Chapter 1

Kinematics, Part 1

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– David Morin

Special relativity is an extremely counterintuitive subject, and in this chapter we will see how its bizarre features come about. We will build up the theory from scratch, starting with the postulates of relativity, of which there are only two. We will be able to derive a surprisingly large number of strange effects from these two easily stated postulates.

The postulate that most people find highly counterintuitive is that the speed of light has the same value in any inertial (that is, non-accelerating) reference frame. This speed, which is about $3 \cdot 10^8$ m/s, is much greater than the speed of everyday objects, so most of the consequences of relativity aren't noticeable. If we instead lived in a world identical to ours except for the speed of light being only 50 mph, then the consequences of relativity would be ubiquitous. We wouldn't think twice about time dilation, length contraction, and so on.

As mentioned in the preface, this chapter is the first of two that cover kinematics. (*Kinematics* deals with lengths, times, speeds, etc., whereas *dynamics* deals with masses, forces, energy, momentum, etc.) The outline of this chapter is as follows. In Section 1.1 we discuss the historical motivations that led Einstein to his theory of special relativity. Section 1.2 covers the two postulates of relativity, from which everything in the theory can be obtained. Section 1.3 is the heart of the chapter, where we derive the three main consequences of the postulates (*loss of simultaneity*, *time dilation*, and *length contraction*). In Section 1.4 we present four instructive examples that utilize the three fundamental effects. Section 1.5 covers the *velocity-addition formula*, which gives the proper correction to the naive Newtonian result (simply adding the velocities). In Chapter 2 we will continue our discussion of kinematics, covering more advanced topics.

1.1 Motivation

Although it was certainly a stroke of genius that led Einstein to his theory of relativity, it didn't just come out of the blue. A number of conundrums in 19th-century physics suggested that something was amiss. Many people had made efforts to explain away these conundrums, and at least a few steps had been taken toward the correct theory. But Einstein was the one who finally put everything together, and he did so in a way that had consequences far beyond the realm of the specific issues that people were trying to understand. Indeed, his theory turned our idea of space and time on its head. But before we get to the heart of the theory, let's look at two of the major problems in late 19th-century physics. (A third issue, involving the addition of velocities, is presented in Problem 1.15.) If you can't wait to get to the postulates (and subsequently the results) of special relativity, you can go straight to Section 1.2. The present section can be skipped on a first reading.

1.1.1 Galilean transformations, Maxwell's equations

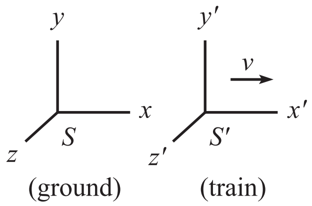

Figure 1.1

Imagine standing on the ground and watching a train travel by with constant speed v in the x direction. Let the reference frame of the ground be labeled S, and let the reference frame of the train be labeled S', as shown in Fig. 1.1. Consider two events that happen on the train. An *event* is defined as something that occurs with definite space and time coordinates (as measured in a given frame). For example, a person might clap her hands; this clap takes place at a definite time and a definite location. Technically, the clap lasts for a nonzero time (a few hundredths of a second), and the hands extend over a nonzero distance (a few inches). But we'll ignore these issues and assume that the clap can be described be unique x, y, z, and t values. Note that a given event isn't associated with one particular frame. The event simply happens, independent of a frame. For any arbitrary frame we then choose to consider, we can describe the event by specifying the coordinates as measured in that frame.

On our train, the two events might be one person clapping her hands and another person stomping his feet. If the space and time separations between these two events in the frame of the train are $\Delta x'$ and $\Delta t'$, what are the space and time separations, Δx and Δt, in the frame of the ground? Ignoring what we'll be learning about relativity in this chapter, the answers are "obvious" (although, as we'll see in Section 2.1 when we derive the Lorentz transformations, obvious things can apparently be incorrect!). The time separation Δt is "clearly" the same as on the train, so we have $\Delta t = \Delta t'$. We know from everyday experience that nothing strange happens with time. When you see people exiting a train station, they're not fiddling with their watches, trying to recalibrate them with a ground-based clock.

The spatial separation is a little more interesting, but still fairly simple. If the train weren't moving, then we would just have $\Delta x = \Delta x'$. This is true because if the train isn't moving, then the only possible difference between the frames is the location of the origin. But the only consequence of this difference is that every x' coordinate in the train is equal to a given fixed number plus the corresponding x coordinate on the ground. This fixed number then cancels when calculating the separation, $\Delta x' \equiv x'_2 - x'_1$.

However, in the general case where the train is moving, everything in the train gets carried along at speed v during the time Δt (which equals $\Delta t'$) between the two events.

So as seen in the ground frame, the person stomping his feet ends up $v\,\Delta t'$ to the right (or left, if v is negative) of where he would be if the train weren't moving. The total spatial separation Δx between the events in the ground frame is therefore the $\Delta x'$ separation that would arise if the train weren't moving, plus the $v\,\Delta t'$ separation due to the motion of the train. That is, $\Delta x = \Delta x' + v\,\Delta t'$, as shown in Fig. 1.2.

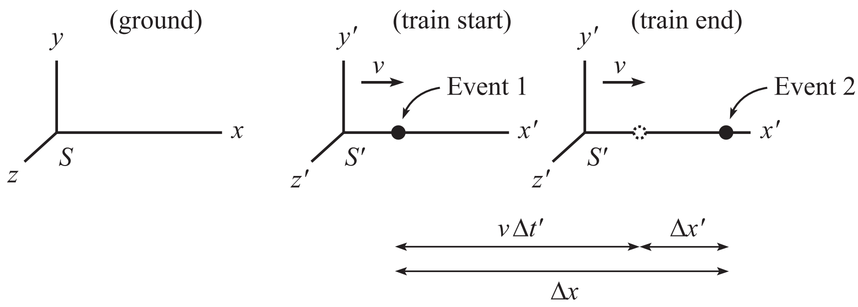

Figure 1.2

The *Galilean transformations* (first written down by Galileo Galilei in 1638) are therefore

$$\boxed{\begin{aligned}\Delta x &= \Delta x' + v\,\Delta t'\\ \Delta t &= \Delta t'\end{aligned}} \tag{1.1}$$

Nothing interesting happens in the y and z directions (assuming the train is traveling in the x direction), so we additionally have $\Delta y = \Delta y'$ and $\Delta z = \Delta z'$. Both of these common values are zero for the events in Fig. 1.2, because the events occur on the x axis. But for general locations of the events, the Δ's will be nonzero.

A special case of Eq. (1.1) arises when the two events occur at the same place on the train, so that $\Delta x' = 0$. In this case we have $\Delta x = v\,\Delta t'$. This makes sense, because the spot on the train where the events occur simply travels a distance $v\,\Delta t$ (which equals $v\,\Delta t'$) by the time the second event occurs.

The principle of *Galilean invariance* says that the laws of physics are invariant under the above Galilean transformations. Alternatively, it says that the laws of physics hold in all inertial (non-accelerating) frames. (It was assumed prior to Einstein that these two statements say the same thing, but we will soon see that they do not. The second statement is the one that remains valid in relativity.) This principle is quite believable. For example, in Galilean (nonrelativistic) physics, Newton's second law, $F = ma$ (or really $F = dp/dt$) holds in all inertial frames, because (1) the force F is the same in all inertial frames, and (2) the constant relative velocity v_{rel} between any two inertial frames implies that the acceleration of a given particle is the same in all inertial frames. Written out explicitly, the velocities v_1 and v_2 in the two frames are related by $v_1 = v_2 + v_{\text{rel}}$, so

$$a_1 \equiv \frac{dv_1}{dt_1} = \frac{d(v_2 + v_{\text{rel}})}{dt_1} = \frac{dv_2}{dt_2} + 0 \equiv a_2, \tag{1.2}$$

where we have used the facts that $t_1 = t_2$ (at least in a Galilean world) and that the derivative of a constant is zero.

Remarks:

1. Note that the Galilean transformations in Eq. (1.1) aren't symmetric in x and t. This isn't necessarily a bad thing, but it turns out that it will in fact be a problem in special relativity, where space and time are treated on a more equal footing. We'll find in Section 2.1 that the Galilean transformations are replaced by the *Lorentz transformations*, and the latter are in fact symmetric in x and t (up to factors of the speed of light, c).

2. Eq. (1.1) deals only with the *differences* in the x and t values between two events, and not with the values of the coordinates themselves of each event. The values of the coordinates of a single event depend on where you pick your origin, which is an arbitrary choice. The coordinate differences between two events, however, are independent of this choice, and this allows us to make the physically meaningful statements in Eq. (1.1). Since it makes no sense for a physical result to depend on your arbitrary choice of origin, the Lorentz transformations we derive in Section 2.1 will also need to involve only differences in coordinates.

3. We've been talking a lot about "events," so just to make sure we're on the same page with the definition of an event, we should give some examples of things that are *not* events. If a train is at rest on the ground (or even if it is moving), and if you look at it at a snapshot in time, then this doesn't describe an event, because the train has spatial extent. There isn't a unique spatial coordinate that describes the train. If you instead consider a specific point on the train at the given instant, then that does describe an event. As another example of a non event, if you look at a pebble on the ground for a minute, then this doesn't describe an event, because you haven't specified the time coordinate. If you instead consider the pebble at a particular instant in time, then that does describe an event. (We'll consider the pebble to be a point object, so that the spatial coordinate is unique.) ♣

We introduced the Galilean transformations above because of their relation (more precisely, their conflict) with *Maxwell's equations*. One of the great triumphs of 19th-century physics was the theory of electromagnetism. In 1864, James Clerk Maxwell wrote down a set of equations that collectively described everything that was known about the subject. These equations involve the electric and magnetic fields through their space and time derivatives. Maxwell's original formulation consisted of a large number of equations, but these were later written more compactly, using vector calculus, as four equations. We won't worry about their specific form here, but it turns out that if you transform the equations from one reference frame to another via the Galilean transformations, they end up taking a different form. That is, if you've written down Maxwell's equations in one frame (where they take their standard nice-looking form), and if you then replace the coordinates in this frame by those in another frame, using Eq. (1.1), then the equations look different (and not so nice).

This different appearance presents a major problem. If Maxwell's equations take a nice form in one frame and a not-so-nice form in every other frame, then why is one frame special? Said in another way, it can be shown that Maxwell's equations imply that light moves with a certain speed c. But which frame is this speed measured with respect to? The Galilean transformations imply that if the speed is c with respect to a given frame, then it is *not* c with respect to any other frame. (You need to add or subtract the relative speed v between the frames.) The proposed special frame where Maxwell's equations are nice and the speed of light is c was called the frame of the *ether*. We'll talk in detail about the ether in the next subsection, but experiments showed that light was surprisingly always measured to move with speed c in every frame, no matter which way the frame was moving through the supposed ether. We say "supposed" because the final conclusion was that the ether simply doesn't exist.

There were thus various possibilities. Something was wrong with either Maxwell's equations, the Galilean transformations, or the way in which measurements of speed were done (see Footnote 2 on page 8). Considering how "obvious" the Galilean transformations are, the natural assumption in the late 19th century was that the problem lay elsewhere. However, after a good deal of effort by many people to make everything else fit with the Galilean transformations, Einstein finally showed that these were in fact the culprit. It was well known that Maxwell's equations were invariant under the Lorentz transformations (in contrast with their non-invariance under the Galilean ones), but Einstein was the first to recognize the full meaning of these transformations. Instead of being relevant only to electromagnetism, the Lorentz transformations replaced the Galilean ones universally.

More precisely, in 1905 Einstein showed why the Galilean transformations are simply a special case of the Lorentz transformations, valid (to a high degree of accuracy) only when the speed involved is much less than the speed of light. As we'll see in Section 2.1, the coefficients in the Lorentz transformations depend on both the relative speed v of the frames and the speed of light c, where the c's appear in various denominators. Since c is quite large (about $3 \cdot 10^8$ m/s) compared with everyday speeds v, the parts of the Lorentz transformations involving c are negligible, for any typical v. The surviving terms are the ones in the Galilean transformations in Eq. (1.1). These are the only terms that are noticeable for everyday speeds. This is why no one prior to Einstein realized that the correct transformations between two frames had anything to do with the speed of light.

As he pondered the long futile fight
To make Galileo's world right,
In a new variation
Of the old transformation,
It was Einstein who first saw the light.

In short, the reasons why Maxwell's equations are in conflict with the Galilean transformations are: (1) The speed of light is what determines the scale at which the Galilean transformations break down, (2) Maxwell's equations inherently involve the speed of light, because light is an electromagnetic wave.

1.1.2 Michelson–Morley experiment

As mentioned above, it was known in the late 19th century, after Maxwell wrote down his equations, that light is an electromagnetic wave and that it moves with a speed of about $3 \cdot 10^8$ m/s.[1] Now, every other wave that people knew about at the time needed a medium to propagate in. Sound waves need air, ocean waves of course need water, waves on a string of course need the string, and so on. It was therefore natural to assume that light also needed a medium to propagate in. This proposed medium was called the *ether*.

However, if light propagates in a given medium, and if the speed in this medium is c, then the speed in a reference frame moving relative to the medium should be different from c. Consider, for example, sound waves in air. If the speed of sound in air is v_{sound}, and if you run toward a sound source with speed v_{you}, then the speed of the sound waves with respect to you (assuming it's a windless day) is $v_{\text{sound}} + v_{\text{you}}$. Equivalently, if you are standing at rest downwind and the speed of the wind is v_{wind}, then the speed of the sound waves with respect to you is $v_{\text{sound}} + v_{\text{wind}}$.

Assuming that the ether really exists (although we'll soon see that it doesn't), a reasonable thing to do is to try to measure one's speed with respect to it. This can be done as follows. We'll frame this discussion in terms of sound waves in air. Let v_s be the speed of sound in air. Imagine two people standing on the ends of a long platform of length L that moves at speed v_p with respect to the reference frame in which the air is at rest. One person claps, the other person claps immediately when he hears the first clap (assume that the reaction time is negligible), and then the first person records the

[1] The exact value of the speed is 299,792,458 m/s. A meter is actually *defined* to be 1/299,792,458 of the distance that light travels in one second in vacuum. So this speed of light is *exact*. There is no need for an error bar because there is no measurement uncertainty.

total time elapsed when she hears the second clap. What is this total time? Well, we can't actually give an answer without knowing which direction the platform is moving. Is it moving parallel to its length, or perpendicular to it (or somewhere in between)? Let's look at these two basic cases. For both of these, we'll view the setup and do the calculation in the frame in which the air is at rest.

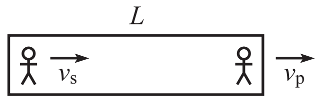

Figure 1.3

- Parallel motion: Consider first the case where the platform moves parallel to its length. In the reference frame of the air, assume that the person at the rear is the one who claps first; see Fig. 1.3. Then if v_s is the speed of sound and v_p is the speed of the platform, it takes a time of $L/(v_s - v_p)$ for the sound from the first clap to travel forward to the front person. This is true because the sound closes the initial gap of L at a relative speed of $v_s - v_p$, as viewed in the frame of the air. (Alternatively, relative to the initial position of the back of the platform, the position of the sound wave is $v_s t$, and the position of the front person is $L + v_p t$. Equating these gives $t = L/(v_s - v_p)$.) This time is longer than the naive answer of L/v_s because the front person is moving away from the rear person, which means that the sound has to travel farther than L.

  By similar reasoning, the time for the sound from the second clap to travel backward to the rear person is $L/(v_s + v_p)$. This time is shorter than the naive answer of L/v_s because the rear person is moving toward the front person, which means that the sound travels less than L.

  Adding the forward and backward times gives a total time of

$$t_1 = \frac{L}{v_s - v_p} + \frac{L}{v_s + v_p} = \frac{2Lv_s}{v_s^2 - v_p^2}. \tag{1.3}$$

  This correctly equals $2L/v_s$ when $v_p = 0$. In this case the platform is at rest, so the sound simply needs to travel forward and backward a total distance of $2L$ at speed v_p. And the result correctly equals infinity when $v_p \to v_s$. In this case the front person is receding as fast as the sound is traveling, so the sound from the first clap can never catch up.

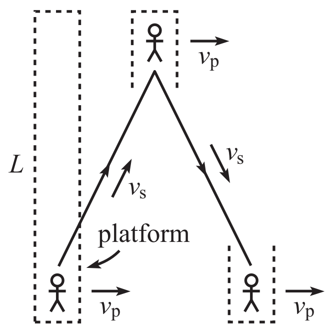

Figure 1.4

- Perpendicular motion: Now consider the case where the platform moves perpendicular to its length. In the reference frame of the air, we have the situation shown in Fig. 1.4. The sound moves diagonally with speed v_s. (The sound actually moves in all directions, of course, but it's only the part of the sound wave that moves in a particular diagonal direction that ends up hitting the other person.) Since the "horizontal" component of the diagonal velocity is the platform's speed v_p, the Pythagorean theorem gives the "vertical" component as $\sqrt{v_s^2 - v_p^2}$, as shown in Fig. 1.5. This is the speed at which the length L of the platform is traversed during both the out and back parts of the trip. So the total time is

$$t_2 = \frac{2L}{\sqrt{v_s^2 - v_p^2}}. \tag{1.4}$$

  Again, this correctly equals $2L/v_s$ when $v_p = 0$, and infinity when $v_p \to v_s$. The vertical component of the velocity is zero in the latter case, because the diagonal path is essentially horizontal.

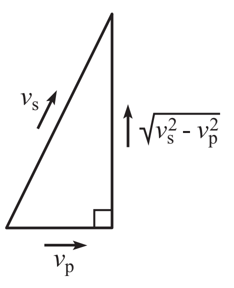

Figure 1.5

The times in Eqs. (1.3) and (1.4) are not equal; you can quickly show that $t_1 \geq t_2$. It turns out that (for given values of v_s and v_p) of all the possible orientations of the platform relative to the direction of motion (which we have been taking to be rightward), the t_1 in Eq. (1.3) is the largest possible time, and the t_2 in Eq. (1.4) is the smallest.

(The proof of this is somewhat tedious, but at least it is believable that if the platform is oriented between the above two special cases, the time lies between the associated times t_1 and t_2.) Therefore, if you are on a large surface that is moving with respect to the air, and if you know the value of v_s, then if you want to figure out what v_p is, all you have to do is repeat the above experiment with someone standing at various points along the circumference of a given circle of radius L around you. (Assume that it doesn't occur to you to toss a little piece of paper in the air, in order to at least determine the direction of the wind with respect to you.) If you take the largest total time observed and equate it with t_1, then Eq. (1.3) will give you v_p. Alternatively, you can equate the smallest total time with t_2, and Eq. (1.4) will yield the same v_p.

In the limiting case where $v_p \ll v_s$, we can make some approximations to the above expressions for t_1 and t_2. These approximations involve the Taylor-series expressions $1/(1-\epsilon) \approx 1+\epsilon$ and $1/\sqrt{1-\epsilon} \approx 1+\epsilon/2$. (See Appendix G for a discussion of Taylor series.) These expressions yield the following approximate result for the difference between t_1 and t_2 (after first rewriting t_1 and t_2 so that a "1" appears in the denominator):

$$\begin{aligned}
\Delta t = t_1 - t_2 &= \frac{2L}{v_s}\left(\frac{1}{1-v_p^2/v_s^2} - \frac{1}{\sqrt{1-v_p^2/v_s^2}}\right) \\
&\approx \frac{2L}{v_s}\left(\left(1+\frac{v_p^2}{v_s^2}\right) - \left(1+\frac{v_p^2}{2v_s^2}\right)\right) \\
&= \frac{Lv_p^2}{v_s^3}\,. \qquad (1.5)
\end{aligned}$$

The difference $t_1 - t_2$ is what we'll be concerned with in the Michelson–Morley experiment, which we will now discuss.

