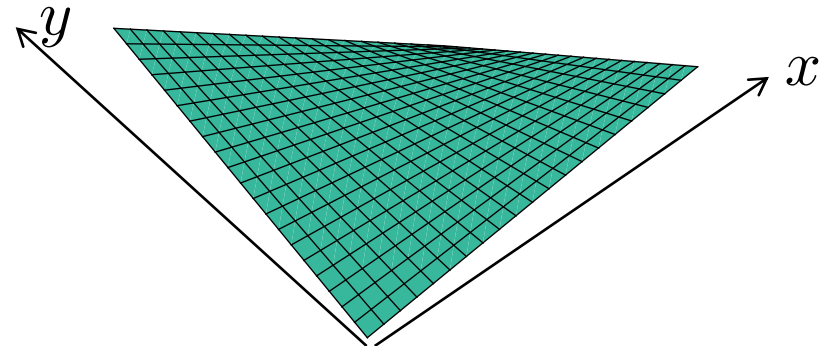
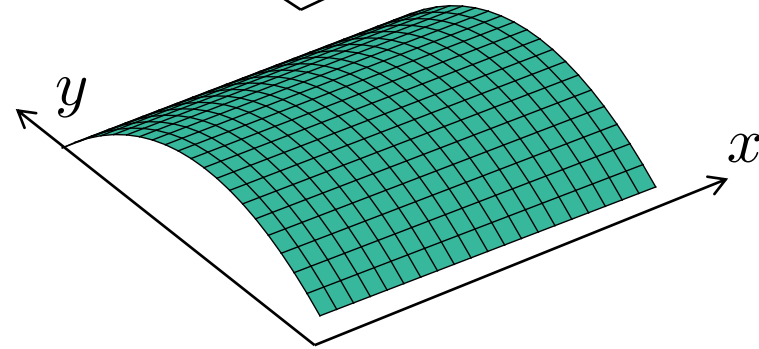
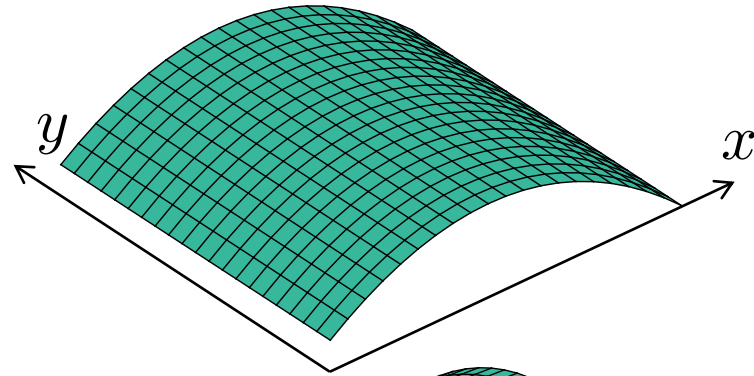
Kinematics

- Curvatures:

$$\kappa_{xx} = -\frac{\partial^2 w(x, y)}{\partial x^2}$$

$$\kappa_{yy} = -\frac{\partial^2 w(x, y)}{\partial y^2}$$

$$\kappa_{xy} = -\frac{\partial^2 w(x, y)}{\partial x \partial y}$$



Strains, Stresses and Moments

- Strains:
$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{Bmatrix} = z \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{Bmatrix} = -z \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

- Stresses:
(isotropic)
$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{Bmatrix}$$

- Moments:
(constant
thickness = h)

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = -\frac{Eh^3}{12(1 - \nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

Integrate stress to calculate moments

Plate Equations

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + q(x, y) = 0$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = -\frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

PDE for Plate Bending

Assumptions:

Isotropic Material

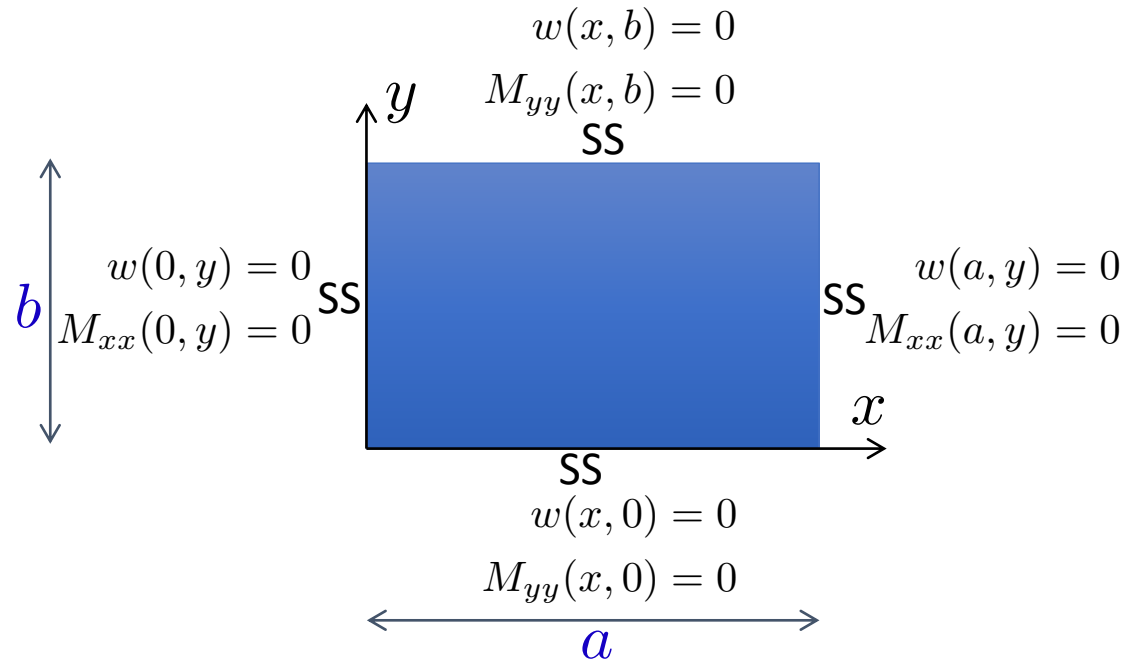
Constant Thickness

$$D \left[\frac{\partial^4 w(x, y)}{\partial x^4} + 2 \frac{\partial^4 w(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x, y)}{\partial y^4} \right] = q(x, y)$$

- Flexural Rigidity: $D = \frac{Eh^3}{12(1-\nu^2)}$

Navier Solution:

Simply-Supported Rectangular Plates with Sinusoidal Loading



$$q(x, y) = q_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

Navier Solution

continued

- assume sinusoidal solution

$$w(x, y) = w_1 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

- substitute in PDE to find w_1

$$w(x, y) = \frac{a^4 b^4}{\pi^4 (a^2 + b^2)^2} \frac{q_0}{D} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

- The solution is exact!

Navier solution derivation

Navier Solution

General loading on SS rectangular plates – series solution

- For general loading represented as a double Fourier series of sinusoidal loads

$$q(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

- Assume solution of the form

$$w(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

- The exact solution is

$$w(x, y) = \frac{1}{\pi^4 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{q_{mn}}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$