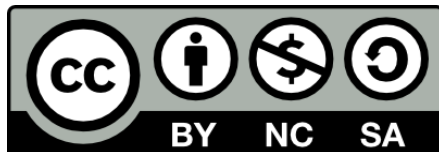


Newton-Euler Method for Rigid Body Dynamics



Dynamics: Newton-Euler Equations of Motion

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Newton's Laws of Motion

- Newton's Laws were published in 1687 in *Principia*
- Newton's 2nd Law

The force applied on a particle is proportional to the time derivative of the particle's (linear) momentum

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = m\vec{a}$$

- Assumptions above
 - Frame is inertial (Newtonian, Galilean)
 - Constant mass (for the equation in terms of acceleration)
 - Valid for particles (not rigid bodies), though everything is relative, e.g., sometimes a planet can be considered as a particle
- Extension of Newton's 2nd law to bodies was done by Leonhard Euler
 - Bodies have size, shape, and mass distribution

Leonhard Euler

- [Euler's laws of motion - Wikipedia](#)
- [Euler angles – Wikipedia](#)
- [Euler–Lagrange equation - Wikipedia](#)
- [Euler equations \(fluid dynamics\) – Wikipedia](#)
- [Euler–Bernoulli beam theory - Wikipedia](#)
- [Euler's critical load \(buckling\) - Wikipedia](#)
- [Euler method \(Numerical ODE\) - Wikipedia](#)
- [Euler's formula – Wikipedia](#)
- Optics, Number Theory, Graph Theory, Logic



Linear Momentum (Vector)

- For a particle i :

$$\vec{L}^i = m^i \vec{v}^i$$

- For a set of particles or a rigid body:

$$\vec{L} = \sum_i m^i \vec{v}^i$$

- In terms of Center of Mass of a set of particles or a rigid body:

$$\vec{L} = \sum_i m^i \vec{v}^i = \sum_i m^i \frac{d\vec{r}^i}{dt} = \frac{d \sum_i m^i \vec{r}^i}{dt} = \frac{d(m\vec{r}^{OC})}{dt} = m\vec{v}^C$$

All derivatives are in an inertial frame!

Center of Mass defined in the next slide ...

Center of Mass (Set of Particles)

- The linear momentum of the particles is related to the position vector of the particles

$$\sum_i m^i \vec{v}^i = \sum_i m^i \frac{d\vec{r}^i}{dt} = \sum_i \frac{d(m^i \vec{r}^i)}{dt} = \frac{d(\sum_i m^i \vec{r}^i)}{dt} = \frac{d(m \vec{r}^{OC})}{dt}$$

... the above assumed that the mass of the particles is not changing

- We thus define the center of mass location and total mass via:

$$m \vec{r}^{OC} = \sum_i m^i \vec{r}^i$$

where,

$$\vec{r}^{OC} = \frac{\sum_i m^i \vec{r}^i}{m}$$
$$m = \sum_i m^i$$

Euler's First Law

- Newton's 2nd law for i^{th} particle in a system of particles

$$\vec{F}^i + \sum_j \vec{f}^{ij} = \frac{d(m^i \vec{v}^i)}{dt}$$

- Summing the equations for all particles

$$\sum_i \vec{F}^i = \sum_i \frac{d(m^i \vec{v}^i)}{dt} = \frac{d \sum_i (m^i \vec{v}^i)}{dt} = \frac{d\vec{L}}{dt}$$

... we used the fact that the interaction forces are equal and opposite (Newton's 3rd Law) and thus do not contribute to the dynamics of the system as a whole

- Euler's First Law in terms of Linear Momentum:

$$\sum \vec{F}^i = \frac{d\vec{L}}{dt} = \frac{d(m\vec{v}_C)}{dt} = m\vec{a}_C$$

... valid for set of particles, rigid bodies, flexible bodies, mechanisms etc.

Extension to Rigid (or flexible) Bodies

- Replace the summations with integrations

$$\sum \vec{F}^i = \frac{d(\int \vec{v} dm)}{dt} \qquad \sum \vec{F}^i = \frac{d(\int \rho \vec{v} dV)}{dt}$$

- Equation for the Center of Mass

$$\vec{r}^{OC} = \frac{\int \vec{r}^{OP} dm}{m} \qquad \vec{r}^{OC} = \frac{\int \rho \vec{r}^{OP} dV}{m}$$
$$m = \int dm \qquad m = \int \rho dV$$

- Euler's First Law of Motion for translation of rigid bodies

$$\sum \vec{F}^i = \frac{d\vec{L}}{dt} = \frac{d(m\vec{v}_C)}{dt} = m\vec{a}_C$$

... same as for the set of particles

Angular Momentum

(vector, depends on the point about which it is calculated)

- For a particle i , angular momentum about ant point P :

$$\vec{H}^{i/P} = \vec{r}^{Pi} \times m^i \vec{v}^i$$

Angular momentum is moment of (linear) momentum!