The strategy in the above sound-in-air setup is the basic idea behind Michelson's and Morley's attempt in 1887 to measure the speed of the earth through the supposed ether. (See Handschy (1982) for the data and analysis of the experiment.) There is, however, a major complication with light that doesn't arise with sound. The speed of light is so large that any time intervals that are individually measured will inevitably have measurement errors that are far larger than the difference between t_1 and t_2. Therefore, individual time measurements give essentially no information. Fortunately, there is a way out of this impasse. The trick is to measure t_1 and t_2 concurrently, as opposed to separately. More precisely, the trick is to measure only the difference $t_1 - t_2$, and not the individual values t_1 and t_2. This can be done as follows.

Consider two of the above "platform" scenarios arranged at right angles with respect to each other, with the same starting point. This can be arranged by having a (monochromatic) light beam encounter a beam splitter that sends two beams off at 90° angles. The beams then hit mirrors and bounce back to the beam splitter where they (partially) recombine before hitting a screen; see Fig. 1.6. The fact that light is a wave, which is what got us into this ether mess in the first place, is now what saves the day. The wave nature of light implies that the recombined light beam produces an interference pattern on the screen. At the center of the pattern, the beams will constructively or destructively interfere (or something in between), depending on whether the two light beams are in phase or out of phase when they recombine. This interference pattern is extremely delicate. The slightest change in travel times of the beams will cause the pattern to noticeably shift. This type of device, which measures the interference between two light beams, is known as an *interferometer*.

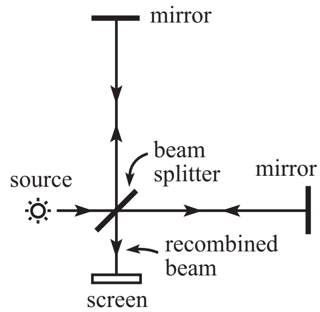

Figure 1.6

If the whole apparatus is rotated around, so that the experiment is performed at various angles, then the maximum amount that the interference pattern changes can be used to determine the speed of the earth through the ether (v_p in the platform setup above). In one extreme case, the time in a given arm is longer than the time in the other arm by Lv^2/c^3. (We have changed notation in Eq. (1.5) so that $v_p \to v$ is the speed of the earth through the supposed ether, and $v_s \to c$ is the speed of light.) But in the other extreme case, the time in the given arm is shorter by Lv^2/c^3. So the maximum shift in the interference pattern corresponds to a time difference of $2Lv^2/c^3$.

However, when Michelson and Morley performed their experiment, they observed no interference shift as the apparatus was rotated around. Their setup did in fact allow enough precision to measure a nontrivial earth speed through the ether, if such a speed existed. So if the ether did exist, their results implied that the speed of the earth through it was zero. This result, although improbable, was technically fine. It might simply have been the case that they happened to do their experiment when the relative speed was zero. However, when they performed their experiment a few months later, when the earth's motion around the sun caused it to be moving in a different direction, they still measured zero speed. It wasn't possible for both of these results to be zero (assuming that the ether exists), without some kind of modification to the physics known at the time.

Many people over the years tried to explain this null result, but none of the explanations were satisfactory. Some led to incorrect predictions in other setups, and some seemed to work fine but were a bit ad hoc.[2] The correct explanation, which followed from Einstein's 1905 theory of relativity, was that the ether simply doesn't exist.[3] In other words, light doesn't need a medium to propagate in. It doesn't move with respect to a certain special reference frame, but rather it moves with respect to whoever is looking at it.

The findings of Michelson–Morley
Allow us to say very surely,
"If this ether is real,
Then it has no appeal,
And shows itself off rather poorly."

Remarks:

1. We assumed above that the lengths of the two arms in the apparatus were equal. However, in practice there is no hope of constructing lengths that are equal, up to errors that are small compared with the wavelength of the light. But fortunately this doesn't matter. We're concerned not with the difference in the travel times associated with the two arms, but rather with the *difference in these differences* as the apparatus is rotated around. Using Eqs. (1.3) and (1.4) with different lengths L_1 and L_2, you can show (assuming $v \ll c$) that the maximum interference shift corresponds to a time of $(L_1 + L_2)v^2/c^3$. This is the generalization of the $2Lv^2/c^3$ result we derived in Eq. (1.5) (in different notation) when the lengths were equal. The measurement errors in L_1 and L_2 therefore need only be small compared with the (macroscopic) lengths L_1 and L_2, as opposed to small compared with the (microscopic) wavelength of light.

2. Assuming that the lengths of the arms are approximately equal, let's plug in some rough numbers to see how much the interference pattern shifts. The Michelson–Morley setup

[2]The most successful explanation was the Lorentz–FitzGerald contraction. These two physicists independently proposed that lengths are contracted in the direction of the motion by precisely the right factor, namely $\sqrt{1 - v^2/c^2}$, to make the travel times in the two arms of the Michelson–Morley setup equal, thus yielding the null result. This explanation was essentially correct, although the reason why it was correct wasn't known until Einstein came along.

[3]Although we've presented the Michelson–Morley experiment here for pedagogical purposes, the consensus among historians is that Einstein actually wasn't influenced much by the experiment, except indirectly through Lorentz's work on electrodynamics. See Holton (1988).

had arms with effective lengths of about 10 m. We'll take v to be on the order of the speed of the earth around the sun, which is about $3 \cdot 10^4$ m/s. We then obtain a maximal time difference of $t = 2Lv^2/c^3 \approx 7 \cdot 10^{-16}$ s. The large negative exponent here might make us want to throw in the towel, thinking that the effect is hopelessly small. However, the distance that light travels in the time t is $ct = (3 \cdot 10^8\text{ m/s})(7 \cdot 10^{-16}\text{ s}) \approx 2 \cdot 10^{-7}$ m, and this happens to be a perfectly reasonable fraction of the wavelength of visible light, which is around $\lambda = 6 \cdot 10^{-7}$ m, give or take. So we have $ct/\lambda \approx 1/3$. This maximal interference shift of about a third of a cycle was well within the precision of the Michelson–Morley setup. So if the ether had really existed, Michelson and Morley definitely would have been able to measure the speed of the earth through it.

3. One proposed explanation of the observed null effect was "frame dragging." What if the earth drags the ether along with it, thereby always yielding the observed zero relative speed? This frame dragging is quite plausible, because in the platform example above, the platform drags a thin layer of air along with it. And more mundanely, a car completely drags the air in its interior along with it. But it turns out that frame dragging is inconsistent with *stellar aberration*, which is the following effect.

   Depending on the direction of the earth's instantaneous velocity as it obits around the sun, it is an experimental fact that a given star might (depending on its location) appear at slightly different places in the sky when viewed at two times, say, six months apart. This is due to the fact that a telescope must be aimed at a slight angle relative to the actual direction to the star, because as the star's light travels down the telescope, the telescope moves slightly in the direction of the earth's motion. We're assuming (correctly) here that frame dragging does *not* exist.

   As a concrete analogy, imagine holding a tube while running through vertically falling rain. If you hold the tube vertically, then the raindrops that enter the tube won't fall cleanly through. Instead, they will hit the side of the tube, because the tube is moving sideways while the raindrops are falling vertically. However, if you tilt the tube at just the right angle, the raindrops will fall (vertically) cleanly through without hitting the side. At what angle θ should the tube be tilted? If the tube travels horizontally a distance d during the time it takes a raindrop to fall vertically a distance h, then the ratio of these distances must equal the ratio of the speeds: $d/h = v_{\text{tube}}/v_{\text{rain}}$ (see Fig. 1.7). The angle θ is then given by $\tan\theta = d/h \implies \tan\theta = v_{\text{tube}}/v_{\text{rain}}$. With respect to your frame as you run along, the raindrops come down at an angle θ; they don't come down vertically.

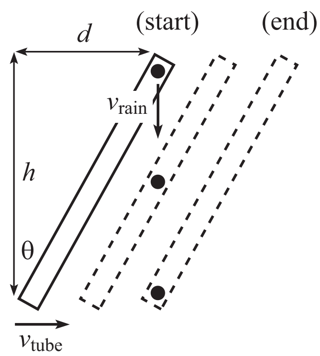

Figure 1.7

   Returning to the case of light, v_{tube} gets replaced with (roughly) the speed v of the earth around the sun, and v_{rain} gets replaced with the speed c of light. The ratio of these two speeds is about $v/c = 10^{-4}$, so the effect is small. But it is large enough to be noticeable, and it has indeed been measured; stellar aberration *does* exist. At two different times of the year, a telescope must be pointed at slightly different angles when viewing a given star.

   Now, if frame dragging *did* exist, then the light from the star would get dragged along with the earth and would therefore travel down a telescope that was pointed directly at the star, in disagreement with the observed fact that the telescope must point at the slight angle mentioned above. (Or even worse, the dragging might produce a boundary layer of turbulence that would blur the stars.) The existence of stellar aberration therefore implies that frame dragging doesn't occur.

4. Note that it is the velocity of the telescope that matters in stellar aberration, and not its position. This aberration effect should not be confused with the *parallax* effect, where the direction of the actual position of an object changes, depending on the *position* of the observer. For example, people at different locations on the earth see the moon at slightly different angles (that is, they see the moon in line with different distant stars). As a more down-to-earth example, two students sitting at different locations in a classroom see the teacher at different angles. Although stellar parallax has been measured for nearby stars (as the earth goes around the sun), its angular effect is much smaller than the angular effect from stellar aberration. The former decreases with the distance to the star, whereas the latter doesn't. For further discussion of aberration, and of why it is only the earth's velocity (or rather, the change in its velocity) that matters, and not also the star's velocity (since

you might think, based on the fact that we're studying relativity here, that it is the relative velocity that matters), see Eisner (1966). ♣

1.2 The postulates

Let's now start from scratch and see what the theory of special relativity is all about. We'll take the route that Einstein took and use two postulates as the foundation of the theory. We'll start with the "relativity postulate" (also called the Principle of Relativity). This postulate is quite believable, so you might just take it for granted and forget to consider it. But like any other postulate, it is crucial. It can be stated in various ways, but we'll write it simply as:

- Postulate 1: *All inertial (non-accelerating) frames are "equivalent."*

This postulate says that a given inertial frame is no better than any other; there is no preferred reference frame. That is, it makes no sense to say that something is moving. It makes sense only to say that one thing is moving with respect to another. This is where the "relativity" in "special relativity" comes from. There is no absolute inertial frame; the motion of any frame is defined only relative to other frames.

This postulate also says that if the laws of physics hold in one inertial frame (and presumably they do hold in the frame in which I now sit), then they hold in all others. (Technically, the earth is spinning while revolving around the sun, and there are also little vibrations in the floor beneath my chair, etc., so I'm not *really* in an inertial frame. But it's close enough for me.) The postulate also says that if we have two frames S and S', then S should see things in S' in exactly the same way as S' sees things in S, because we can just switch the labels of S and S'. (We'll get our money's worth out of this statement in the next few sections.) It also says that empty space is homogeneous (that is, all points look the same), because we can pick any point to be, say, the origin of a coordinate system. It also says that empty space is isotropic (that is, all directions look the same), because we can pick any axis to be, say, the x axis of a coordinate system.

Unlike the second postulate below (the speed-of-light postulate), this first one is entirely reasonable. We've gotten used to having no special places in the universe. We gave up having the earth as the center, so let's not give any other point a chance, either.

Copernicus gave his reply
To those who had pledged to deny.
"All your addictions
To ancient convictions
Won't bring back your place in the sky."

The first postulate is nothing more than the familiar principle of Galilean invariance, assuming that this principle is written in the "The laws of physics hold in all inertial frames" form, and not in the form that explicitly mentions the Galilean transformations in Eq. (1.1), which are inconsistent with the second postulate below.

Everything we've said here about the first postulate refers to empty space. If we have a chunk of mass, then there is certainly a difference between the position of the mass and a point a meter away. To incorporate mass into the theory, we would have to delve into the subject of general relativity. But we won't have anything to say about that in this chapter. We will deal only with empty space, containing perhaps a few observant souls sailing along in rockets or floating aimlessly on little spheres. Although that might sound boring at first, it will turn out to be anything but.

The second postulate of special relativity is the "speed-of-light" postulate. This one is much less intuitive than the relativity postulate.

- POSTULATE 2: *The speed of light in vacuum has the same value c (approximately* $3 \cdot 10^8$ m/s*) in any inertial frame.*

This statement certainly isn't obvious, or even believable. But on the bright side, at least it's easy to understand what the postulate says, even if you think it's too silly to be true. It says the following. Consider a train moving along the ground with constant velocity. Someone on the train shines a light from one point on the train to another. The speed of the light with respect to the train is c. Then the above postulate says that a person on the ground also sees the light move at speed c.

This is a rather bizarre statement. It doesn't hold for everyday objects. If a baseball is thrown forward with a given speed on a train, then the speed of the baseball is different in the ground frame. An observer on the ground must add the velocity of the ball (with respect to the train) to the velocity of the train (with respect to the ground) in order to obtain the velocity of the ball with respect to the ground. Strictly speaking, this isn't quite true, as the velocity-addition formula in Section 1.5 shows. But it's true enough for the point we're making here.

The truth of the speed-of-light postulate cannot be demonstrated from first principles. No statement with any physical content in physics (that is, one that isn't purely mathematical, such as, "two apples plus two apples gives four apples") can be proven. In the end, we must rely on experiment. And indeed, all of the consequences of the speed-of-light postulate have been verified countless times during the past century. As discussed in the previous section, the most well-known of the early experiments on the speed of light was the one performed by Michelson and Morley. The zero interference shift they always observed implied that the v_p speed in Eq. (1.5) was always zero. This in turn implies that no matter what (inertial) frame you are in, you are always at rest with respect to a frame in which the speed of light is c. In other words, the speed of light is the same in any inertial frame.

In more recent years, the consequences of the second postulate have been verified continually in high-energy particle accelerators, where elementary particles reach speeds very close to c. The collection of all the data from numerous experiments over the years allows us to conclude with near certainty that our starting assumption of an invariant speed of light is correct (or is at least the limiting case of a more correct theory).

REMARK: Given the first postulate, you might wonder if we even need the second. If all inertial frames are equivalent, shouldn't the speed of light be the same in any frame? Well, no. For all we know, light might behave like a baseball. A baseball certainly doesn't have the same speed in all inertial frames, and this doesn't ruin the equivalence of the frames.

It turns out (see Section 2.7) that nearly all of special relativity can be derived by invoking *only* the first postulate. The second postulate simply fills in the last bit of necessary information by stating that *something* has the same finite speed in every frame. It's actually not important that this thing is light. It could be mashed potatoes or something else, and the theory would still come out the same. (Well, the thing has to be massless, as we'll see in Chapter 3, so we'd need to have massless potatoes, but whatever.) The second postulate can therefore be stated more minimalistically as, "There is something that has the same speed in any inertial frame." It just so happens that in our universe this thing is what allows us to see.

To go a step further, it's not even necessary for there to exist something that has the same speed in any frame. The theory will still come out the same if we write the second postulate as, "There is a limiting speed of an object in any frame." (See Section 2.7 for a discussion of this.) There is no need to have something that actually travels at this speed. It's conceivable to have a theory that contains no massless objects, in which case everything travels slower than the limiting speed. ♣

Let's now see what we can deduce from the above two postulates. There are many different ways to arrive at the various kinematical consequences. Our road map for the initial part of the journey (through Section 2.2) is shown in Fig. 1.8. Additional kinematics topics are covered in Sections 2.3 through 2.7.

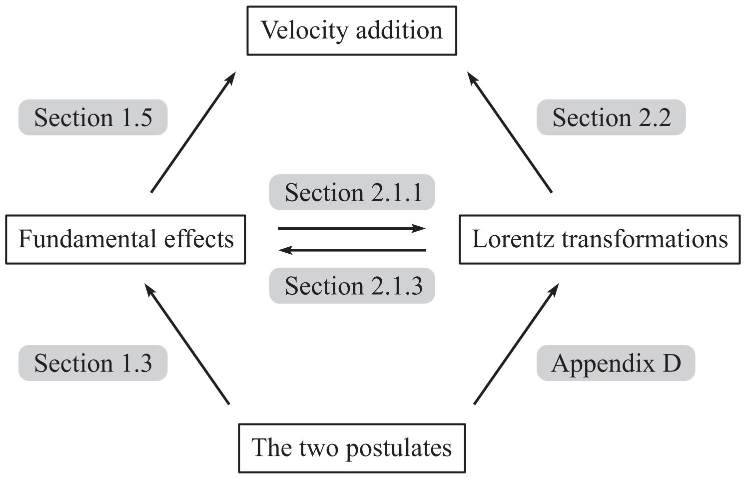

Figure 1.8

1.3 The fundamental effects

The most striking effects of the two postulates of relativity are (1) the *loss of simultaneity* (equivalently, the *rear-clock-ahead* effect), (2) *time dilation*, and (3) *length contraction*. In this section, we'll discuss these three effects by using some time-honored concrete setups. In Chapter 2, we'll use these three effects to derive the Lorentz transformations.

1.3.1 Loss of simultaneity

The basic effect

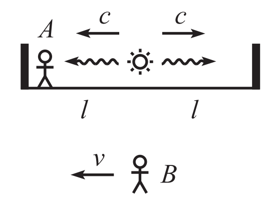

Figure 1.9

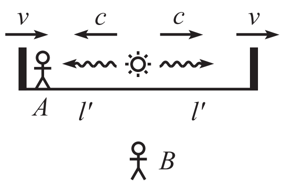

Figure 1.10

Consider the following setup. In person A's reference frame, a light source is placed midway between two receivers, a distance ℓ from each (see Fig. 1.9). The light source emits a flash. In A's reference frame, the light hits the two receivers at the same time, ℓ/c seconds after the flash. So if Event 1 is "light hitting the left receiver" and Event 2 is "light hitting the right receiver," then the two events are simultaneous in A's frame.

Now consider another observer, B, who travels to the left at speed v. In B's reference frame, does the light hit the receivers at the same time? That is, are Events 1 and 2 simultaneous in B's frame? We will show that surprisingly they are not.

In B's reference frame, the situation looks like that in Fig. 1.10. If you want, you can think of A as being on a train, and B as standing on the ground. With respect to B, the receivers (along with everything else in A's frame) move to the right with speed v. Additionally, with respect to B (and this is where the strangeness of relativity comes into play), the light travels in both directions at speed c, as indicated in the figure. Why is this the case? Because the speed-of-light postulate says so!

Note that everyday objects do *not* behave this way. Consider, for example, a train (A's frame) moving at 30 mph with respect to the ground (B's frame). If A stands in the middle of the train and throws two balls forward and backward, each with speed 50 mph *with respect to the train*, then the speeds of the two balls *with respect to the ground* are $50 - 30 = 20$ mph (backward) and $50 + 30 = 80$ mph (forward). (We're ignoring the minuscule corrections from the velocity-addition formula discussed in Section 1.5.)

These two speeds are *different*. In contrast with everyday objects like these balls, light has the bizarre property that its speed is always c (when viewed in an arbitrary inertial frame), independent of the speed of the source. Strange, but true.

