Angular Velocity

….. and Acceleration

mpatil@gatech.edu

Angular Velocity

- Angular velocity is a vector!

  … even though the general 3D orientation cannot be represented as a vector.

- Angular velocity is derived from small changes in the angle

  … small (infinitesimal) angles are additive and commutative and can be represented as a vector

- For small-angle, linearized, dynamics, we may not need representation of general 3D orientation

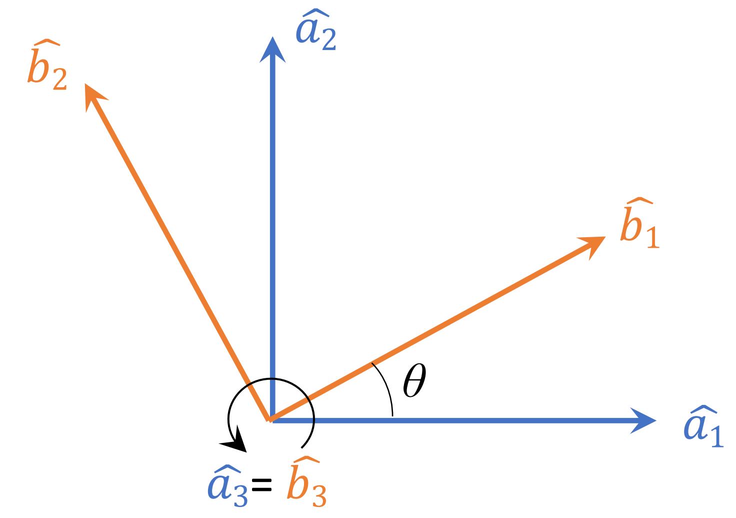
  e.g., linearized attitude dynamic of aircraft including linearized stability analysis

Simple Rotation and Simple Angular Velocity

- Let's consider two reference frames A and B
- If there is a rotation of frame B relative to frame A, about a fixed axis in both frames (simple rotation), say the z-axis, of θ

$$\vec{\omega} = \dot{\theta}\hat{k}$$

- This is simple angular velocity of B in A for a rotation about a fixed axis in A and B



Mathematical Definition of Angular Velocity

$$
\begin{aligned}
{}^A\vec{\omega}^B &\triangleq \hat{b}_1 \frac{{}^A d\hat{b}_2}{dt} \cdot \hat{b}_3 + \hat{b}_2 \frac{{}^A d\hat{b}_3}{dt} \cdot \hat{b}_1 + \hat{b}_3 \frac{{}^A d\hat{b}_1}{dt} \cdot \hat{b}_2 \\
&= \left(\frac{{}^A d\hat{b}_2}{dt} \cdot \hat{b}_3 \right) \hat{b}_1 + \left(\frac{{}^A d\hat{b}_3}{dt} \cdot \hat{b}_1 \right) \hat{b}_2 + \left(\frac{{}^A d\hat{b}_1}{dt} \cdot \hat{b}_2 \right) \hat{b}_3
\end{aligned}
$$

- The angular velocity is written in terms of the rate of change of unit vectors of B in A
- All the angular velocity expression can be derived from the above equation
- The expression can also be used to derive other very useful relations

Addition Theorem for Angular Velocities

- Lets consider two reference frames A and B and auxiliary or intermediate frames $A_1, A_2, ..., A_{n-1}, A_n$

$$^{A}\vec{\omega}^{B} = {}^{A}\vec{\omega}^{A_1} + {}^{A_1}\vec{\omega}^{A_2} + \dots {}^{A_{n-1}}\vec{\omega}^{A_n} + {}^{A_n}\vec{\omega}^{B}$$

- Thus:

$$^{B}\vec{\omega}^{A} = -{}^{A}\vec{\omega}^{B}$$

- We will use this in calculation of the aircraft angular velocity in terms of intermediate frames C and D
- For later, angular acceleration does not satisfy any addition theorem!