- For a set of particles or a rigid body:

$$\vec{H}^P = \sum_i (\vec{r}^{Pi} \times m^i \vec{v}^i)$$

- Angular momentum about any point P is related to angular momentum about the center of mass C and the moment of linear momentum about P :

$$\vec{H}^P = \vec{H}^C + \vec{r}^{PC} \times \vec{L}$$

Euler's Second Law

- Newton's 2nd law for i^{th} particle in a system of particles:

$$\vec{F}^i + \sum_j \vec{f}^{ij} = \frac{d(m^i \vec{v}^i)}{dt}$$

- Moment about any point P :

$$\vec{r}^{Pi} \times \vec{F}^i + \vec{r}^{Pi} \times \sum_j \vec{f}^{ij} = \vec{r}^{Pi} \times \frac{d(m^i \vec{v}^i)}{dt}$$

- Summing the equations for all particles:

$$\sum_i \vec{r}^{Pi} \times \vec{F}^i = \sum_i \vec{r}^{Pi} \times \frac{d(m^i \vec{v}^i)}{dt}$$

$$\sum_i \vec{M}^{i/P} = \sum_i \vec{r}^{Pi} \times \frac{d(m^i \vec{v}^i)}{dt}$$

... is the term on the right related to angular momentum?

Yes, for particular choice for point P !

Euler's Second Law in terms of Angular Momentum about a Point Fixed in an Inertial Frame

- If the moment is taken about a point O , which is fixed in an inertial frame, the angular momentum is:

$$\vec{H}^O = \sum_i (\vec{r}^{Oi} \times m^i \vec{v}^i)$$

- The rate of change of angular momentum is:

$$\frac{d\vec{H}^O}{dt} = \sum_i \left(\frac{d\vec{r}^{Oi}}{dt} \times m^i \vec{v}^i + \vec{r}^{Oi} \times \frac{d(m^i \vec{v}^i)}{dt} \right)$$

$$\frac{d\vec{H}^O}{dt} = \sum_i \left(\vec{v}^i \times m^i \vec{v}^i + \vec{r}^{Oi} \times \frac{d(m^i \vec{v}^i)}{dt} \right) \Rightarrow \frac{d\vec{H}^O}{dt} = \sum_i \vec{r}^{Oi} \times \frac{d(m^i \vec{v}^i)}{dt}$$

- Thus, if the point O is fixed in an inertia frame, Euler's Second Law becomes:

$$\sum_i \vec{M}^{i/O} = \frac{d\vec{H}^O}{dt}$$

Euler's Second Law in terms of Angular Momentum about Center of Mass

- Euler's Second Law for angular momentum about O

$$\sum \vec{M}^{i/O} = \frac{d\vec{H}^O}{dt}$$

- Angular momentum about C is related to angular momentum about O :

$$\vec{H}^O = \vec{H}^C + \vec{r}^{OC} \times \vec{L}$$

- The applied moment about C is related to applied moment about O :

$$\vec{M}^{i/O} = \vec{M}^{i/C} + \vec{r}^{OC} \times \vec{F}^i$$

- Thus, if the point C is the center of mass, Euler's Second Law becomes:

$$\sum_i \vec{M}^{i/C} = \frac{d\vec{H}^C}{dt}$$

Angular Momentum forms of Euler's Second Law

- Euler's Second Law for set of particles or for a rigid body can be simplified and written in terms of the angular momentum about
 - point fixed in inertial frame
 - For a rigid body, it could be a pivot point fixed in inertial frame and the rigid body (but not be it's center of mass), e.g., rotation about a fixed axis not passing through the center of mass in 2D
 - center of mass of the set of particle or of the rigid body (even if moving)

$$\sum_i \vec{M}^{i/O} = \frac{d\vec{H}^O}{dt}$$

$$\sum_i \vec{M}^{i/C} = \frac{d\vec{H}^C}{dt}$$

... valid for set of particles, rigid bodies, flexible bodies, mechanisms etc.

- Euler's Second Law cannot be as easily written in terms of any general point:

$$\sum_i \vec{M}^{i/P} = \sum_i \vec{r}^{Pi} \times \frac{d(m^i \vec{v}^i)}{dt} \neq \frac{d\vec{H}^P}{dt}$$