Returning to our setup with the light beams and receivers, we can say that because B sees both light beams move with speed c, the *relative* speed (as viewed by B) of the light and the left receiver is $c + v$, and the *relative* speed (as viewed by B) of the light and the right receiver is $c - v$.

Remark: Yes, it is legal to simply add or subtract these speeds to obtain the relative speeds *as viewed by* B. The reasoning here is the same as in the discussion of Fig. 1.3 in Section 1.1.2, where we obtained relative speeds of $v_s \pm v_p$. As a concrete example, if the v here equals $2 \cdot 10^8$ m/s, then in one second the left receiver moves $2 \cdot 10^8$ m to the right, while the left ray of light moves $3 \cdot 10^8$ m to the left. This means that they are now $5 \cdot 10^8$ m closer than they were a second ago. In other words, the relative speed (as measured by B) is $5 \cdot 10^8$ m/s, which is $c + v$ here. This is the rate at which the gap between the light and the left receiver closes. So in addition to calling it the "relative" speed, you can also call it the "gap-closing" speed. Note that the above reasoning implies that it is perfectly legal for the relative speed of two things, as measured by a third, to take any value up to $2c$.

Likewise, the relative speed between the light and the right receiver is $c - v$. The v and c in these results are measured with respect to the *same* person, namely B, so our intuition involving simple addition and subtraction works fine. Even though we're dealing with a relativistic speed v here, we don't need to use the velocity-addition formula from Section 1.5, which is relevant in a different setting. This remark is included just in case you've seen the velocity-addition formula and think it's relevant in this setup. But if it didn't occur to you, then never mind.

Note that the speed of the right photon[4] is *not* $c - v$. (And likewise the speed of the left photon is not $c + v$.) The photon moves at speed c, as always. It is the *relative* speed (as measured by B) of the photon and the front of the train that is $c - v$. No *thing* is actually moving with this speed in our setup. This speed is just the rate at which the gap closes. And a gap isn't an actual moving thing. ♣

Let ℓ' be the distance from the light source to each of the receivers, as measured by B.[5] Then in B's frame, the gap between the light beam and the left receiver starts with length ℓ' and subsequently decreases at a rate $c + v$. The time for the light to hit the left receiver is therefore $\ell'/(c + v)$. Similar reasoning holds for the right receiver along with the relative speed of $c - v$. The times t_L and t_R at which the light hits the left and right receivers are therefore given by

$$t_L = \frac{\ell'}{c + v} \qquad \text{and} \qquad t_R = \frac{\ell'}{c - v}\,. \tag{1.6}$$

These two times are not equal if $v \neq 0$. (The one exception is when $\ell' = 0$, in which case the two events happen at the same place and same time in all frames.) Since $t_L < t_R$, we see that in B's frame, the light hits the left receiver before it hits the right receiver. We have therefore arrived at the desired conclusion that the two events (light hitting back, and light hitting front) are *not* simultaneous in B's frame.

The moral of this is that it makes no sense to say that one event happens at the same time as another, unless you also state which frame you're dealing with. Simultaneity depends on the frame in which the observations are made.

[4]Photons are what light is made of. So "speed of the photon" means the same thing as "speed of the light beam." Sometimes it's easier to talk in terms of photons.

[5]We'll see in Section 1.3.3 that ℓ' is not equal to the ℓ in A's frame, due to length contraction. But this won't be important for what we're doing here. The only fact we need for now is that the light source is equidistant from the receivers, as measured by B. This is true because space is homogeneous, which implies that any length-contraction factor we eventually arrive at must be the same everywhere. More on this in Section 1.3.3.

Of the many effects, miscellaneous,
The loss of events, simultaneous,
Allows B to claim
There's no pause in A's frame,

Remarks:

1. The strangeness of the $t_L < t_R$ loss-of-simultaneity result can be traced to the strangeness of the speed-of-light postulate. We entered the bizarre world of relativity when we wrote the c's above the photons in Fig. 1.10. The $t_L < t_R$ result is a direct consequence of the nonintuitive fact that light moves with the same speed c in *every* inertial frame.
2. The invariance of the speed of light led us to the fact that the relative speeds between the photons and the left and right receivers are $c + v$ and $c - v$. If we were talking about baseballs instead of light beams, then the relative speeds wouldn't take these general forms. If v_b is the speed at which the baseballs are thrown in A's frame, then as we noted above in the case where $v = 30$ mph and $v_b = 50$ mph, B sees the balls move with speeds of (essentially, ignoring the tiny corrections due to the velocity-addition formula) $v_b - v$ to the left (assuming $v_b > v$) and $v_b + v$ to the right. These are not equal (in contrast with what happens with light). By the same "gap-closing" reasoning we used above, the relative speeds (as viewed by B) between the balls and the left and right receivers are then $(v_b - v) + v = v_b$ and $(v_b + v) - v = v_b$. These are equal, so B sees the balls hit the receivers at the same time, as we know very well from everyday experience.
3. As explained in the remark prior to Eq. (1.6), it is indeed legal to obtain the times in Eq. (1.6) by simply dividing ℓ' by the relative speeds, $c + v$ and $c - v$. The gaps start with length ℓ' and then decrease at these rates. But if you don't trust this, you can use the following reasoning. In B's frame, the position of the right photon (relative to the initial position of the light source) simply equals ct, and the position of the right receiver (which has a head start of ℓ') equals $\ell' + vt$. The photon hits the receiver when these two positions are equal. Equating them gives

$$ct = \ell' + vt \quad \Longrightarrow \quad t_R = \frac{\ell'}{c - v}. \tag{1.7}$$

   Similar reasoning with the left photon gives $t_L = \ell'/(c + v)$.
4. There is always a difference between the time that an event *happens* and the time that someone *sees* the event happen, because light takes time to travel from the event to the observer. What we calculated above were the times t_L and t_R at which the events *actually happen* in B's frame. (These times are independent of *where* B is standing at rest in the frame.) If we wanted to, we could calculate the times at which B *sees* the events occur. (These times *do* depend on where B is standing at rest in the frame.) But such times are rarely important, so in general we won't be concerned with them. They can easily be calculated by adding on a (distance)$/c$ time for the photons to travel to B's eye. Of course, if B actually did the above experiment to find t_L and t_R, she would do it by writing down the times at which she sees the events occur, and then subtracting off the relevant (distance)$/c$ times, to find when the events actually happened.

   To sum up, the $t_L \neq t_R$ result in Eq. (1.6) is due to the fact that the events truly occur at different times in B's frame. The $t_L \neq t_R$ result *has nothing to do with the time it takes light to travel to B's eye.* ♣

Where this last line is not so extraneous.

The "rear clock ahead" effect

We showed in Eq. (1.6) that t_L is not equal to t_R, that is, the light hits the receivers at different times in B's frame. Let's now be quantitative and determine the degree to which two events that are simultaneous in one frame are not simultaneous in another frame.

Given the times t_L and t_R that we found in Eq. (1.6), the simplest quantitative number that we can produce, as a measure of the non-simultaneity, is the difference $t_\mathrm{R} - t_\mathrm{L}$. This tells us how *un*simultaneous the events are in B's frame (the ground frame), given that they are simultaneous in A's frame (the train frame). The interpretation of the resulting expression for $t_\mathrm{R} - t_\mathrm{L}$ is the task of Problem 1.1 (which relies on time dilation and length contraction, discussed below). But let's take a slightly different route here, which will end up being a little more useful. This route will lead us to the *rear-clock-ahead* effect, which is the standard quantitative statement of the loss of simultaneity.

Consider a setup where two clocks are positioned at the ends of a train of length L (as measured in its own frame). The clocks are synchronized in the train frame. That is, they have the same reading at any given instant, as observed in the train frame, as you would naturally expect. (Throughout this book, we will always assume that clocks are synchronized in the frame in which they are at rest.) The train travels past you at speed v. It turns out that if you observe the clocks at simultaneous times in *your* frame, the readings will *not* be the same. You will observe the rear clock showing a higher reading than the front clock, as indicated in Fig. 1.11.

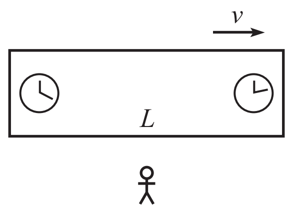

Figure 1.11

We'll explain why this is true momentarily, but first let us note that a nonzero difference in the readings is certainly a manifestation of the loss of simultaneity. To see why, consider a given instant in your (ground) frame when the rear and front clocks read, say, 12:01 and 12:00. (We'll find that the actual difference depends on L and v, but let's just assume it's one minute here, for concreteness.) Assume that you hit both clocks simultaneously (in your ground frame) with paintballs when they show these readings. Then in the train frame, the front clock gets hit when it reads 12:00, and then a minute later the rear clock gets hit when it reads 12:01. The simultaneous hits in your frame are therefore *not* simultaneous in the train frame. We have used the fact that the reading on a clock when a paintball hits it is frame independent. This is true because you can imagine that a clock breaks when a ball hits it, so that it remains stuck at a certain value. Everyone has to agree on what this value is.

Let's now find the exact difference in the readings on the two train clocks in Fig. 1.11. To do this, we will (as we did above in Fig. 1.9) put a light source on the train. But we'll now position it so that the light hits the clocks at the ends of the train at the same time in *your (ground) frame.* As in the discussion of Fig. 1.10, the relative speeds of the photons and the clocks are $c + v$ and $c - v$ (as viewed in your frame). We therefore need to divide the train into lengths in this ratio, in your frame, if we want the light to hit the ends at the same time. Now, because length contraction (discussed below in Section 1.3.3) is independent of position, the ratio in the train frame must also be $c + v$ to $c - v$. You can then quickly show that two numbers that are in this ratio, and that add up to L, are $L(c + v)/2c$ and $L(c - v)/2c$. (Mathematically, you're solving the system of equations, $x/y = (c + v)/(c - v)$ and $x + y = L$.) Dividing the train into these two lengths (in the train frame, as shown in Fig. 1.12) causes the light to hit the ends of the train simultaneously in the *ground* frame.

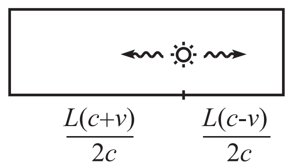

Figure 1.12

Let's now examine what happens during the process in the *train* frame. Compared with the forward-moving light, Fig. 1.12 tells us that the backward-moving light must travel an extra distance of $L(c+v)/2c - L(c-v)/2c = Lv/c$. The light travels at speed c (as always), so the extra time is Lv/c^2. The rear clock therefore reads Lv/c^2 more when it is hit by the backward photon, compared with what the front clock reads when it is hit by the forward photon. (Remember that the clocks are synchronized in the train frame.) This difference in readings has a frame-independent value, because the readings on the clocks when the photons hit them are frame independent, by the same reasoning as with the paintballs above.

Finally, let's switch back to your (ground) frame. Let the instant you look at the clocks be the instant the photons hit them. (That's why we constructed the setup with the hittings being simultaneous in *your* frame.) Then from the previous paragraph, we conclude that you observe the rear clock reading more than the front clock by an amount Lv/c^2:

$$\text{The rear clock is ahead by } Lv/c^2. \qquad (1.8)$$

This result is important enough to spell out in full and put in a box:

> REAR-CLOCK-AHEAD: If a train with length L moves with speed v relative to you, then you observe the rear clock reading Lv/c^2 more than the front clock, at any given instant.

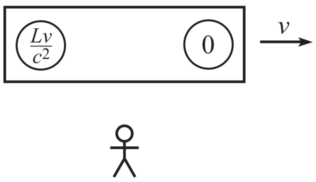

Figure 1.13

This statement corresponds to Fig. 1.13. For concreteness, we have chosen the front clock to read zero. But if the front clock reads, say, 9:47, then the rear clock reads 9:47 plus Lv/c^2. There is of course no need to have an actual train in the setup. In general, all we need are two clocks separated by some distance L and moving with the same speed v. But we'll often talk in terms of trains, since they're easy to visualize.

Note that the L in the Lv/c^2 result is the length of the train *in its own frame*, and not the shortened length that you observe in your frame (see Section 1.3.3). Appendix B gives a number of other derivations of Eq. (1.8), although they rely on material we haven't covered yet.

Example (Clapping first): Two people stand a distance L apart along an east-west road. They clap simultaneously in the ground frame. In the frame of a car driving eastward along the road, which person claps first?

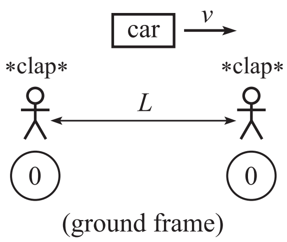

Figure 1.14

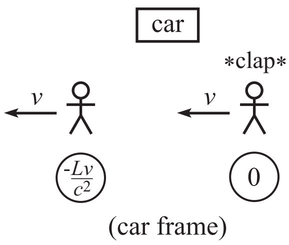

Figure 1.15

Solution: The eastern person claps first, for the following reason. Without loss of generality, let's assume that clocks on the two people read zero when the claps happen. Then a snapshot in the ground frame at the instant the claps happen is shown in Fig. 1.14. We've drawn the car in the middle as it travels past, but its exact location is irrelevant.

Now consider a snapshot in the car frame at the instant the eastern (right) person claps. This person's clock reads zero when he claps, because that is a frame-independent fact. Now, we can imagine that the two people are on a westward-traveling train, which means that the western person is the front person. By the rear-clock-ahead effect, the western (left) person's clock is *behind* by Lv/c^2. So it reads only $-Lv/c^2$, as shown in Fig. 1.15. (As we will see many times throughout this chapter, drawing pictures is extremely helpful when solving relativity problems!) Since this clock hasn't hit zero, the western person hasn't clapped yet. The eastern person therefore claps first, as we claimed.

REMARK: In the car frame, the distance between the two people is actually less than L (as we have indicated in Fig. 1.15), due to the length-contraction result we'll derive in Section 1.3.3. But this doesn't affect the result that the eastern person claps first. Similarly, the time-dilation result that we'll derive in Section 1.3.2 is relevant if we want to determine exactly how long the car observer needs to wait for the western person to clap. (It will turn out to be longer than Lv/c^2.) We'll talk about these matters shortly. ♣

REMARKS:

1. The Lv/c^2 result has nothing to do with the fact that the rear clock passes you at a later time than the front clock passes you. The train could already be past you, or it could even be moving directly toward or away from you. The rear clock will still be ahead by Lv/c^2, as observed in your frame.

2. The Lv/c^2 result does *not* say that you see the rear clock ticking at a faster *rate* than the front clock. They run at the same rate. (They both have the same time-dilation factor relative to you; see Section 1.3.2.) The rear clock is simply always a fixed time ahead of the front clock, as observed in your frame.
3. In the train setup (with the off-centered light source) that led to Eq. (1.8), the fact that the rear clock is *ahead* of the front clock in the ground frame means that in the train frame the light hits the rear clock *after* it hits the front clock.
4. The L in Eq. (1.8) is the separation between the clocks in the *longitudinal* direction, that is, the direction of the velocity of the train (or more generally, the velocity of the clocks, if we don't have a train). The height in the train doesn't matter; all clocks along a given line perpendicular to the train's velocity have the same reading at any given instant in the ground frame.
5. For everyday speeds v, the Lv/c^2 effect is extremely small. If $v = 30\,\text{m/s}$ (about 67 mph) and if $L = 100\,\text{m}$, then $Lv/c^2 \approx 3 \cdot 10^{-14}\,\text{s}$. This is completely negligible on an everyday scale.
6. What if we have a train that doesn't contain the above setup with a light source and two light beams? That is, what if the given events have nothing to do with light? The Lv/c^2 result still holds, because we *could* have built the light setup if we wanted to (arranging for the light-hitting-end events to coincide with the given events). It doesn't matter if the light setup actually exists.
7. It's easy to forget which of the clocks is the one that is ahead. But a helpful mnemonic for remembering "rear clock ahead" is that both the first and fourth letters in each word form the same acronym, "rca," which is an anagram for "car," which is sort of like a train. Sure. ♣

1.3.2 Time dilation

We showed above that if two clocks are separated by a distance L in the horizontal (that is, longitudinal) direction on a train, and if the train is moving with respect to you, then you observe different readings on the clocks, at any given instant in your frame. Note that this result relates the readings on two *different* clocks at a given *instant* in your frame. It says nothing about the *rate* at which a *single* clock runs in your frame. This is what we will now address.

We will demonstrate that if a given clock is moving with respect to you, then you will observe the clock running slowly. That is, if you use a stopwatch to measure how long it takes a given train clock to tick off 10 seconds, your stopwatch might read 20 seconds. The exact time on your watch depends on the speed v of the train, as we'll see shortly. But in any case, your clock will always read *more* than 10 seconds in this setup. (Or it will read exactly 10 seconds if $v = 0$ and the train is sitting at rest.) This effect is called *time dilation*. The name is appropriate, because the word "dilate" means to become larger. Since the moving clock runs slow (as viewed by you), a time T that ticks off on it takes *more* than a time T on your watch.

We will derive this time-dilation result by presenting a classic example of a light beam traveling in the vertical (that is, transverse) direction on a train. Let there be a light source on the floor of the train, and let there be a mirror on the ceiling, which is a height h above the floor. Let observer A be at rest on the train, and let observer B be at rest on the ground. The speed of the train with respect to the ground is v.[6] A flash of light is emitted upward. The light travels up to the mirror, bounces off it, and then

[6]Technically, the words "with respect to . . . " should always be included when talking about speeds, because there is no absolute reference frame, and hence no absolute speed. But in the future, when it is clear what we mean (as in the case of a train moving with respect to the ground), we'll occasionally be sloppy and drop the "with respect to . . . ."

heads back down. Assume that right after the light is emitted, we replace the source with a mirror on the floor, so that the light keeps bouncing up and down indefinitely.

In A's frame, the train is at rest, so the path of the light is simple. It just goes straight up and straight down, as shown in Fig. 1.16. The light travels at speed c, so it takes a time of h/c to reach the ceiling and then a time of h/c to return to the floor. The roundtrip time in A's frame is therefore

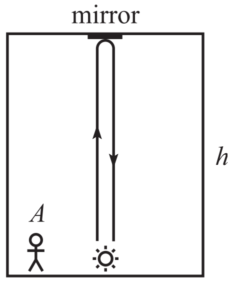

Figure 1.16

$$t_A = \frac{2h}{c} \, . \tag{1.9}$$

There's nothing fancy going on here. All we have used is the fact that rate times time equals distance.

Now consider the setup in B's frame, where the train moves at speed v. In this frame, the path of the light is diagonal, as shown in Fig. 1.17. It is indeed diagonal, because in addition to moving upward, the light also gets carried rightward along with the train. You can imagine that the light travels up and down a vertical tube on the train. Since the light remains in the tube in the train frame (let's imagine that it's a laser beam that doesn't spread out), it also remains in the tube in the ground frame. ("The light remains in the tube" is a frame-independent statement. We could have a setup where a paint bomb explodes if the light touches the side of the tube. All observers must agree on whether the train is covered in paint.) Therefore, since in the ground frame the tube moves rightward along with the train, the light must also move rightward.

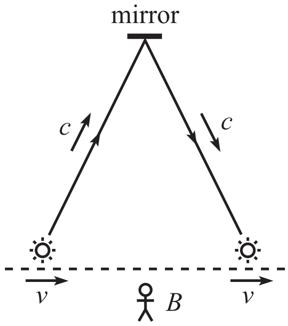

Figure 1.17

The crucial fact that we will now invoke is that the speed of light in B's frame is still c. Why? Because the speed-of-light postulate says so! The light therefore travels along its diagonally upward path in Fig. 1.17 at speed c. Since the horizontal component of the light's velocity is v, the vertical component must be $\sqrt{c^2 - v^2}$, as shown in Fig. 1.18. The horizontal component is in fact v, because the light always remains in the hypothetical vertical tube mentioned above, and this tube moves horizontally with speed v.