Angular Velocity in terms of Body Axis Measure Numbers

- It is common to write angular velocity as

$$\vec{\omega} = p\,\hat{b}_1 + q\,\hat{b}_2 + r\,\hat{b}_3$$

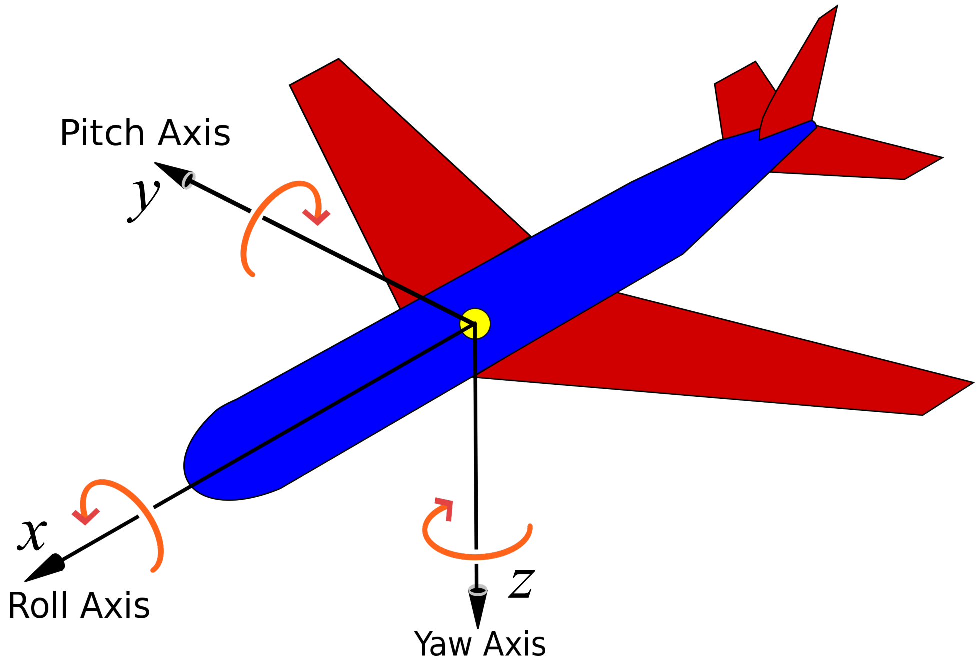
- For small angles:

$$p = \dot{\overline{\delta\phi}}$$

$$q = \dot{\overline{\delta\theta}}$$

$$r = \dot{\overline{\delta\psi}}$$

- Above angular velocities cannot be integrated to get corresponding angles for finite rotations



Angular Velocity in terms of Body Axis Measure Numbers – in terms of finite angles

- Consider angles (ϕ,θ,ψ) representing (Body 3-2-1) orientation from Earth frame A to aircraft body frame B:
  - Rotate about Earth z-axis by ψ to get intermediate reference frame C
  - Rotate about ref frame C y-axis by θ to get intermediate ref frame D
  - Rotate about ref frame D x-axis by ϕ to get aircraft body frame B
  - There are singularities but not if we restrict the angles

$$\begin{aligned}\vec{\omega} &= \dot{\phi}\,\hat{d}_1 + \dot{\theta}\,\hat{c}_2 + \dot{\psi}\,\hat{a}_3 \\ &= \dot{\phi}\,\hat{b}_1 + \dot{\theta}\,\hat{d}_2 + \dot{\psi}\,\hat{c}_3\end{aligned}$$

$$\vec{\omega} = p\,\hat{b}_1 + q\,\hat{b}_2 + r\,\hat{b}_3$$

$$p = -\dot{\psi}\sin\theta + \dot{\phi}$$

$$q = \dot{\psi}\cos\theta\sin\phi + \dot{\theta}\cos\phi$$

$$r = \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi$$

Differentiation in Two Reference Frames

- For any vector

$$\frac{{}^{A}d\vec{v}}{dt} = \frac{{}^{B}d\vec{v}}{dt} + {}^{A}\vec{\omega}^{B} \times \vec{v}$$

- This is an important relation used frequently
- Angular velocity is key to the differentiation of vectors in rotating frames

  … in addition to giving information about the angular motion of the body
- If vector is fixed in B

$$\frac{{}^{A}d\vec{v}}{dt} = {}^{A}\vec{\omega}^{B} \times \vec{v}$$

Angular Acceleration

- Lets consider two reference frames A and B, the angular acceleration of B in A is given by:

$$ {}^{A}\vec{\alpha}^{B} = \frac{{}^{A}d\,{}^{A}\vec{\omega}^{B}}{dt} $$

- We can also differentiate the angular velocity in the B frame to get the angular acceleration

$$ \frac{{}^{A}d\,{}^{A}\vec{\omega}^{B}}{dt} = \frac{{}^{B}d\,{}^{A}\vec{\omega}^{B}}{dt} + {}^{A}\vec{\omega}^{B} \times {}^{A}\vec{\omega}^{B} $$

- For simple rotation and angular velocity, we have

$$ \vec{\omega} = \dot{\theta}\hat{k} \qquad \vec{\alpha} = \ddot{\theta}\hat{k} $$