Kane’s Method

Motion Variables/Generalized Speeds

Partial Velocities

Generalized Active/Inertial Forces

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Motion Variables!

- Motion variables are generalized velocities variables defined by Kane

$$u_r = \sum_s Y_{rs}(q_1, q_2, q_3, \cdots, t)\dot{q}_s + Z_r(q_1, q_2, q_3, \cdots, t)$$

$$\dot{q}_s = \sum_r F_{sr}(q_1, q_2, q_3, \cdots, t)u_r + Q_s(q_1, q_2, q_3, \cdots, t)$$

  - these are called the kinematic differential equations
  - **linear** relation between u_r and $\dot{q}_s$ (can be nonlinear in q_i)
  - the above set of equations have to be **invertible** (one is the inverse of the other)
- This is where Kane's methods start to diverge from Newton's or Lagrange's or Hamilton's
  - Motion variables lead to partial velocities
  - Partial velocities help us calculate generalized forces (active and inertial)
  - Generalized forces leads to (Kane's) equations of motion
- Selection of motion variables is up to the analyst
  - The equations of motion and thus the complexity of final equations of motion depends on the choice of the motion variables
  - The motion variables are directions along which the equations of motion are derived

Selection and Use of Motion Variable

- A default set of motion variables could be:

$$u_i = \dot{q}_i$$

  - this would lead to equations similar to Lagrange's Equations
  - it is better to uncouple the two – the choice of motion variables and choice of generalized coordinates – for optimal (simplest) equations
- Select the motion variables such that your velocities and angular velocities of interest can be written as simply as possible
  - As far as the relationship between the motion variables and the derivatives of generalized coordinates is invertible

Generalized Speeds

- Generalized speeds are the independent motion variables:
  - For holonomic systems, the motion variables are the generalized speeds
  - For non-holonomic systems, constraints can be written in terms of the motion variables – and will thus reduce the number of generalized speeds
- The use of the generalized speeds leads to simplification of the expressions for the velocities and angular velocities and thus simplified calculation and expressions of the acceleration and angular acceleration (in terms of derivatives of generalized speeds)
  - The equations of motion can be written in terms in terms of the generalized coordinates, generalized speeds, and derivatives of the generalized speeds
  - The kinematic differential equations are written in terms of the generalized coordinates, motion variables, and derivatives of the generalized coordinates
  - Any non-holonomic constraints are written in terms of motion variables and generalized coordinates

Partial Angular Velocities & Partial Velocities

- Partial Velocities and Angular Velocities are key to Kane's method
- Partial Velocities and Angular Velocities are vectors!
- Any velocity and angular velocity in the system can be written in terms of the generalized speeds
  - The coefficients are the partial (angular) velocities
- Holonomic System:

$$\vec{\omega} = \sum_{r=1}^{n} \vec{\omega}_r u_r + \vec{\omega}_t$$

$$\vec{v} = \sum_{r=1}^{n} \vec{v}_r u_r + \vec{v}_t$$

  - Partial velocities are vectors while generalized speeds are scalars
  - All velocities and angular velocities have n partial velocities and partial angular velocities corresponding to n generalized speeds
  - Many of the partial velocities may be zero
- Non-Holonomic: Later

Generalized Active Forces: Particle (due to a resultant force)

- We have n generalized speeds
- Velocity of any particle will have n partial velocities
  - Though it is possible that many of these partial velocities are zero – depends on which generalized speeds affect the velocity of that particle
- There will be a resultant force on the particle $\vec{R}^i$
- The n generalized active forces (due to this particular ith resultant force) can be calculated as

$$F_r^i = \vec{v}_r^{P^i} \cdot \vec{R}^i$$

where, $\vec{v}^{P^i}$ is the velocity of the particle (where the resultant force is applied), and $\vec{v}_r^{P^i}$ is the corresponding partial velocity

- Generalized forces are scalars
- Units of the generalized force depends on the units of the generalized speed
- This is the generalized force that goes into the rth equations of motion
- You will have contributions due to all such resultant forces on the particles

Generalized Active Forces: Rigid Body (due to resultant force and torque)

- The system has n generalized speeds
- Velocity at a point on the body and angular velocity of the body will have n partial velocities
- The n generalized active force contributions due to forces on the rigid body can be calculated from the resultant force (with line of action passing through Q^i) and torque (at Q^i) on the rigid body as

$$F_r^i = \vec{\omega}_r \cdot \vec{T}^{Q^i} + \vec{v}_r^{Q^i} \cdot \vec{R}^i$$

  - This gives you the effect of the applied forces (via the corresponding resultant and torque) on the equation of motion (corresponding to the rth generalized speed)
  - You will have contributions due to all such resultant forces/torques on various rigid bodies

What does all this mean?