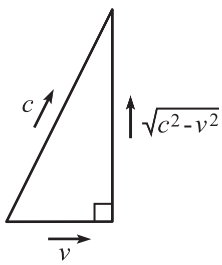

Figure 1.18

The Pythagorean theorem is indeed valid in Fig. 1.18, because it is valid for distances, and because speeds are just distances divided by time. Note that the vertical component of the light's velocity is *not* c, as would be the case if light behaved like a baseball. If you throw a baseball with speed v_b vertically on train, then its velocity with respect to the ground is shown in Fig. 1.19; the vertical speed remains v_b. The difference between this picture and the one in Fig. 1.18 is that for a baseball, the *vertical* speed is the thing that remains the same when shifting to the ground frame, and this leads to a larger *total* speed. But for light, it is the *total* speed that remains the same, and this leads to a smaller *vertical* speed. As with the c's in Fig. 1.10, we entered the bizarre world of relativity when we wrote the c's next to the diagonal paths in Fig. 1.17.

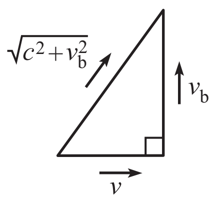

Figure 1.19

Having established that the vertical speed in B's frame is $\sqrt{c^2 - v^2}$, it follows that the time it takes the light to travel upward a height h to reach the mirror is $h/\sqrt{c^2 - v^2}$. Likewise for the downward trip. The roundtrip time is therefore

$$t_B = \frac{2h}{\sqrt{c^2 - v^2}} \, . \tag{1.10}$$

In this reasoning, we have assumed that the height of the train in B's frame is still h. Although we'll see in Section 1.3.3 that there is length contraction along the direction of motion, there is none in the direction perpendicular to the motion; we'll show this at the end of Section 1.3.3. So the height is indeed still h in B's frame.

Dividing Eq. (1.10) by Eq. (1.9) gives

$$\frac{t_B}{t_A} = \frac{c}{\sqrt{c^2 - v^2}} = \frac{1}{\sqrt{1 - v^2/c^2}} \, . \tag{1.11}$$

If we define γ by

$$\boxed{\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}} \tag{1.12}$$

then we arrive at

$$\boxed{t_B = \gamma t_A} \qquad \text{(time dilation)} \tag{1.13}$$

The γ factor here is ubiquitous in special relativity. We'll occasionally add a subscript with the associated velocity v (like γ_v) if a setup involves more than one velocity (and hence γ factor), to avoid any confusion. In these cases we'll sometimes just use whatever fraction of c the speed is, as the subscript. For example, the γ factor associated with the speed $c/2$ is $\gamma_{1/2} = 2/\sqrt{3}$.

Note that γ is always greater than or equal to 1. This means that the roundtrip time in the above setup is *longer* in B's frame than in A's frame. The one exception occurs when $v = 0 \implies \gamma = 1$, in which case the two times are equal. But in this case A is at rest with respect to B, so they are both in the same frame, which isn't very interesting.

What are the implications of Eq. (1.13)? For concreteness, let $v/c = 3/5$, which yields $\gamma = 5/4$. (The numbers work out nicely here, because 3-4-5 is a Pythagorean triple.) We may then say the following. If A is standing at rest on the train next to the light source, and if B is standing on the ground, and if A claps his hands at $t_A = 4$ second intervals according to his watch, then B observes A's claps happening at $t_B = 5$ second intervals according to her watch. (As usual, it is understood that B subtracts off the time it takes the light to travel to her eye, to determine when the claps actually happen in her frame.) This is true because both A and B must agree on the number of roundtrips the light beam completes between claps. If we assume, for convenience, that a roundtrip takes one second in A's frame (yes, that would be a tall train), then Eq. (1.13) tells us that the four roundtrips between successive claps take five seconds in B's frame. B therefore sees A's clock running slow, by a factor $4/5$.

We just made the claim that both A and B must agree on the number of roundtrips between successive claps. However, since A and B disagree on so many things (whether two events are simultaneous, the rate at which clocks tick, and the length of things, as we'll see below), you might be wondering if there's *anything* they agree on. Yes, there are still frame-independent statements we can hang on to. We noted on page 15 that the reading on a clock when a paintball hits it is frame independent. As another example, if a bucket of paint flies past you and dumps paint on your head, then everyone agrees that you are covered with paint. Likewise, if A is standing next to the light clock and claps when the light reaches the floor, then everyone agrees on this. If the light is actually a strong laser pulse, and if A's clapping motion happens to bring his hands over the mirror right when the pulse gets there, then everyone agrees that his hands get burned by the laser.

What if we have a train that doesn't contain one of our special light clocks? It doesn't matter. We *could* have built one if we wanted to, so the same results concerning the claps must still hold. Therefore, light clock or no light clock, B observes A moving strangely slowly. From B's point of view, A's heart beats slowly, his blinks are lethargic, and his sips of coffee are slow enough to suggest that he needs another cup.

> The effects of dilation of time
> Are magical, strange, and sublime.
> In your frame, this verse,
> Which you'll see is not terse,
> Can be read in the same amount of time it takes someone
> else in another frame to read a similar sort of rhyme.

Our assumption that A is at rest on the train was critical in the above derivation. If A is moving with respect to the train, then Eq. (1.13) doesn't hold, because we *cannot* say that both A and B must agree on the number of roundtrips the light beam takes between claps, because there is now an issue with simultaneity. More precisely, if A is at rest on the train right next to the light source, then there is no issue with simultaneity, because the distance L in Eq. (1.8) is zero. And if A is at rest at a fixed distance from the source, then consider a person A' at rest on the train right next to the source. The distance L between A and A' is nonzero, so from the rear-clock-ahead effect, B sees their two clocks differ by Lv/c^2. But this difference is *constant*, so B sees A's clock tick at the same rate as A''s clock. Basically, since A and A' represent the same reference frame, there is again no issue with simultaneity. (More precisely, there *is* a loss of simultaneity, but it has no consequence here, because it is constant.) Equivalently, we can just build a second light clock next to A, and it will have the same speed v (and thus yield the same γ factor) as the original clock.

However, if A is moving with respect to the train, then we have a problem. If A' is again at rest on the train next to the source, then the distance L between A and A' is *changing*, so B can't use the reasoning in the previous paragraph to conclude that A's and A''s clocks tick at the same rate. And in fact they do not, because as above, we can build another light clock and have A hold it. In this case, A's speed is what goes into the γ factor in Eq. (1.12), and this speed is different from A''s speed (which is the speed of the train).

Remarks:

1. The speed v needs to be fairly large in order for the γ factor in Eq. (1.12) to differ appreciably from 1. If $v = c/10$ (which is still quite fast), we only have $\gamma_{1/10} \approx 1.005$. A few other values are: $\gamma_{1/2} \approx 1.15$, $\gamma_{9/10} \approx 2.3$, and $\gamma_{99/100} \approx 7$.

2. The time-dilation result in Eq. (1.13) is a bit strange, no doubt, but there doesn't seem to be anything downright incorrect about it until we look at the situation from A's point of view. A sees B flying by at a speed v in the other direction. The ground frame is no more fundamental than the train frame, so the same reasoning we used above also applies to A's frame. Equivalently, we can just switch all the A and B labels in the above derivation. The time-dilation factor, γ, doesn't depend on the sign of v, so A sees the same time-dilation factor that B sees. That is, A sees B's clock running slow. But how can this be? Are we claiming that A's clock is slower than B's, and also that B's clock is slower than A's? Well . . . yes and no.

   Remember that the above time-dilation reasoning applies only to a situation where something is motionless in the appropriate frame. In the second situation (where A sees B flying by), the statement $t_A = \gamma t_B$ holds only for two events (say, two ticks on B's clock) that happen at the same place in B's frame. But two such events are certainly not at the same place in A's frame, so the $t_B = \gamma t_A$ result in Eq. (1.13) does *not* hold. The conditions of being motionless in each frame can never both hold in a given setup (unless $v = 0$, in which case $\gamma = 1$ and $t_A = t_B$). So the answer to the question at the end of the previous paragraph is "yes" if you ask the questions in the appropriate frames, and "no" if you think the answer should be frame independent.

3. Concerning the fact that A sees B's clock run slow, *and* B sees A's clock run slow, consider the following statement. "This is a contradiction. It is essentially the same as saying, 'I have two apples on a table. The left one is bigger than the right one, and the right one is bigger than the left one.' " How would you respond to this statement?

   Well, it is not a contradiction. Observers A and B are using *different coordinates* to measure time. The times measured in each of their frames are quite different things. The seemingly contradictory time-dilation result is really no stranger than having two people run away from each other into the distance, and having them both say that the other person looks smaller. In short, we are not comparing apples and apples. We are comparing apples and oranges. A more correct analogy would be the following. An apple and an orange sit on a table. The apple says to the orange, "You are a much uglier apple than I am," and the orange says to the apple, "You are a much uglier orange than I am."

4. One might view the statement, "A sees B's clock running slow, and also B sees A's clock running slow," as somewhat unsettling. But in fact it would be a complete disaster for the theory if A and B viewed each other in different ways. A critical ingredient in the theory of relativity is that A sees B in exactly the same way that B sees A.
5. In everything we've done so far, we've assumed that A and B are in inertial frames, because these are the frames that the postulates of special relativity deal with. However, it turns out that the time-dilation result in Eq. (1.13) holds even if A is accelerating, as long as B isn't. In other words, if you are looking at a clock that is undergoing a complicated accelerated motion, then to figure out how fast it is ticking in your frame at a given instant, all you need to know is its speed at that instant; its acceleration is irrelevant. (This has plenty of experimental verification. Perhaps the quickest theoretical argument involves using a Minkowski diagram; see Section 2.4 and the third remark in the solution to Problem 2.12.) If, however, *you* are accelerating, then all bets are off, and it isn't valid for you to use the time-dilation result when looking at a clock. But it's still possible to get a handle on such situations, as we'll see in Chapter 5. Problem 2.12 also deals with this issue. ♣

In the second remark above, we noted that the time-dilation result in Eq. (1.13) holds in setups where two events happen at the same place in one of the frames. The γ factor in Eq. (1.13) appears on the side of the equation associated with the frame in which the two events happen at the same place. An equivalent (and simpler) way of stating how to properly use time dilation is: *If you look at a moving clock, you observe it running slowly.* This has the "happen at the same place" requirement built into it, because two ticks on a clock certainly happen at the same place in the clock's frame, since they both happen at the clock. (Well, the hands on an analog clock necessarily move a little bit, but let's ignore that.)

To summarize, we can state the time-dilation result with this equation:

$$\boxed{t_{\text{observed}} = \gamma t_{\text{proper}}} \qquad \text{(time dilation)} \tag{1.14}$$

The *proper time* is the time that elapses between two ticks on a given clock in the frame of the clock. (More generally, the proper time is the time between two events, as measured in the frame where the events happen at the same place.) The observed time in Eq. (1.14) is the time that elapses between the two ticks, as measured in the frame of an observer who is looking at the clock. We can also state the time-dilation result with these words:

> Time dilation: If you look at a clock moving with speed v relative to you, then you observe the clock running slowly by a factor $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$.

When relating the times in different frames, it is easy to get confused about where to put the γ factor. The safest way to proceed is to (1) start with the fact that if you look at a moving clock, it runs slow, then (2) identify which time is larger/smaller, and then (3) put the γ factor (which is always larger than 1) where it needs to be so that the relative size of the times is correct. For example, let's say that during a time T that elapses on your watch, you want to determine how much time elapses on a clock that is flying past you. Since you see the clock running slow, the time on it must be less than T, which means that the answer is T/γ; we must divide by γ.

A common trap to fall into when applying time dilation is the following. (Every physics student is bound to make this error at least once.) Let's say that at a given moment, you look at a clock flying by (say, a clock at the back of a train). And then a little later you look at a *different* clock (say, a clock at the front of the train). You take the difference in the readings and then multiply this difference by γ (because you see the train clocks run slow), to find the time elapsed on your watch in the ground frame. This strategy is incorrect, because it uses the readings on two *different* clocks. (The Lv/c^2 rear-clock-ahead result is what messes things up. This is evident in the examples in Section 1.4.) To apply time dilation correctly, you must take the difference in the readings on a *single* clock. Remember, all that time dilation says is, "If you look at a moving clock, you observe it running slowly." The word "clock" here is singular.

Note well that it is *elapsed times* that get dilated, and not *readings* on clocks. If you look at a clock on a moving train and observe that it has a reading of t_1, then there isn't much you can do with that. But if you then look at the same clock later on and observe that it has a reading of t_2, then you can say something. You can say that the time that elapses on your own clock between your two observations (during which a time of $t_2 - t_1$ elapses on the train clock) equals $\gamma(t_2 - t_1)$, because you see the train clock run slow. Since time dilation deals only with elapsed times and not with actual readings, we should technically be writing Δt's instead of t's in Eq. (1.14), that is,

$$\Delta t_{\text{observed}} = \gamma\,\Delta t_{\text{proper}}. \tag{1.15}$$

But we'll usually drop the Δ's, for simplicity.

Let's now do two classic examples involving time dilation.

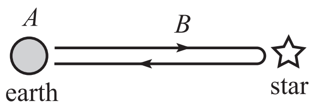

Figure 1.20

Example 1 (Twin paradox): Twin A stays on the earth, while twin B flies quickly to a distant star and back (see Fig. 1.20). After B returns, are the twins the same age? If not, who is younger?

Solution: From A's point of view, B's clock (and heartbeat, cell aging, and everything else) is running slow by a factor γ on both the outward and return parts of the trip. Therefore, B is younger than A when they meet up again. This is the answer, and that's that. So if getting the right answer is all we care about, then we can pack up and go home. But our reasoning leaves one large point unaddressed. The "paradox" part of this example's title comes from the following alternative reasoning. Someone might say that in B's frame, A's clock is running slow by a factor γ, so A should be younger than B when they meet up again.

It's definitely true that when the two twins are standing next to each other after B's journey concludes (that is, when they are eventually in the same frame), we can't have B younger than A, and also A younger than B. So what is wrong with the reasoning at the end of the preceding paragraph? The error lies in the fact that B doesn't remain in a single inertial frame. Her inertial frame for the outward trip is different from her inertial frame for the return trip. The derivation of our time-dilation result requires a single inertial frame.

Said in a different way, B accelerates when she turns around, and our time-dilation result holds only from the point of view of an *inertial* observer. The symmetry in the problem is broken by the acceleration. If both A and B are blindfolded, they can still tell who is doing the traveling, because B will feel the acceleration at the turnaround. Constant velocity cannot be felt, but acceleration can be. (However, see Chapter 5 on general relativity. Gravity complicates things.) For the entire outward and return parts of the trip, B *does* observe A's clock running slow, but enough strangeness occurs during the turning-around period (from B's point of view) to make A end up older.

The above paragraphs show what is wrong with the "A is younger" reasoning, but they don't show how to modify it quantitatively to obtain the correct answer. There are many different ways of doing this, and you can tackle some of them in the problems; see Exercises 1.30 and 2.32, Problems 1.21 and 2.11, and various problems in Chapter 5. Also, Appendix C gives a list of all the possible resolutions to the twin paradox that I can think of, although some rely on material we haven't covered yet.

Example 2 (Muon decay): Elementary particles called *muons* (which are identical to electrons, except that they are about 200 times as massive, and they decay) are created in the upper atmosphere when cosmic rays (energetic protons, mostly) collide with air molecules. The muons have an average lifetime of about $2 \cdot 10^{-6}$ seconds (this is the proper lifetime, that is, the lifetime as measured in the frame of the muon). They then decay into other particles (electrons and neutrinos). The muons move at nearly the speed of light. Assume for simplicity that a particular muon is created at a height of 20 km, moves straight downward, has speed $v = (0.9999)c$, decays in exactly $T = 2 \cdot 10^{-6}$ seconds, and doesn't collide with anything on the way down.[7] Will the muon reach the earth before it (the muon!) decays?

Solution: The naive thing to say is that the distance traveled by the muon is $d = vT \approx (3 \cdot 10^8\,\text{m/s})(2 \cdot 10^{-6}\,\text{s}) = 600\,\text{m}$, and that this is less than 20 km, so the muon doesn't reach the earth. This reasoning is incorrect, because of time dilation. We must remember that in the earth frame the muon lives longer by a factor of γ, which is $\gamma = 1/\sqrt{1 - v^2/c^2} \approx 70$ here. (You can imagine that the muon has a little clock, and when the clock hits $T = 2 \cdot 10^{-6}$ seconds, the muon decays.) So the actual lifetime in the earth frame is $\gamma T = (70)(2 \cdot 10^{-6}\,\text{s}) = 1.4 \cdot 10^{-4}\,\text{s}$. The correct distance traveled in the earth frame is therefore $v(\gamma T)$. This is $\gamma \approx 70$ times the $vT \approx 600\,\text{m}$ distance we found above, so we end up with 40 km. Hence, the muon travels the 20 km, with room to spare. The real-life fact that we actually do detect muons reaching the surface of the earth in the predicted abundances is one of the many experimental tests that support special relativity. The naive $d = vT$ reasoning would predict that we shouldn't see any (or at most a very small number, if we had based our calculation on more realistic assumptions).

1.3.3 Length contraction

Having discussed the loss-of-simultaneity (rear-clock-ahead) and time-dilation effects, we now come to the third of the fundamental effects of special relativity, namely *length contraction*. We will derive this effect by again looking at how a light beam travels in a train, except that now we will shine the light in the longitudinal (horizontal) direction instead of the transverse (vertical) direction. There is actually a much quicker derivation of length contraction than the one we will give here; see the third remark below. But we'll work through the present derivation because the calculation is instructive.

Consider the following setup. Person A is at rest on a train that he measures to have length L_A, and person B is at rest on the ground. The train moves at speed v with respect to the ground. A light source is located at the back of the train, and a mirror is located at the front. The source emits a flash of light that heads to the mirror, bounces off, then heads back to the source. By looking at how long this process takes in each of the two reference frames, we can determine the length of the train as measured by B. In A's frame (see Fig. 1.21), the light travels a total distance of $2L_A$ at speed c, so the roundtrip time is simply

$$t_A = \frac{2L_A}{c}\,. \tag{1.16}$$

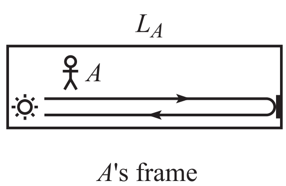

Figure 1.21

[7] In the real world, the muons are created at various heights, move in different directions, have different speeds, decay in lifetimes that vary according to a standard half-life formula, and may very well bump into air molecules. So technically we've got everything wrong here. But that's no matter. Our assumptions are good enough for the present purpose!

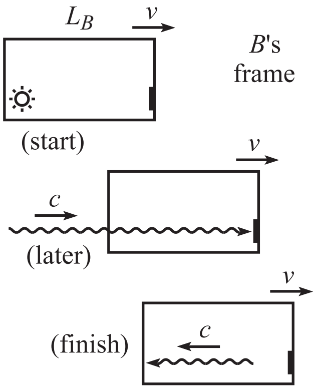

Figure 1.22

Things are a little more complicated in B's frame; see Fig. 1.22. Let the length of the train as measured by B be L_B. For all we know at this point, L_B might be equal to L_A, but we'll soon find that it is not. During the first part of the trip, the relative speed (as measured by B) of the light and the mirror at the front of the train is $c - v$. Since the initial gap between the light and the mirror is L_B, the time it takes to close this gap down to zero is $L_B/(c - v)$. This is the same type of reasoning we used on various occasions in Section 1.3.1.

Similarly, during the second part of the trip, the relative speed (as measured by B) of the light and the back of the train is $c + v$. Since the initial gap between the light and the back of the train is again L_B, the time it takes to close this gap down to zero is $L_B/(c + v)$. The total roundtrip time in B's frame is therefore

$$t_B = \frac{L_B}{c-v} + \frac{L_B}{c+v} = \frac{2cL_B}{c^2-v^2} = \frac{2L_B/c}{1-v^2/c^2} = \gamma^2\frac{2L_B}{c}. \tag{1.17}$$

But we also know from Eq. (1.13) that

$$t_B = \gamma t_A. \tag{1.18}$$

This is a valid statement, because the two events we are concerned with (light leaving back, and light returning to back) happen at the same place in the train frame (A's frame), so it is legal to use the time-dilation result in Eq. (1.13). The γ factor goes on the side of the equation associated with the frame in which the two events happen at the same place, which is A's frame here. Equivalently, just imagine a clock ticking at the back of the train; B sees this clock run slow.

Substituting the results for t_A and t_B from Eqs. (1.16) and (1.17) into Eq. (1.18), we find

$$\gamma^2\frac{2L_B}{c} = \gamma\cdot\frac{2L_A}{c} \quad\Longrightarrow\quad \boxed{L_B = \frac{L_A}{\gamma}} \qquad \text{(length contraction)} \tag{1.19}$$

Note that we could not have used this setup to derive length contraction if we had not already derived time dilation in Eq. (1.13).

Since $\gamma \geq 1$, we see that B measures the train to be shorter than A measures (or equal, if $v = 0$). The term *proper length* is used to describe the length of an object in its rest frame. So L_A is the proper length of the train, and the length L_B in any other frame is less than or equal to L_A. This length contraction is often called the *Lorentz–FitzGerald contraction,* for the reason given in Footnote 2.

Relativistic limericks have the attraction
Of being shrunk by a Lorentz contraction.
But for readers, unwary,
The results may be scary,
When a fraction . . .

Remarks:

1. The length-contraction result in Eq. (1.19) holds for lengths in the direction of the relative velocity between the two frames (the longitudinal direction). There is no length contraction in the perpendicular direction (the transverse direction), as we'll show at the end of this section.

2. As with time dilation, length contraction is a bit strange, but there doesn't seem to be anything actually paradoxical about it, until we look at things from A's point of view. To make a nice symmetrical situation, let's say B is standing on an identical train, which is at

rest with respect to the ground. Then A sees B flying by at speed v in the other direction. Neither train is any more fundamental than the other, so the same reasoning we used above also applies here. (Just switch all the A and B labels in the derivation.) We conclude that A sees the same length-contraction factor that B sees. That is, A measures B's train to be short. But how can this be? Are we claiming that A's train is shorter than B's, and also that B's train is shorter than A's? Is the actual setup the one shown in Fig. 1.23, or is it the one shown in Fig. 1.24? Well . . . it depends.

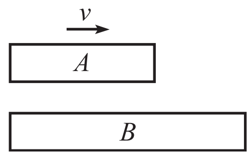

Figure 1.23

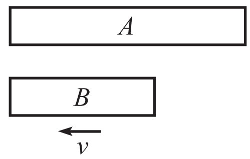

Figure 1.24

As with time dilation, it makes no sense to say what the length of a train really *is*. It makes sense only to say what the length is *in a given frame*. The situation doesn't really look like one thing in particular. The look depends on the frame in which the looking is being done.

Let's be a little more specific. How do you measure a length? You write down the position coordinates of the ends of something measured *simultaneously*, and then you take the difference between these coordinates. But the word "simultaneously" here should send up all sorts of red flags. Simultaneous events in one frame are not simultaneous in another. Stated more precisely, here is what we are claiming: Let B write down simultaneous coordinates of the ends of A's train, and also simultaneous coordinates of the ends of her own train. Then the difference between the former is smaller than the difference between the latter. Likewise, let A write down simultaneous coordinates of the ends of B's train, and also simultaneous coordinates of the ends of his own train. Then the difference between the former is smaller than the difference between the latter. There is no contradiction here, because the times at which A and B are writing down the coordinates don't have much to do with each other, due to the loss of simultaneity. We'll be quantitative about this in the second example in Section 1.4. As with time dilation, we are comparing apples and oranges.

3. As we mentioned at the beginning of this section, there is a quick argument that demonstrates why time dilation implies length contraction, and vice versa. Let A stand on the ground, next to a stick with (proper) length L. Let B fly past the stick at speed v; see Fig. 1.25. In A's frame, it simply takes B a time of L/v to traverse the length of the stick. Therefore (assuming that we have demonstrated the time-dilation result), since A sees B's clock run slow, a watch on B's wrist will advance by a time of only $L/\gamma v$ while he traverses the length of the stick.

   How does B view the situation? He sees A and the stick fly by at speed v. The time between the two ends passing him is $L/\gamma v$, because we found above that this is the time elapsed on his watch. (This is a frame-independent value. Imagine a switch that starts his watch when he coincides with one end of the stick, and stops it when he coincides with the other end.) To obtain the length of the stick in his frame, B simply multiplies the speed of the stick times the time. So he measures the length to be $(L/\gamma v)v = L/\gamma$. This is the desired length contraction. The same argument in reverse shows conversely that length contraction implies time dilation. In short, any theory that has one of these effects must have the other.

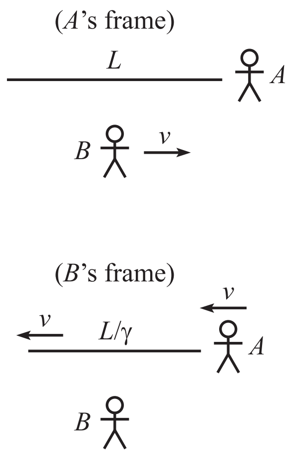

Figure 1.25

4. As mentioned in Footnote 5, the length-contraction factor γ is independent of the position on the train. That is, all parts of the train are contracted by the same factor. This follows from the fact that all points in space are equivalent. Equivalently, we could put a large number of small replicas of the above source-mirror system along the length of the train. They would all produce the same value of γ (because they all have the same v), independent of the position on the train.

5. If you still want to ask, "Is length contraction actually *real*?" then consider the following hypothetical undertaking. Imagine a sheet of paper moving sideways past the Mona Lisa, skimming the surface of the painting. A standard sheet of paper is plenty large enough to cover her face, so if the paper is moving slowly, and if you take a photograph at the appropriate time, then in the photo her entire face will be covered by the paper. However, if the sheet is flying by sufficiently fast, and if you take a photograph at the appropriate time, then in the photo you'll see a thin vertical strip of paper covering only a small fraction of her face. So you'll still see her smiling at you. ♣

To summarize, we can state the length-contraction result with this equation:

$$\boxed{L_{\text{observed}} = \frac{L_{\text{proper}}}{\gamma}} \qquad \text{(length contraction)} \tag{1.20}$$

As mentioned above, the proper length is the length measured in the frame of the stick (or whatever). The observed length is the length measured in any other frame. We can also state the length-contraction result with these words:

> LENGTH CONTRACTION: If you look at a stick moving longitudinally with speed v relative to you, then you observe the stick to be short by a factor $1/\gamma = \sqrt{1 - v^2/c^2}$.

As with time dilation, when relating distances in different frames, it is easy to get confused about where to put the γ factor. The safest way to proceed is to (1) start with the fact that if you look at a moving stick, it is short, then (2) identify which length is longer/shorter, and then (3) put the γ factor (which is always larger than 1) where it needs to be so that the relative size of the lengths is correct. For example, let's say that you want to determine the proper length of a moving train that has length L in your frame. Since you see the train as length contracted, its proper length must be longer than L (because that length is contracted down to the L that you observe). So the proper length is γL; we must multiply by γ.

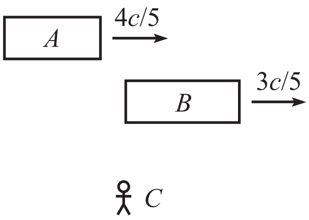

Figure 1.26

Example 1 (Passing trains): Two trains, A and B, each have proper length L and move in the same direction. A's speed is $4c/5$, and B's speed is $3c/5$. A starts behind B; see Fig. 1.26. How long, as measured by person C on the ground, does it take for A to overtake B? By this we mean the time between the front of A passing the back of B, and the back of A passing the front of B.

Solution: Relative to C on the ground, the γ factors associated with A and B are $5/3$ and $5/4$, respectively. Therefore, their lengths in the ground frame are $3L/5$ and $4L/5$. The overtaking begins when the back of A is a distance $7L/5$ (the sum of the lengths of the trains) behind the front B. The overtaking ends when the back of A reaches the front of B. So we need the initial gap of $7L/5$ to decrease to zero. The gap decreases at a rate of $c/5$ (the difference of the speeds in the ground frame). The overtaking therefore takes a time in the ground frame equal to

$$t_C = \frac{7L/5}{c/5} = \frac{7L}{c}\,. \tag{1.21}$$

Example 2 (Muon decay, again): Consider the "Muon decay" example in Section 1.3.2. From the muon's point of view, it lives for a time of $T = 2 \cdot 10^{-6}$ seconds, and the earth is speeding toward it at $v = (0.9999)c$. How, then, does the earth (which travels only $d = vT \approx 600\,\text{m}$ before the muon decays) reach the muon?

Solution: The important point here is that in the muon's frame, the distance to the earth is contracted by a factor $\gamma \approx 70$. Therefore, the earth starts only $(20\,\text{km})/70 \approx 300\,\text{m}$ away. (You can imagine that the muons are created next to the top of a hypothetical tower with height 20 km. This tower is at rest in the earth frame, so it is length contracted in the muon frame.) Since the earth can travel a distance of 600 m during the muon's lifetime, the earth collides with the muon, with room to spare.

As stated in the third remark above, time dilation and length contraction are intimately related. We can't have one without the other. In the earth's frame, we saw in the example in Section 1.3.2 that the muon's arrival at the earth is explained by time dilation. In the muon's frame, we just saw in the present example that the earth's arrival at the muon is explained by length contraction.

Observe that for muons created,
The dilation of time is related
To Einstein's insistence
Of shrunken-down distance
In the frame where decays aren't belated.

Example 3 (Two distances): In the ground frame, two people stand a distance L apart, and they clap simultaneously in the ground frame; see Fig. 1.27. A train moves to the right at speed v. In the train frame, what is the distance between the *people*, and what is the distance between the clapping *events*?

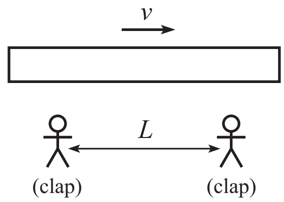

Figure 1.27

Solution: The train sees the proper distance L between the people as length contracted, so the people are only L/γ apart in the train frame.

Let the claps somehow make marks on the train. These marks are L apart in the ground frame, so they must be γL apart in the train frame, because from the ground's point of view, this distance is what is length contracted down to the distance L in the ground frame. The events are therefore γL apart in the train frame.

The non equality of the above two answers (L/γ and γL) is a consequence of the loss of simultaneity. In the train frame, as the people fly by to the left, the right person claps first. The people then travel leftward for some time before the left person claps. In the train frame, the distance between the events is therefore greater than the distance between the people. We'll be quantitative about this in the second example in Section 1.4, where we explain how the L/γ and γL distances are consistent with each other.

No transverse length contraction

We have mentioned a few times that there is no length contraction in the direction perpendicular to the relative velocity of two frames (that is, the transverse direction). We'll now show why this is true, with the following setup. Consider two meter sticks, A and B, that move past each other as shown in Fig. 1.28. Stick A has paint brushes on its ends. If the paint brushes touch B, they leave marks on B. We can use this setup to show that in the frame of one stick, the other stick still has a length of one meter.

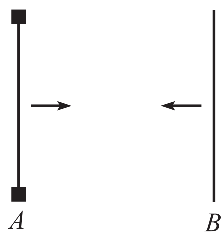

Figure 1.28

The key fact that we need to invoke is the first postulate of relativity, which says that all inertial frames are equivalent. In particular, the frames of the two sticks are equivalent. This implies that if A sees B shorter than (or longer than, or equal to) itself, then B *also* sees A shorter than (or longer than, or equal to) itself. The contraction factor must be the same when going each way between the frames. At first glance, this might seem backwards. After all, everyday life is full of statements such as, "If Alice is taller than Sue, then Sue is shorter than Alice," where the word "taller" is replaced with "smaller" (its opposite). But we're dealing with an entirely different thing here. We're talking about rules that hold in different reference frames. It would be a complete disaster for the theory if different frames had different rules. If the rule in one frame were "moving sticks are long" and the rule in another frame were "moving sticks are short," then this would be a violation of the fact that all inertial frames are equivalent.

Let's assume (in search of a contradiction) that A sees B shortened. Then B won't extend out to the ends of A, so there will be no paint marks on B; see the top picture

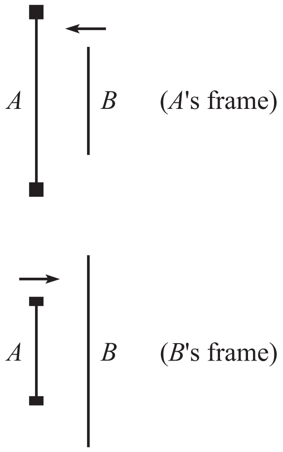

Figure 1.29

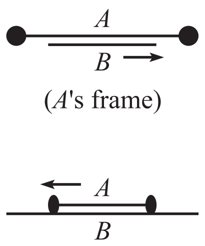

Figure 1.30

in Fig. 1.29. But in this case, B must *also* see A shortened, so there *will* be marks on B; see the bottom picture in Fig. 1.29. This is a contradiction, because the existence or nonexistence of marks on B is a frame-independent fact. Everyone must agree on whether or not there are marks on B. Likewise, if we assume that A sees B lengthened, we also reach a contradiction. We are therefore left with the only other possibility, namely that each stick sees the other stick as exactly one meter long. There is therefore no transverse length contraction.

Remark: Having used the above scenario to show that there is no *transverse* length contraction, you might wonder why we can't use the same kind of reasoning to show that there is no *longitudinal* length contraction. We had better *not* be able to use it, because we showed in Eq. (1.19) that longitudinal length contraction *does* exist. To see why we can't use the same kind of reasoning, first note that if the sticks are aligned longitudinally, then the paint brushes on A will each simply leave a streak along the entire length of B, independent of whether or not there is length contraction. This particular setup therefore can't be used to conclude anything about length contraction.

A possible improvement in the setup is to replace the paint brushes with paint bombs, so that they deposit paint only at a single instant. At first glance this seems to accomplish the task of producing a contradiction, because if the rule is that moving sticks are short, then the top picture in Fig. 1.30 shows no marks on B, whereas the bottom picture shows two marks. However, these two pictures don't describe the same scenario, because in the top picture the bombs explode simultaneously in A's frame, whereas in the bottom picture they explode simultaneously in B's frame. These two scenarios are inconsistent with each other, due to the loss of simultaneity. We can have one or the other picture, but not both. There is therefore no contradiction. Of course, this lack of a contradiction doesn't prove that longitudinal length contraction *does* in fact exist. But our goal here was only to show that the above type of reasoning can't be used to show that longitudinal length contraction *doesn't* exist. If you want to make some quantitative statements about how the loss of simultaneity relates to longitudinal length contraction, see the second example in Section 1.4 below.

In the above transverse case, there was no issue with simultaneity, because there was no extent in the longitudinal direction, which meant that the L in Lv/c^2 was zero. So we did in fact end up with a contradiction that ruled out transverse length contraction. ♣

1.3.4 A few other important topics

We have completed our treatment of the three fundamental effects, but let's discuss a few other important things before moving on.

Lattice of clocks and meter sticks

In everything we've done so far, we've taken the route of having observers sitting in various frames, making various measurements. But as mentioned earlier, this can cause some ambiguity, because you might think that the time when light reaches the observer is important, whereas what we are generally concerned with is the time when something actually happens.

A way to avoid this ambiguity is to remove the observers and define each frame by filling up space with a large rigid lattice of meter sticks and synchronized clocks, all at rest in the given frame. Different frames are defined by different lattices; assume that the lattices of different frames can somehow pass freely through each other. All of the meter sticks in a given frame are at rest with respect to all the others, so we don't have to worry about issues of length contraction within each frame. Likewise, we don't have to worry about time dilation within each frame. However, with respect to a given frame, the lattice of a different frame is squashed in the direction of its motion, because all the meter sticks in that direction are contracted. Likewise, all the clocks in the moving lattice run slow.

To measure the length of an object in a given frame, we just need to determine where the ends are (at simultaneous times, as measured in that frame) with respect to the lattice. As far as the synchronization of the clocks within each frame goes, this can be accomplished by putting a light source midway between any two clocks and sending out signals, and then setting the clocks to a certain value when the signals hit them. Alternatively, a more straightforward method of synchronization is to start with all the clocks synchronized right next to each other, and then move them very slowly to their final positions. Any time-dilation effects can be made arbitrarily small by moving the clocks sufficiently slowly. This is true because the time-dilation factor γ is second order in v, whereas the time it takes a clock to reach its final position is only first order in $1/v$; see Problem 1.3.

This lattice way of looking at things emphasizes that observers are not important, and that a frame is defined simply as a lattice of space and time coordinates. Anything that happens (an "event") is automatically assigned a space and time coordinate in every frame, independent of any observer. Just record the spatial coordinates of the lattice point where the event is located, along with the reading on the clock at that point. You can assume that the lattice spacing is arbitrarily small, constructed with, say, millimeter sticks instead of meter sticks.

Frame independence

The three fundamental effects that we derived above tell us that many "truths" from everyday life must be thrown out the window. We can no longer count on two people in different frames agreeing on matters of simultaneity, time, or length. In fact, so much has been thrown out the window, that you might be wondering if there's *anything* two people in different frames can agree on.

Fortunately, there are still some things you can hang on to, that is, things that are *frame independent*. We have encountered many examples of frame-independent statements in the above sections, and we have presented various arguments for why these statements were in fact frame independent. Let's revisit one example that came up – the reading on a clock when a ball hits it. All observers, no matter what frame they are in, must agree on what this reading is. If one person says noon, then everyone says noon. There are two ways to see why this is the case.

First, you can imagine that the ball breaks the clock, so that the clock is stuck on whatever value it had when the ball hit it. Everyone will agree on what this broken value is, because the clock can be arranged to eventually sit at rest next to any given person. Second, the two events, "ball hitting clock" and "clock reading noon (or whatever)," happen at the *same* location in any frame, namely, at the clock. (We'll assume that the clock has negligible spatial extent.) The two events are therefore separated by a distance of $L = 0$, which means that the Lv/c^2 rear-clock-ahead effect is zero. In other words, we don't have to worry about any issues involving the loss of simultaneity. In short, the two events are really just *one* event, the "ball hitting clock and clock reading noon" event, described by specific space and time coordinates. Note that we are talking here about the *reading* on the clock, and not the *time elapsed* on it. An elapsed time is the difference between two readings.