- To find the effect of an applied force (or torque) on an equation of motion which corresponds to a generalized speed
  - Find the velocity (or angular velocity) of the point (or rigid body)
  - Find the n partial velocities (or partial angular velocities)
    - This gives the dependence of the motion at that point (or rigid body) on the n generalized speeds
  - Take the dot product of the force (or torque) with the the rth partial velocity (or partial angular velocity)
    - This is the effect of the force (or torque) on the rth equation of motion (corresponding to the rth equation partial velocity or partial angular velocity)
- Note if the rth partial velocity (or partial angular velocity) of a point (or rigid body) is zero, it means that a force (or torque) on that point (or rigid body) does not affect the rth generalized speed directly and does not show up in the rth equation of motion

Generalized Inertia Forces: Particle

- The inertia force on a particle can be written as:

$$\vec{R}^{\star} = -m\vec{a}^{P}$$

where, the star indicates that it is related to the inertial forces

- The generalized inertia force can be written as:

$$F_r^{\star} = \vec{v}_r^{P} \cdot \vec{R}^{\star}$$

Generalized Inertia Forces: Rigid Body

- The inertia force and torque on a rigid body can be written using the center of mass as the reference point:

$$\vec{R}^\star = -m\vec{a}^C \qquad \vec{T}^\star = -\vec{\vec{I}}^C \cdot \vec{\alpha} - \vec{\omega} \times \left(\vec{\vec{I}}^C \cdot \vec{\omega} \right)$$

- The generalized inertia force can be written as:

$$F_r^\star = \vec{\omega}_r \cdot \vec{T}^\star + \vec{v}_r^C \cdot \vec{R}^\star$$

Kane's Equations of Motion (Holonomic)

- The equations of motion of a holonomic system S with n degrees of freedom can be written in terms of the generalized active forces and generalized inertial forces as:

$$\sum_i F_r^i + \sum_i F_r^{\star i} = 0$$

- The generalized active forces and generalized inertial forces are calculated in N
  - The partial velocities and partial angular velocities are in the inertial frame
  - The accelerations and angular accelerations are in the inertial frame
- The generalized active forces include any applied forces on particles and rigid bodies, as well as any forces due to gravity, friction, and interactions between bodies/particles
- The generalized inertial forces include the forces on all particles and rigid bodies

Steps in Deriving the Equations of Motion (Holonomic System)

- Start with configuration variables
- Determine any Holonomic Constraints
- Choose generalized coordinates that satisfy the Holonomic Constraints
- Derive expressions for the angular velocities of bodies and velocities of particles and bodies at significant points
- Choose motion variables – write the kinematic differential equations
- For holonomic system, motion variables are the independent generalized speeds
- Calculate all the resultant forces on particles and resultant forces/torques on rigid bodies
- Determine the partial angular velocities and partial velocities
- Derive the angular acceleration and acceleration in terms of the generalized speeds, time derivative of generalized speeds, and generalized coordinates
- Determine the generalized active forces and generalized inertial forces
- Write the equations of motion

Number of Coordinates and Speeds (Holonomic)

- Number of Configuration Variables: N
- Number of Holonomic Constraints: M
  - Assumed satisfied by the generalized coordinates for Kane's method
- Number of Generalized Coordinates: $n = N - M$
- Number of Motion Variables: n
- Number of Non-Holonomic Constraints: 0
- Number of Generalized Speeds: n
- Number of kinematic differential equations: n
- Number of equations of motion: n
- Number of equations equal to the number of unknowns: $2n$
  - n *(gen coord)* + n *(gen speed)*