What other kinds of frame-independent statements can we make? Well, if a paint bomb explodes and leaves a mark on an object at the location of a dent that was already there, then everyone agrees that there is a mark, and that it is located at the dent. Another example is: If a person departs from one clock when it reads T_1 and arrives at another clock when it reads T_2, then the difference in these readings, $T_2 - T_1$, is frame independent. This is true because each reading is frame independent. Note that we are *not* saying that the time elapsed between these two events (person departing from one clock, person arriving at the other) is frame independent. The time elapsed *does* depend on the observer's frame (there will be a time-dilation factor involved), because the time elapsed in a given frame is the difference in readings on the clock of the *observer* (who is at rest in the given frame), and not on the two given clocks. Apples and oranges.

Draw pictures, and stick to a frame

When solving problems using the fundamental effects, a very important strategy is to draw a picture of the setup in whatever frame you have chosen to work in. You should draw a picture at every moment when something of importance happens, as we did above in Fig. 1.22, for example. Once we drew those pictures, it was reasonably clear what we needed to do. But without the pictures, we almost certainly would have gotten confused. This problem-solving strategy is so important that we'll display it in a box:

DRAW PICTURES!

When drawing pictures at different times in a process, it is most informative to draw one picture above the other, with the x-axis origins of the two (or more) pictures vertically aligned. This way, objects that are at rest in the given frame are vertically aligned. See, for example, Figs. 1.32 and 1.34 below.

The importance of drawing pictures in relativity is analogous to the importance of drawing free-body diagrams when using $F = ma$ in mechanics problems. In both cases, the problem is usually easy once you draw the picture/diagram, but often hopeless if you don't (except in very simple cases).

A related strategy is to plant yourself in a frame and *stay there*. The only thoughts running through your head should be what *you* observe. That is, don't try to use reasoning like, "Well, the person I'm looking at in this other frame sees such-and-such." This will almost certainly cause an error somewhere along the way, because you will inevitably end up writing down an equation that combines quantities that are measured in different frames, which is a no-no. Or you might end up using time dilation backwards by putting the γ factor in the wrong place. The strategy of drawing pictures helps you avoid this kind of error, because when drawing a picture you necessarily have to pick a frame.

Of course, you might want to solve another part of the problem by working in another frame, or you might want to redo the whole problem in another frame. That's fine, but once you decide which frame you're going to use for a given line of reasoning, make sure you put yourself there and stay there. If you are drawing a picture in a train frame, then *be* the train. If you are drawing a picture in a ball frame, then *be* the ball. Sticking to a single frame is another problem-solving strategy that is so important that we'll display it in a box:

CHOOSE A FRAME AND STICK WITH IT!

"Seeing" things

We mentioned above in the fourth remark on page 14 that there is always a difference between the time an event *happens* and the time someone *sees* the event happen, because light takes time to travel from the event to the observer. We will generally be concerned with the former and not the latter. You can avoid the issue of "seeing" if you use the lattice of clocks and meter sticks introduced above. The time associated with an event is the time on the clock at the location of the event. With a lattice setup, we never have to worry about the travel time of light.

In this chapter, we will often be a little sloppy and use language such as, "What time does B see event Q happen?" But we don't really mean, "When do B's eyes register that Q happened?" Instead, we mean, "What time does B *know* that event Q happened in her frame?" If we ever want to use "see" in the former sense, we will explicitly say so. Two such examples are the "Rotated square" setup in Problem 1.4 and the Doppler effect in Section 2.5.

1.4 Four instructive examples

We'll now present four examples that integrate everything we've done so far. Each of these examples involves all three of the fundamental effects we've discussed: rear clock ahead, time dilation, and length contraction. The first two examples address the paradoxical issues with time dilation and length contraction.

You should try to solve each of these problems on your own before looking at the solution. If you can't solve it right away, set it aside for a while and come back to it later. It will still be there when you come back. In the end, you'll find that these four problems are all quite similar. There are only three fundamental effects, so there are only so many ways to combine them!

Example 1 (Explaining time dilation): Two planets, A and B, are at rest with respect to each other, a distance L apart, with synchronized clocks. A spaceship flies with speed v past planet A toward planet B. Right when it passes A, it synchronizes its clock with A's; they both set their clocks to zero. The spaceship eventually flies past B and compares its clock with B's. We know, from working in the planets' frame, that when the spaceship reaches B, B's clock simply reads L/v. Additionally, the spaceship's clock reads $L/\gamma v$, because it runs slow by a factor of γ when viewed in the planets' frame. See Fig. 1.31.

How would someone on the spaceship quantitatively explain why B's clock reads L/v (which is *more* than its own $L/\gamma v$) when the spaceship and B coincide, considering that the spaceship sees B's clock running *slow*? Shouldn't a spaceship person conclude that B's clock reads only $(L/\gamma v)/\gamma = L/\gamma^2 v$?

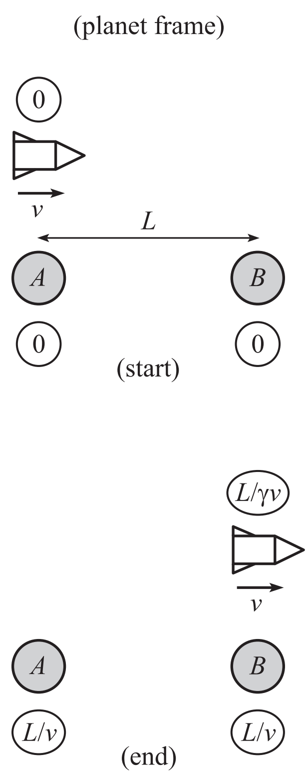

Figure 1.31

Solution: First note that if you want to work entirely in the spaceship's frame and not obtain the above $L/\gamma v$ result by using time dilation from the planets' point of view, you can use the fact that the spaceship says the distance between the planets is L/γ due to length contraction. Since the planets travel at speed v, the process therefore takes a time of $L/\gamma v$ on the spaceships's clock.

The resolution to the apparent paradox is the "head start" that B's clock has over A's clock, as seen in the spaceship frame. From Eq. (1.8), we know that in the spaceship frame, B's clock reads Lv/c^2 more than A's, because B is the rear person as they move leftward past the spaceship. See Fig. 1.32.

Therefore, what a person on the spaceship says is: "My clock advances by $L/\gamma v$ during the whole process. I see B's clock running slow by a factor γ, so I see B's clock advance by only $(L/\gamma v)/\gamma = L/\gamma^2 v$. However, B's clock started not at zero but at Lv/c^2. The final reading on B's clock when it reaches me is its initial reading of Lv/c^2 plus the elapsed time of $L/\gamma^2 v$, which gives

$$\frac{Lv}{c^2} + \frac{L}{\gamma^2 v} = \frac{L}{v}\left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right) = \frac{L}{v}\left(\frac{v^2}{c^2} + \left(1 - \frac{v^2}{c^2}\right)\right) = \frac{L}{v}, \tag{1.22}$$

as we wanted to show." The final reading on A's clock is only $L/\gamma^2 v$, but that isn't relevant here.

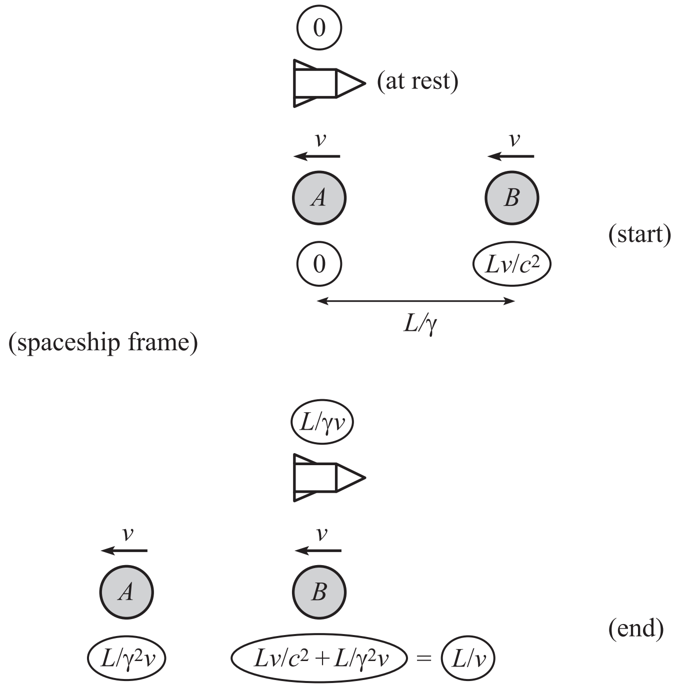

Figure 1.32

Note that in the sentence preceding Eq. (1.22), we used the phrase "on B's clock when it reaches me," as opposed to "on B's clock when I reach it." That latter would be incorrect, because we are working in the spaceship frame, where the spaceship is (of course) at rest; the two spaceships are vertically aligned in Fig. 1.32. Since the spaceship isn't moving, it therefore can't do any "reaching." B, however, is moving, so it can reach the spaceship.

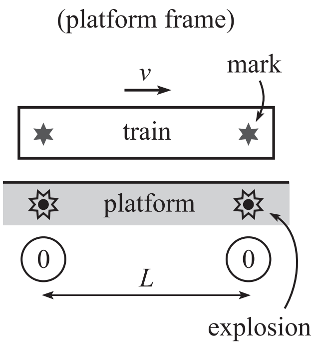

Figure 1.33

Example 2 (Explaining length contraction): Two paint bombs lie on a train platform, a distance L apart. As a train moves rightward at speed v, the paint bombs explode simultaneously (in the platform frame) and leave marks on the train; see Fig. 1.33. Due to the length contraction of the train, we know that the marks on the train are a distance γL apart when viewed in the train frame, because this distance is what is length-contracted down to the given distance L in the platform frame.

How would someone on the train quantitatively explain why the marks are a distance γL apart, considering that in the train frame the paint bombs are only a distance L/γ apart due to length contraction? (This example is the quantitative treatment of Example 3 on page 27.

Solution: The resolution to the apparent paradox is that the explosions do not occur simultaneously in the train frame. As the platform rushes past the train, the "rear" (right) paint bomb explodes before the "front" (left) one explodes.[8] The front one then gets to travel farther by the time it explodes and leaves its mark. The distance between the marks is therefore larger than the L/γ distance that you might naively expect. Let's be quantitative about this, to show that it all works out.

[8]Since we'll be working in the train frame here, we'll use the words "rear" and "front" in the way that someone on the train uses them as she watches the platform rush by. That is, if the train is heading east with respect to the platform, then from the point of view of the train, the platform is heading west. So the western paint bomb on the platform is the front one, and the eastern paint bomb is the rear one. They therefore have the opposite orientation compared with the way that someone on the platform labels the rear and front of the train. Using the same orientation would entail writing the phrase "front clock ahead" below, which would make me cringe.

For concreteness, let the two paint bombs contain clocks that read zero when they explode (they are synchronized in the platform frame), as shown above in Fig. 1.33. Then in the train frame, the front bomb's clock reads only $-Lv/c^2$ when the rear bomb explodes at the instant it reads zero; see Fig. 1.34. This is the rear-clock-ahead result from Eq. (1.8). The front bomb's clock must therefore advance by a time of Lv/c^2 before it explodes, because it is a frame-independent fact that it explodes when it reads zero. However, the train sees the bombs' clocks running slow by a factor γ, so in the train frame the front bomb explodes a time $\gamma Lv/c^2$ after the rear bomb explodes. (Note that it is the *elapsed* time of $0 - (-Lv/c^2)$ that gets dilated, and not the *reading* of Lv/c^2. Elapsed times get dilated, not readings.) During the time of $\gamma Lv/c^2$ in the train frame, the platform moves a distance $(\gamma Lv/c^2)v = \gamma Lv^2/c^2$ relative to the train, as shown in Fig. 1.34.

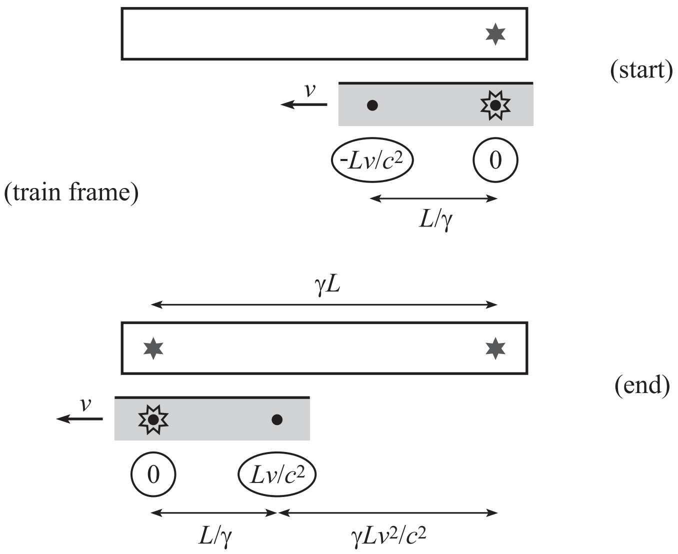

Figure 1.34

Therefore, what a person on the train says is: "Due to length contraction, the distance between the paint bombs is L/γ. The front (left) bomb is therefore a distance L/γ ahead of the rear (right) bomb when the latter explodes. The front bomb then travels an additional distance of $\gamma Lv^2/c^2$ by the time it explodes, at which point it is a distance of

$$\frac{L}{\gamma} + \frac{\gamma Lv^2}{c^2} = \gamma L\left(\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right) = \gamma L\left(\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\right) = \gamma L \qquad (1.23)$$

ahead of the rear bomb's mark, as we wanted to show."

Example 3 (A passing stick): A stick with proper length L moves past you at speed v, as shown in Fig. 1.35. There is a time interval between the front end coinciding with you and the back end coinciding with you. What is this time interval in:

(a) your frame? (Calculate this by working in your frame.)

(b) your frame? (Work in the stick's frame.)

(c) the stick's frame? (Work in your frame. This is the tricky one.)

(d) the stick's frame? (Work in the stick's frame.)

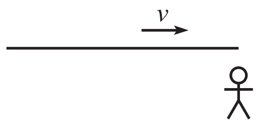

Figure 1.35

Solution:

(a) The stick has length L/γ in your frame, and it moves with speed v. Therefore, the time taken in your frame to cover the distance L/γ is $(L/\gamma)/v = L/\gamma v$.

(b) The stick sees you fly by at speed v. The stick has length L in its own frame, so the time elapsed in the stick frame is L/v. During this time, the stick sees the watch on your wrist run slow by a factor γ. Therefore, a time of only $(L/v)/\gamma = L/\gamma v$ elapses on your watch, in agreement with part (a).

Logically, the two solutions in parts (a) and (b) differ in that one uses length contraction while the other uses time dilation. Mathematically, they differ simply in the order in which the divisions by γ and v occur.

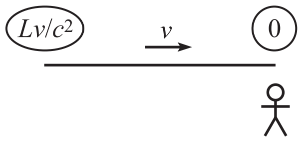

Figure 1.36

(c) Due to the rear-clock-ahead effect, you see the rear clock on the stick showing a time of Lv/c^2 more than the front clock. If we assume for concreteness that the front clock on the stick reads zero when it passes you, then at this same instant (in your frame), the rear clock on the stick reads Lv/c^2; see Fig. 1.36. In addition to this initial reading on the rear clock, more time will of course elapse on it by the time it reaches you. As we found in part (a), the time in your frame is $L/\gamma v$, because the stick has length L/γ in your frame and travels at speed v. But the stick's clocks run slow, so a time of only $(L/\gamma v)/\gamma = L/\gamma^2 v$ elapses on the rear clock by the time it reaches you. The final reading on the rear clock when it passes you is its initial reading plus the time elapsed on it, which gives

$$\frac{Lv}{c^2} + \frac{L}{\gamma^2 v} = \frac{L}{v}\left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right) = \frac{L}{v}\left(\frac{v^2}{c^2} + \left(1 - \frac{v^2}{c^2}\right)\right) = \frac{L}{v}. \tag{1.24}$$

(This is the same calculation as in Eq. (1.22).) Since the two clocks are synchronized (that is, they show the same time at any given instant) in the stick frame, the difference between the initial reading on the front clock (which is zero) and the final reading on the rear clock (which we just found to be L/v) is the time elapsed in the stick frame. So the time elapsed is L/v, in agreement with the quick calculation that follows in part (d).

(d) The stick sees you fly by at speed v. The stick has length L in its own frame, so the time elapsed in the stick frame is simply L/v. (Of course, we already knew this from solving part (b).)

Example 4 (Photon on a train): A train with proper length L has clocks at the front and back. A photon is fired from the back to the front. Working in the train frame, we can easily say that if the photon leaves the back of the train when a clock there reads zero, then it arrives at the front when a clock there reads L/c.

Now consider this setup in the ground frame, where the train travels by at speed v. Rederive the above frame-independent result (namely, if the photon leaves the back of the train when a clock there reads zero, then it arrives at the front when a clock there reads L/c) by working *only* in the ground frame.

Solution: In the ground frame the train has length L/γ, so the photon starts the process a distance L/γ behind the front of the train. It must close this gap at a relative speed of $c - v$, because the front of the train is receding at speed v. (Simple subtraction of these speeds is valid because they are both measured with respect to the ground, and we are looking for the relative speed as viewed by the ground.) The time elapsed in the ground frame is therefore $(L/\gamma)/(c - v)$. But the ground frame sees the train clocks run slow, so only $(L/\gamma^2)/(c - v)$ elapses on any given train clock.

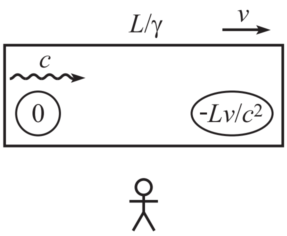

Figure 1.37

As viewed in the ground frame, when the photon is released next to the back clock when it reads zero, the front clock reads $-Lv/c^2$ due to the rear-clock-ahead effect. Fig. 1.37 shows the initial picture. The reading on the front clock when the photon hits it is the initial reading of $-Lv/c^2$ plus the time elapsed on it, which we found above to be $(L/\gamma^2)/(c - v)$. The final reading on the front clock is therefore

$$-\frac{Lv}{c^2} + \frac{L}{\gamma^2(c-v)} = -\frac{Lv}{c^2} + \frac{L\left(1 - \frac{v^2}{c^2}\right)}{c\left(1 - \frac{v}{c}\right)} = -\frac{Lv}{c^2} + \frac{L}{c}\left(1 + \frac{v}{c}\right) = \frac{L}{c}, \tag{1.25}$$

as desired.

At this point, you might want to look at the "Qualitative relativity questions" in Appendix A, just to make sure there aren't any misconceptions lingering in your mind. The first half of the collection (through Question 25) deals with material we've covered so far.

1.5 Velocity addition

It's now time to derive the *velocity-addition formula*, which we have mentioned a few times in this chapter. If you want, you can consider the formula to be a fourth fundamental effect, in addition to rear clock ahead, time dilation, and length contraction.

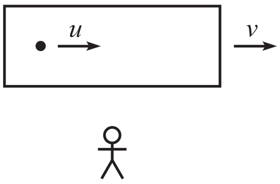

Figure 1.38

Consider the following setup. A ball moves at speed u with respect to a train, and the train moves at speed v with respect to the ground (in the same direction as the motion of the ball; see Fig. 1.38). What is the speed V of the ball with respect to the ground? The result is the desired velocity-addition formula.