Interesting Problem

Non-Holonomic Constraints

- In terms of generalized coordinates and derivatives of generalized coordinates

$$f_j(q_i, \dot{q}_i) = 0 \qquad 1 \leq j \leq m$$

- In terms of motion variables

$$f_j(q_i, u_i) = 0 \qquad 1 \leq j \leq m$$

- Simple non-holonomic constraint (linear in motion variables)

$$\sum_i f_{ji}(q_k)u_i + f_j(q_k) = 0 \qquad 1 \leq j \leq m$$

  - Explicitly for m dependent motion variables in terms of the p generalized speeds

$$u_j = \sum_{i=1}^{p} g_{ji}(q_k)u_i + g_j(q_k) \qquad p+1 \leq j \leq n$$

Partial Angular Velocities & Partial Velocities

- Holonomic:

$$\vec{\omega} = \sum_{r=1}^{n} \vec{\omega}_r u_r + \vec{\omega}_t$$

$$\vec{v} = \sum_{r=1}^{n} \vec{v}_r u_r + \vec{v}_t$$

- Non-Holonomic:
  - $u_{p+1}, u_{p+2}, \ldots u_n$, do not show up the expressions as they have been replaced using the m non-holonomic constraint equations
  - Possible for simple non-holonomic constraints (linear in motion variables)

$$\vec{\omega} = \sum_{r=1}^{p} \tilde{\vec{\omega}}_r u_r + \tilde{\vec{\omega}}_t$$

$$\vec{v} = \sum_{r=1}^{p} \tilde{\vec{v}}_r u_r + \tilde{\vec{v}}_t$$

where, tilde indicates that the partial velocity is derived for a non-holonomic system

(it does not indicate a dual matrix!)

Generalized Forces: Non-Holonomic Systems

- We can derive the generalized active forces on a non-holonomic system by
    - using the corresponding reduced set of p generalized speeds
    - which lead to a reduced set of p partial (angular) velocities
    - and reduced set of p generalized active forces
    - which goes into a reduced set of p equations of motions in terms of derivatives of the p generalized speeds

$$\tilde{F}_r = \tilde{\vec{v}}_r^P \cdot \vec{R}$$

$$\tilde{F}_r = \tilde{\vec{\omega}}_r \cdot \vec{T} + \tilde{\vec{v}}_r^Q \cdot \vec{R}$$

- Generalized inertial force are similarly derived

$$\tilde{F}_r^\star = \tilde{\vec{v}}_r^P \cdot \vec{R}^\star$$

$$\tilde{F}_r^\star = \tilde{\vec{\omega}}_r \cdot \vec{T}^\star + \tilde{\vec{v}}_r^C \cdot \vec{R}^\star$$

Kane's Equations of Motion (Non-Holonomic System)

- For non-holonomic system, we will get a reduced set of p equations of motions

$$\tilde{F}_r + \tilde{F}_r^\star = 0$$

Steps in Deriving the Equations of Motion (Non-Holonomic System)

- Start with configuration variables
- Determine any Holonomic Constraints
- Choose generalized coordinates that satisfy the Holonomic Constraints
- Derive expressions for the angular velocities of bodies and velocities of particles and bodies at significant points
- Choose motion variables – write the kinematic differential equations
- For non-holonomic system, derive the motion constraints and choose the reduced set of independent generalized speeds (possible for simple non-holonomic systems)
- Express all angular velocities and velocities in terms of the (independent) generalized speeds
- Calculate all the resultant forces on particles and resultant forces/torques on rigid bodies
- Determine the partial angular velocities and partial velocities
- Derive the angular acceleration and acceleration in terms of the generalized speeds, time derivative of generalized speeds, and generalized coordinates
- Determine the generalized active forces and generalized inertial forces
- Write the equations of motion

Number of Coordinates and Speeds (Non-Holonomic System)

- Number of Configuration Variables: N
- Number of Holonomic Constraints: M
  - Assumed satisfied by the generalized coordinates
- Number of Generalized Coordinates: $n = N - M$
- Number of Motion Variables: n
- Number of (Simple) Non-Holonomic Constraints: m
- Number of Generalized Speeds: $p = n - m$
- Number of kinematic differential equations: n
- Number of equations of motion: p
- Number of differential equations equal to the number of unknowns: $n + p$
  - n *(gen coord)* $+ p$ *(gen speed)*
  - The motion constraints (algebraic) can be used to post-process the other m motion variables