In the nonrelativistic limit (that is, when u and v are small compared with c), V is simply equal to $u + v$, as we know very well from everyday experience (although technically this result isn't *exactly* correct, as we'll see below). But the simple $u + v$ answer can't be correct for larger speeds, because if, for example, u and v are both equal to $(0.9)c$, then $u + v = (1.8)c$. This is certainly incorrect, because it is larger than c. The fact of the matter is that it is impossible for an object (or at least any object we can interact with) to move faster than c. There are various ways to demonstrate this. One is that it would require an infinite amount of energy to accelerate an object up to speed c. We'll see why in Chapter 3.

To find the correct general expression for V (that is, to derive the velocity-addition formula), let's consider a concrete setup where we look at the time it takes a ball to travel from the back of a train to the front. Let the train have proper length L. Our strategy for finding V will be to generate two different expressions for the time of this process in the ground frame. Equating these two expressions will allow us to solve for V. The setup is shown in Fig. 1.39. We've put the ball outside the train to emphasize that the speed V is with respect to the ground.

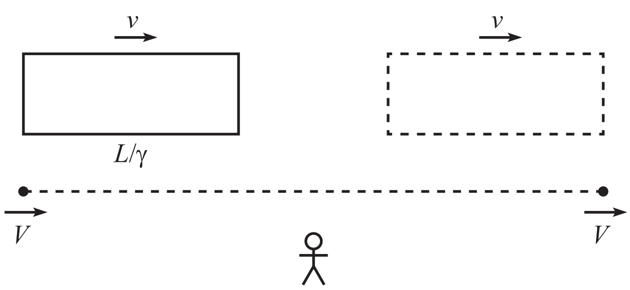

Figure 1.39

The first expression for the time (for the ball to go from the back of the train to the front) in the ground frame is found by noting that (a) the initial gap between the ball and the front of the train is L/γ due to length contraction, and (b) this gap is closed at a rate $V - v$, because this is the relative speed of the ball and the front of the train, as viewed in the ground frame. As we've seen a number of times in this chapter, it is legal to simply subtract these speeds, because they are both measured with respect to the same frame (the ground frame). The time in the ground frame for the ball to reach the front of the train is therefore

$$t_g = \frac{L/\gamma}{V - v}, \tag{1.26}$$

where V is not yet known. The γ factor here is associated with the speed v of the train (that is, not with V or u).

The second expression for the time in the ground frame is found by looking at a particular clock on the train and using time dilation. Assume that a clock at the back of the train reads zero when the ball is thrown. Then by working in the frame of the train, we quickly see that a clock at the front reads L/u when the ball gets there. (No relativity needed for this.) Now look at things in the ground frame. The readings we just mentioned are frame independent, so the starting and ending readings in the ground frame must be the ones shown in Fig. 1.40. At the start, the rear clock reads zero (frame independent), so the front clock reads $-Lv/c^2$, due to the rear-clock-ahead effect. And at the finish, the front clock reads L/u (frame independent), so the rear clock reads $L/u + Lv/c^2$, due to the rear-clock-ahead effect.

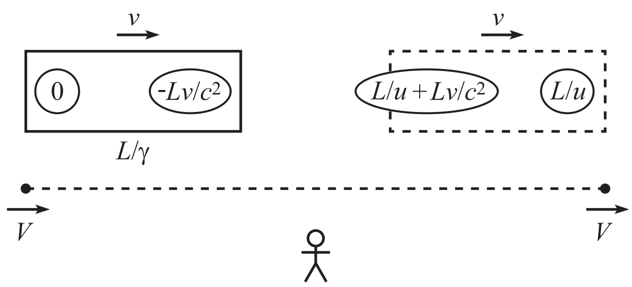

Figure 1.40

Looking at the rear clock (the front clock works just as well), we see that the time elapsed on this clock is $L/u + Lv/c^2$. Since we are looking at a single clock, it is legal to use time dilation, which tells us that the time elapsed on the ground is longer by a factor γ. (Again, this γ factor is associated with the speed v of the train.) So our second expression for the time in the ground frame is

$$t_g = \gamma\left(\frac{L}{u} + \frac{Lv}{c^2}\right). \tag{1.27}$$

Note that we *cannot* say that since the time elapsed on the train is L/u, the time elapsed on the ground should be $\gamma(L/u)$. This is incorrect because in the ground frame, L/u isn't the time elapsed on a *single* clock. Fig. 1.40 tells us that L/u is the final reading on the *front* clock minus the initial reading on the *rear* clock. It isn't legal to use time dilation when comparing the readings on two different clocks. In the above correct reasoning, we compared the readings on a *single* clock (either the front clock *or* the rear clock).

Now all we have to do is equate the two times in Eqs. (1.26) and (1.27) and then solve for V. There's a little algebra involved, but it isn't so bad. We have

$$\frac{L/\gamma}{V-v} = \gamma\left(\frac{L}{u} + \frac{Lv}{c^2}\right) \quad\Longrightarrow\quad \frac{1}{\gamma^2}\frac{1}{\frac{1}{u}+\frac{v}{c^2}} = V - v$$

$$\Longrightarrow\quad V = \frac{1-\frac{v^2}{c^2}}{\frac{1}{u}+\frac{v}{c^2}} + v = \frac{1+\frac{v}{u}}{\frac{1}{u}+\frac{v}{c^2}}$$

$$\Longrightarrow\quad \boxed{V = \frac{u+v}{1+\frac{uv}{c^2}}} \qquad \text{(velocity-addition formula)} \tag{1.28}$$

This is the *longitudinal* velocity-addition formula, relevant when adding velocities that are parallel to each other. There is also a *transverse* velocity-addition formula, relevant when the two given velocities are perpendicular. We'll derive that in Section 2.2.2. Note that we used all three of the fundamental effects (rear clock ahead, time dilation, and length contraction) in the above derivation.

Given a velocity v, the letter β is used to denote the ratio of v to c. That is, $\beta \equiv v/c$. Sometimes a subscript is added, as in β_v or β_1, if there is ambiguity about which speed is being referred to. In terms of β's, the velocity-addition formula for the speeds v_1 and v_2 takes the form,

$$\beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\,. \tag{1.29}$$

This expression is often simpler to work with than Eq. (1.28), because there are no c's to clutter things up. Alternatively, we will often simply drop the c's in calculations, to keep things from getting too messy. Eq. (1.28) then becomes $V = (u+v)/(1+uv)$. The c's can always be put back in at the end, by figuring out where they need to go in order to make the units correct (which is usually easy to see). Equivalently, you can just pretend that V, u, and v are actually β_V, β_u, and β_v.

Two other derivations of Eq. (1.28) are given in Exercises 1.47 and 1.48. They are similar to the above derivation, in the following way. There are (at least) three different methods for finding the time (for the ball to travel from the back of the train to the front) in the ground frame: (1) the front of the train has an initial head start over the ball, and we can find the time it takes the ball to close this gap down to zero, (2) we can apply time dilation to a given train clock, or (3) we can apply time dilation to the ball's clock. The above derivation used the first and second of these, Exercise 1.47 uses the first and third, and Exercise 1.48 uses the second and third.

Let's look at some of the properties of the V in Eq. (1.28).

- V is symmetric in u and v. We'll explain why when we discuss Fig. 1.41 below.
- If u and v are small compared with c, then $V \approx u + v$, because the uv/c^2 term in the denominator is negligible. This result makes sense, because we know very well that V equals $u + v$ for everyday speeds.
- If $u = c$ or $v = c$, then we obtain $V = c$. This is correct, because anything that moves with speed c in one frame moves with speed c in another.
- The maximum (or minimum) of V in the square region defined by the two inequalities, $-c \le u \le c$ and $-c \le v \le c$, equals c (or $-c$). This is due to the fact that the partial derivatives $\partial V/\partial u$ and $\partial V/\partial v$ are never zero in the interior

of the region (or anywhere, for that matter), as you can verify. The extrema must therefore occur on the boundary, and you can show that V always takes on the value of c or $-c$ there. (There are two exceptions at the two $u = -v = \pm c$ corners of the region, where V takes on the undefined value of 0/0.)

The last of these bullet points tells us that if we take any two velocities that are less than c and add them according to Eq. (1.28), then we will obtain a velocity that is again less than c. If you don't want to rely on partial derivatives, you can alternatively demonstrate this fact with the following inequalities:

$$\frac{u+v}{1+uv/c^2} < c \iff u+v < c+uv/c \iff u(1-v/c) < c-v$$
$$\iff u < \frac{c-v}{(c-v)/c} \iff u < c. \qquad (1.30)$$

Assuming that $c - v$ isn't zero (because otherwise we would have divided by zero), all of the above steps are reversible. So if we start with both $u < c$ and $v < c$, then we end up with $(u + v)/(1 + uv/c^2) < c$. This means that no matter how much you keep accelerating an object (that is, no matter how many times you give the object a speed u with respect to the frame moving at speed v that it was just in), you can't bring the speed up to c. As mentioned earlier, this also follows from energy considerations, as we'll see in Chapter 3.

For a bullet, a train, and a gun,
Adding the speeds can be fun.
Take a trip down the path
Paved with Einstein's new math,
Where a half plus a half isn't one.

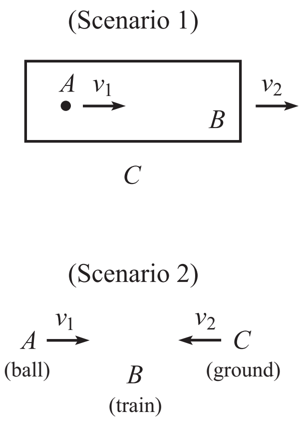

Figure 1.41

In addition to applying to the above "ball on train on ground" setup, there is another common scenario where the velocity-addition formula applies. Consider the two scenarios shown in Fig. 1.41. The first is the original "ball on train on ground" setup. These two scenarios are actually identical. The only difference is that the first one is shown in C's frame, while the second one is shown in B's frame. They are indeed identical, because in the first scenario B sees A approach him rightward at speed v_1 (in agreement with the second scenario). Similarly, in the first scenario, B sees C approach him leftward at speed v_2 (again in agreement with the second scenario). Basically, in the second scenario, A is the ball, B is the train, and C is the ground. Therefore, if the goal is to find the velocity of A with respect to C in the second scenario, then since we showed above that the velocity-addition formula applies in the first scenario, we conclude that it also applies to the second scenario.

We noted above that the V in Eq. (1.28) is symmetric in u and v, or equivalently in v_1 and v_2 in the present notation. This is clear in the second scenario in Fig. 1.41, because switching v_1 and v_2 is equivalent to switching A and C, and because the speed of A as viewed by C is the same as the speed of C as viewed by A (see the first remark below). Therefore, since the two scenarios in Fig. 1.41 are equivalent, V must also be symmetric in u and v in the original setup in Fig. 1.38.

In the second scenario in Fig. 1.41 (as in the first), the velocity-addition formula tells us the answer to the question, "What is the relative speed of A and C, *as viewed by C*?" The answer is $(v_1 + v_2)/(1 + v_1 v_2/c^2)$. However, the formula does *not* apply if we ask the more mundane question, "What is the relative speed of A and C, *as viewed by B*?" The answer to this is simply $v_1 + v_2$. In short, if the two velocities are given with respect to the *same* observer, B, and if you are asking for the relative velocity as measured by B, then you just have to add the velocities. But if you are asking for the relative velocity as measured by A or C, then you have to use the velocity-addition formula. Equivalently, in the first scenario in Fig. 1.41, it makes no sense to naively add velocities that are measured with respect to different observers. Doing so would involve adding things that are measured in different coordinate systems, which is meaningless. Taking the velocity of A with respect to B and adding it to the velocity of B with respect to C, hoping to obtain the velocity of A with respect to C, is invalid.

We see that relativistically the question, "What is the relative speed of A and C?" is ambiguous. We have to finish it with ". . . as measured by such and such a person." Nonrelativistically, though, there is no ambiguity. The answer is simply $v_1 + v_2$ in any frame.

Note that the $v_1 + v_2$ relative speed of A and C, as viewed by B, in Fig. 1.41 can certainly be greater than c. If I see a ball heading toward me at $(0.9)c$ from the right, and another one heading toward me at $(0.9)c$ from the left, then the relative speed of the balls in my frame is $(1.8)c$. In the frame of one of the balls, however, Eq. (1.28) gives the relative speed as $(1.8/1.81)c \approx (.9945)c$. This is correctly less than c, because you (or one of the balls, in this case) can never see another object (the other ball) move with a speed greater than c. This restriction doesn't rule out the above result of $(1.8)c$, because this is simply the rate at which the gap between the balls is closing. A gap isn't an actual object, so there is nothing wrong with the length of the gap decreasing at a rate faster than c. In the extreme, if one photon heads rightward and another one heads leftward, then as measured by you, their relative speed is $2c$. That is the rate at which the gap between them is closing in your frame.

Remarks:

1. If two people, A and B, are moving with respect to each other in one dimension, why is the speed of B as viewed by A equal to the speed of A as viewed by B? This equality seems quite reasonable, of course, but how do we prove it rigorously from basic principles (that is, ignoring what we've derived about velocity addition)? The proof follows directly from the first postulate of relativity – that all inertial frames are equivalent (which implies that there is no preferred location or direction in space). Let's assume that the relative speed measured by the left person is larger than the relative speed measured by the right person. This implies that there is a preferred direction in space; apparently people on the left always measure a larger speed. This violates the first postulate. Likewise if the left person measures a smaller speed. The two speeds must therefore be equal.

2. Strictly speaking, the signs in the numerator and denominator in Eq. (1.28) are always plus signs, assuming that v_1 and v_2 are the signed *velocities* of the objects in the first scenario in Fig. 1.41. However, in practice it is often more convenient to let v_1 and v_2 be *speeds* (which are always positive). In this case, the same sign appears in the numerator and denominator in Eq. (1.28), and the correct choice of sign is determined by the sign you would use in the simple nonrelativistic case. For example, the nonrelativistic speed of A with respect to C in the first scenario in Fig. 1.41 is simply the sum of the speeds, $v_1 + v_2$, which means we must use positive signs in Eq. (1.28). If the ball is instead thrown backward on the train, then the nonrelativistic speed of A with respect to C is the difference of the speeds, $-v_1 + v_2$ (or $|-v_1 + v_2|$ if this quantity is negative), which means we must use negative signs in Eq. (1.28). In any case, the numerator in the relativistic case is always the naive nonrelativistic answer. If you get confused about the signs, it's best to just plug is some actual numbers for v_1 and v_2, to see what's going on.

3. For everyday speeds, the nonrelativistic and relativistic results for the speed of A as viewed by C in Fig. 1.41 are essentially the same. If $v_1 = 50\,\text{m/s}$ and $v_2 = 30\,\text{m/s}$, then the nonrelativistic result is simply $v_1 + v_2 = 80\,\text{m/s}$, while Eq. (1.28) gives the relativistic result as $(80\,\text{m/s})(1 - 1.67 \cdot 10^{-14})$. The nonrelativistic result is therefore incorrect by less than 2 parts in 10^{14}. That's plenty good for me.

4. The sum $v_1 + v_2$ doubles as the answer to two different questions concerning Fig. 1.41. It is the *approximate* (nonrelativistic) answer to the question, "What is the relative speed of A and C, as viewed by C?" It is also the *exact* (relativistic or nonrelativistic) answer to the question, "What is the relative speed of A and C, as viewed by B?" ♣

Let's now do two examples. The second one has a little bit of everything we've done so far – rear clock ahead, time dilation, length contraction, and velocity addition.

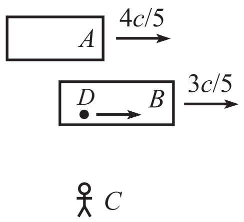

Figure 1.42

Example 1 (Passing trains, again): Consider again the scenario in the "Passing trains" example in Section 1.3.3.

(a) How long, as viewed by A and as viewed by B, does it take for A to overtake B?

(b) Let event E_1 be "the front of A passing the back of B", and let event E_2 be "the back of A passing the front of B." Person D walks at constant speed from the back of B to the front (see Fig. 1.42), such that he coincides with both events, E_1 and E_2. How long does the "overtaking" process take, as viewed by D?

Solution:

(a) First consider B's point of view. From the velocity-addition formula, B sees A move with speed

$$u = \frac{\frac{4c}{5} - \frac{3c}{5}}{1 - \frac{4}{5} \cdot \frac{3}{5}} = \frac{5c}{13}. \tag{1.31}$$

This expression involves minus signs because the naive nonrelativistic relative speed involves subtracting the speeds. The γ factor associated with the speed u is $\gamma_{5/13} = 13/12$, as you can check. Therefore, B sees A's train contracted to a length $12L/13$. During the overtaking, A must travel a distance equal to the sum of the lengths of the trains in B's frame (see Fig. 1.43), which is $L + 12L/13 = 25L/13$. Since A moves with speed $5c/13$, the total time in B's frame is

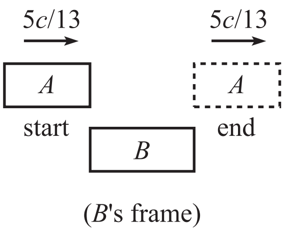

Figure 1.43

$$t_B = \frac{25L/13}{5c/13} = \frac{5L}{c}. \tag{1.32}$$

The exact same reasoning holds from A's point of view, so we have $t_A = t_B = 5L/c$.

(b) Look at things in D's frame. D is at rest, and the two trains move with equal and opposite speeds v (to be determined), because this causes the second event E_2 to be correctly be located at D; see Fig. 1.44. Our setup is equivalent to the second scenario in Fig. 1.41, so the speed of A as viewed by B is the relativistic addition of v with itself. But from part (a), we know that this relative speed equals $5c/13$. Therefore (with $\beta \equiv v/c$),

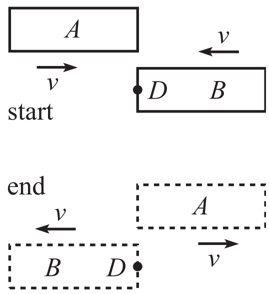

Figure 1.44

$$\frac{v + v}{1 + v^2/c^2} = \frac{5c}{13} \implies 5\beta^2 - 26\beta + 5 = 0$$

$$\implies (5\beta - 1)(\beta - 5) = 0 \implies \beta = \frac{1}{5}, \tag{1.33}$$

which gives $v = c/5$. We have ignored the unphysical solution, $\beta = 5 \implies v = 5c$, because v can't exceed c. The γ factor associated with $v = c/5$ is $\gamma_{1/5} = 5/2\sqrt{6}$. So D sees both trains contracted to a length $2\sqrt{6}L/5$. During the process, each train must travel a distance equal to its length (as shown in Fig. 1.44) because both events, E_1 and E_2, take place right at D. The time in D's frame is therefore

$$t_D = \frac{2\sqrt{6}L/5}{c/5} = \frac{2\sqrt{6}L}{c}. \tag{1.34}$$

Remarks: There are a few double checks we can perform. The speed of D with respect to the ground can be obtained either via B's frame by relativistically adding

$3c/5$ and $c/5$, or via A's frame by relativistically subtracting $c/5$ from $4c/5$. You can check that these methods give the same answer (as they must), namely $5c/7$. (The $v = c/5$ speed can actually be determined by demanding that these methods give the same answer, instead of by Eq. (1.33).) The γ factor between the ground and D is therefore $\gamma_{5/7} = 7/2\sqrt{6}$. We can then use time dilation to say that someone on the ground sees the overtaking take a time of $(7/2\sqrt{6})t_D$. (We can say this because both events happen right at D.) Using the t_D from Eq. (1.34), this gives a ground-frame time of $7L/c$, in agreement with our earlier result in Eq. (1.21). Likewise, since the γ factor between D and either train is $\gamma_{1/5} = 5/2\sqrt{6}$, the time of the overtaking as viewed by either A or B is $(5/2\sqrt{6})t_D = 5L/c$, in agreement with Eq. (1.32).

Note that we *cannot* use simple time dilation to relate the time on the ground to the time on either train, because the two events don't happen at the same place in either of the train frames (or in the ground frame). But since both events happen at the same place in D's frame, namely right at D, it is legal to use time dilation to go from D's frame to any other frame. And when doing so, the relevant γ factor always appears in front of t_D. ♣

Example 2 (Clock readings on trains):

(a) Two trains each have proper length L and travel on parallel tracks. They both move with speed v with respect to the ground, one rightward and one leftward. You notice that clocks at the fronts of the trains both read zero when the fronts coincide, as shown in Fig. 1.45. What do clocks at the backs of the trains read when the backs eventually coincide? Answer this by working in the ground frame.

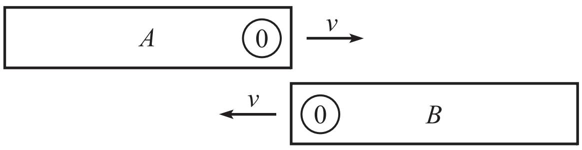

Figure 1.45

(b) Again find the readings on the two clocks at the backs of the trains when the backs coincide, but now by working in the frame of train B. To keep things from getting messy, you can let v take on the particular value of $c/2$ for this part.

Solution:

(a) At the given initial instant in the ground frame, a ground observer sees both of the rear clocks reading Lv/c^2, due to the rear-clock-ahead effect. The backs of the trains eventually coincide at the same location the fronts coincided, after each train has traveled a distance L/γ; this is the contracted length of each train in the ground frame. The time elapsed in the ground frame is therefore $(L/\gamma)/v$. But the ground observer sees the train clocks (in particular, the rear clocks) running slow by a factor γ. So the time elapsed on each rear clock is $(L/\gamma v)/\gamma$. The final reading on each rear clock is its initial reading of Lv/c^2 plus the elapsed time of $L/\gamma^2 v$. The final reading is therefore (as we've seen a few times in other examples)

$$\frac{Lv}{c^2} + \frac{L}{\gamma^2 v} = \frac{Lv}{c^2} + \left(1 - \frac{v^2}{c^2}\right)\frac{L}{v} = \frac{L}{v}. \tag{1.35}$$

Remark: A quick way to see why this result is so simple is the following. Imagine a person standing at rest on the ground, at the initial location of the fronts of the trains. Since the person is at rest, the backs of the trains will be located at the person when they eventually coincide. Therefore, in the frame of one of the trains, the person simply travels the length L of the train (by the time the backs and the person all coincide), at speed v (because that is the relative speed of the ground and a train). So the time in the train frame is L/v. This elapsed time is the desired reading on the back clock, because the clock started at zero in the train frame, since clocks on a given train are synchronized in that train's frame. ♣

(b) The given setup is equivalent to the second scenario in Fig. 1.41, so the speed V of one train as viewed by the other is the relativistic addition of $v \equiv c/2$ with itself:

$$V = \frac{\frac{c}{2} + \frac{c}{2}}{1 + \frac{1}{2} \cdot \frac{1}{2}} = \frac{4c}{5}. \tag{1.36}$$

Since $\gamma_{4/5} = 5/3$, the contracted length of A in B's frame is $3L/5$. The setup in B's frame is therefore shown in Fig. 1.46 (ignore A's rear clock for now).

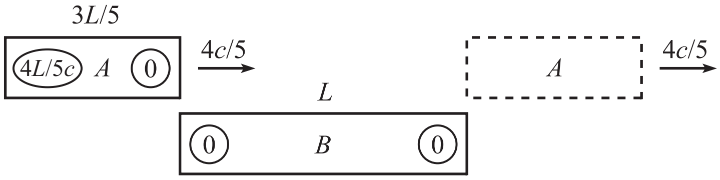

Figure 1.46

Let's first find the final reading on B's rear (right) clock when A's rear (left) clock reaches it. (B's right clock starts at zero, as shown, because B's clocks are synchronized in B's frame.) This is obtained by noting that A must travel a distance equal to the sum of the lengths of the trains. It does this at speed $4c/5$, so the final reading on B's right clock is

$$0 + \frac{3L/5 + L}{4c/5} = \frac{2L}{c}. \tag{1.37}$$

This agrees with the result in part (a) when $v = c/2$, as it must, because the reading is frame independent.

To find the final reading on A's left clock, we must remember that it starts the process reading $LV/c^2 = 4L/5c$, due to the rear-clock-ahead effect. It then advances by $(2L/c)/\gamma_{4/5}$, due to time dilation and due to the fact that $2L/c$ is the time that elapses in B's frame. (B sees A's clocks run slow, so we must divide by γ.) The final reading on A's left clock is therefore

$$\frac{4L}{5c} + \frac{3}{5} \cdot \frac{2L}{c} = \frac{2L}{c}. \tag{1.38}$$

Again this agrees with the result in part (a) when $v = c/2$. Note that while there is a symmetry between A's and B's rear clocks in the ground frame in part (a), there is no such symmetry between the clocks here in B's frame. The calculation for A's clock here is therefore different from the calculation for B's clock (although we know from part (a) that the readings must end up the same).

General problem-solving strategies

We'll end this chapter by collecting all of our problem-solving strategies in one place. We'll encounter additional kinematics strategies in Chapter 2, but for the types of problems we've solved in this chapter, things generally boil down to the following ingredients. If you look back at the examples we've done, you can verify that these strategies pretty much have everything covered. You should therefore keep this checklist on the tip of your mind.

1. Loss of simultaneity (rear clock ahead): As viewed in the ground frame, the rear clock on a train reads Lv/c^2 more than the front clock.

2. Time dilation: If you look at a moving clock, it runs slow by a factor γ.

3. Length contraction: If you look at a stick moving longitudinally, it is short by a factor γ.

4. Velocity-addition formula: This gives the speed of A as viewed by C in the two scenarios in Fig. 1.41.

5. Gap-closing speed: $v_1 + v_2$ (or $v_1 - v_2$, depending on directions and sign conventions) is the relative speed of A and C, as viewed by B in the second scenario (and in the first) in Fig. 1.41.

6. Frame-independent statements: If information is given in one frame, but you want to work out the problem in another frame, you need to identify which statements remain true in the new frame.

7. Draw pictures and stick to a frame! Drawing pictures in a given frame helps you properly implement the above strategies. You should draw a picture whenever anything of importance happens, labeling all speeds, lengths, clock readings, etc.

1.6 Summary

In this chapter we learned about the basics of special relativity. In particular, we learned:

- A few puzzles in 19th-century physics suggested that something was amiss. These puzzles included the inconsistency of the Galilean transformations with Maxwell's equations, and the null result of Michelson and Morley in their search for the ether. Einstein's theory of relativity solved these puzzles by showing that the Lorentz transformations replace the Galilean transformations, and that light waves propagate without the need for an ether medium.

- The theory of special relativity rests on two postulates:

    1. All inertial (non-accelerating) frames are equivalent; there is no preferred reference frame.
    2. The speed of light in vacuum has the same value in any inertial frame.

- The above postulates lead to many counterintuitive effects, the most fundamental of which are:

    1. Loss of simultaneity (rear-clock-ahead): As viewed in the ground frame, the rear clock on a train reads Lv/c^2 more than the front clock, where L is the proper length of the train.
    2. Time dilation: If you look at a moving clock, it runs slow by a factor γ:

    $$t_{\text{observed}} = \gamma t_{\text{proper}}. \tag{1.39}$$

    More generally, if you are considering the time between two events, the γ goes on the side of the equation associated with the frame in which the two events happen at the same place (which is automatically the case for two ticks on a clock). The time in this frame is by definition the proper time. Remember that it is *elapsed times* that get dilated, and not *readings* on clocks.

3. Length contraction: If you look at a stick moving longitudinally, it is short by a factor γ:

$$L_{\text{observed}} = \frac{L_{\text{proper}}}{\gamma}. \tag{1.40}$$

The proper length of an object is the length as measured in the frame in which the object is at rest. In any frame, the length of an object is defined to be the distance between the ends, measured simultaneously in that frame. There is no transverse length contraction.

- A reference frame can be defined by a lattice of meter sticks and clocks. The coordinates of an event are the spatial coordinates of the lattice point where the event is located, along with the reading on the clock at that point.

- The velocity-addition formula,

$$V = \frac{u+v}{1+\frac{uv}{c^2}}, \tag{1.41}$$

gives the speed of A as viewed by C in the two scenarios in Fig. 1.41 (with $u \leftrightarrow v_1$ and $v \leftrightarrow v_2$). When written in terms of β's, the formula takes the form in Eq. (1.29).

The velocity addition formula does *not* apply when finding the relative speed of A and C, as viewed by B, in the second scenario (and also in the first) in Fig. 1.41. This gap-closing speed is simply $v_1 + v_2$.

- The various problem-solving strategies we have used throughout this chapter are listed just before this summary.

1.7 Problems

Section 1.3: The fundamental effects

1.1. **Consistency with Lv/c^2** $*$

Show that the $t_R - t_L$ difference of the times in Eq. (1.6) (where ℓ' is half the length of the train in the ground frame) is consistent with the Lv/c^2 rear-clock-ahead result (where L is the proper length of the train).

1.2. **Here and there** $*$

A train with proper length L travels past you at speed v. A person on the train stands at the front, next to a clock that reads zero. At this moment (as measured by you), a clock at the back of the train reads Lv/c^2, due to the rear-clock-ahead effect. How would you respond to the following statement:

"In the train frame, the person at the front of the train can leave the front right after the clock there reads zero, and then run to the back and get there right before the clock there reads Lv/c^2. You (on the ground) will therefore see the person simultaneously at *both* the front and the back of the train when the clocks there read zero and Lv/c^2, respectively."

1.3. **Synchronizing clocks** $*$

Two synchronized clocks, A and B, are at rest in a given frame, a distance L apart. A third clock, C, is initially located right next to A. All three clocks have initial readings of zero, and then C is moved very slowly (with speed $v \ll c$) from A to B. Show that its final reading can be made arbitrarily close to B's, by making v be sufficiently small. (The Taylor series $\sqrt{1-\epsilon} \approx 1 - \epsilon/2$ will come in handy.)

1.4. **Rotated square** $**$

A square with sides of proper length L flies past you at speed v, in a direction parallel to two of its sides. You stand in the plane of the square. When you see the square at its nearest point to you (see Fig. 1.47), show that it *looks* to you like it is rotated instead of contracted, and find the apparent angle of rotation. Assume that L is small compared with the distance between you and the square. (This setup is one of the few cases where we are actually concerned with the time it takes light to travel to your eye.)

Figure 1.47

1.5. **Deriving length contraction** $**$

The derivation of length contraction in Section 1.3.3 relied on time dilation. This problem gives a derivation that is independent of time dilation.

Assume that the rule for (longitudinal) length contraction is: "If a stick with proper length L moves at speed v with respect to you, then its length in your frame is $a_v L$." (The subscript v signifies the possible dependence of a on v.) Your eventual goal is to show that $a_v = 1/\gamma_v$, but for all you know at the moment, a_v might be larger than, less than, or equal to 1. A critical point here is that the first postulate of relativity says that all inertial frames are equivalent. So the same $a_v L$ rule must apply to everyone.

Consider the following setup. A train with proper length L moves with speed v. When the back of the train passes a tree, a photon is fired from the back toward the front. It arrives at the front when the front passes a house. What is the distance between the tree and the house (in the ground frame)?

Now look at things in the train frame. Using the tree-house proper distance you just found, write down the relation that expresses the fact that the house meets the front of the train at the same time the photon does. This will allow you to solve for a_v.

1.6. **Pole in barn** $*$

A pole with proper length L moves rightward with speed v through a barn, also with proper length L. Assume that initially the door at the left end of the barn is open and the door at the right end is closed. Just after the left end of the pole enters the barn, the left door closes. And just before the right end of the pole leaves the barn, the right door opens. How would you respond to the following question: "Is the pole ever completely inside the barn (with both doors closed)?"

1.7. **Train in a tunnel** $**$

A train and a tunnel both have proper length L. The train moves toward the tunnel at speed v. A bomb is located at the front of the train. The bomb is designed to explode when the front of the train passes the far end of the tunnel. A deactivation sensor is located at the back of the train. When the back of the train passes the near end of the tunnel, the sensor sends a signal to the bomb, telling it to disarm itself. Does the bomb explode?

1.8. **Bouncing stick** $**$

A stick, oriented horizontally, falls and bounces off the ground. Qualitatively, what does this setup look like in the frame of someone running by at speed v?

1.9. **Seeing behind the board** $**$

A ruler is positioned perpendicular to a wall, and you stand at rest with respect to the ruler and the wall. A board with proper length L moves to the right with speed

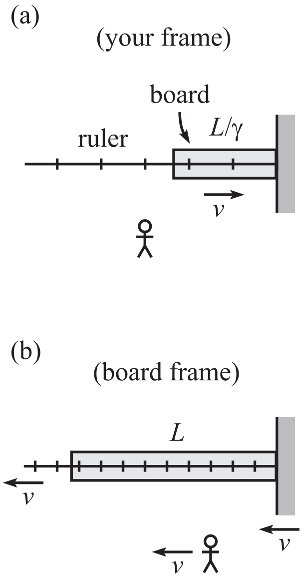

Figure 1.48

Figure 1.49

v. It travels in front of the ruler, so that it obscures part of the ruler from your view. The board eventually hits the wall. Which of the following two reasonings is correct (and what is wrong with the incorrect one)?

In your reference frame, the board is shorter than L, due to length contraction. Therefore, right before it hits the wall, you are able to see a mark on the ruler that is less than L units from the wall; see Fig. 1.48(a).

In the board's frame, the marks on the ruler are closer together, due to length contraction. Therefore, the closest mark to the wall that you will ever be able to see on the ruler is greater than L units; see Fig. 1.48(b).

1.10. **Cookie cutter** $**$

Cookie dough (chocolate chip, of course) lies on a conveyor belt that moves with speed v. A horizontal circular cookie cutter stamps out cookies as the dough rushes by beneath it. When you buy these cookies in a store, what shape are they? That is, are they squashed in the direction of the belt, stretched in that direction, or circular?

1.11. **Getting shorter** $**$

Two balls move with speed v along a line toward two people standing along the same line. The proper distance between the balls is γL, and the proper distance between the people is L. Due to length contraction, the people measure the distance between the balls to be L, so the balls pass the people simultaneously (as measured by the people), as shown in Fig. 1.49. Assume that the people's clocks both read zero at this time. If the people catch the balls, then the resulting proper distance between the balls becomes L, which is shorter than the initial proper distance of γL. Your task: By working in the frame in which the balls are initially at rest, explain how the proper distance between the balls decreases from γL to L. Do this in the following way.

(a) Draw the beginning and ending pictures for the process. Indicate the readings on both clocks in the two pictures, and label all relevant lengths.

(b) Using the *distances* labeled in your pictures, how far do the people travel? Using the *times* labeled in your pictures, how far do the people travel? Show that these two methods give the same result.

(c) Explain in words how the proper distance between the balls decreases.

1.12. **Transforming the length** $*$

A stick moves rightward with speed $3c/5$ with respect to the ground. The length of the stick *in the ground frame* is L. You move rightward with speed $c/2$ with respect to the ground. What is the length of the stick in your frame?

1.13. **Magnetic force** $***$

This problem demonstrates how the magnetic force arises from the combination of the electric force and length contraction. The interpretation of this problem (discussed in the solution) requires a familiarity with basic concepts of electricity and magnetism, although the problem itself does not.

Consider a current-carrying straight wire. The wire is neutral, that is, it has the same number of negatively charged electrons and positively charged protons in any given volume, on average. (If it weren't neutral, it would attract or repel electrons, thereby producing neutrality.) In any current-carrying wire, the current is caused by electrons moving along the wire; the protons are bolted down. So we have the situation shown in Fig. 1.50. (Ignore the charge q for a moment.) In reality, the electrons and protons are distributed throughout the wire, but we have drawn them separated for clarity. The equal and opposite charge densities $\pm\lambda_0$ (charge per meter) are indicated. The electrons' speed is v_0.

Now consider an electric charge q near the wire. If the charge q is at rest, then since the wire is neutral, q will feel no force. But let's assume that the charge q is moving to the right with speed v, as shown in Fig. 1.50. What are the charge densities of the protons and electrons in q's rest frame? What then is the net charge density of the wire in q's frame?

You will find that the charge density is nonzero, which means that the wire exerts a force on q in q's frame (and hence also in the original frame). Returning to the original frame, we conclude that an electric charge q that is *at rest* near a neutral current-carrying wire feels no force, whereas a charge q that is *moving* near a neutral current-carrying wire *does* feel a force. This force is known as the *magnetic force*. We will discuss it in the solution.

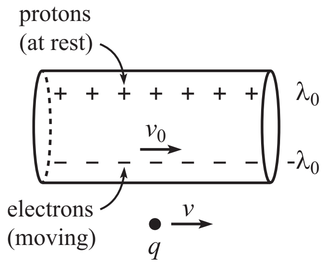

Figure 1.50

Section 1.5: Velocity addition

1.14. **Pythagorean triples** $*$

Let (a, b, h) be a Pythagorean triple. (We'll use h to denote the hypotenuse, instead of c, for obvious reasons.) Consider the relativistic addition or subtraction of two speeds with β values of $\beta_1 = a/h$ and $\beta_2 = b/h$. ($\beta \equiv v/c$ is a speed's fraction of the speed of light.) Show that the numerator and denominator of the result are the leg and hypotenuse of another Pythagorean triple, and find the other leg. What is the associated γ factor?

1.15. **Fizeau experiment** $**$

The second postulate of relativity says that the speed of light in vacuum is always c (in an inertial frame). However, the speed of light in a medium (such as water) is given by c/n, where n is the *index of refraction* of the medium. For water, n is about $4/3$.

Imagine aiming a beam of light rightward into a pipe of water moving rightward with speed v. Naively, the speed of the light with respect to the ground should be $c/n + v$. Find the correct speed by using the velocity-addition formula. Then in the case where $v \ll c$ (certainly a valid approximation in the case of moving water), show that to leading order in v, the speed takes the form of $c/n + Av$. What is the value of A?

1.16. **Equal speeds** $*$

A and B travel at $4c/5$ and $3c/5$ with respect to the ground, as shown in Fig. 1.51. How fast should C travel so that she sees A and B approaching her at the same speed? What is this speed?

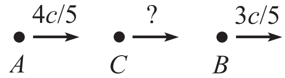

Figure 1.51

1.17. **More equal speeds** $**$

A travels at speed v with respect to the ground, and B is at rest, as shown in Fig. 1.52. How fast should C travel so that she sees A and B approaching her at the same speed? In the ground frame (B's frame), what is the ratio of the distances CB and AC? (Assume that A and C arrive at B at the same time.) The answer to this is nice and clean. Can you think of a simple intuitive explanation for the result?

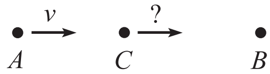

Figure 1.52

1.18. Many velocity additions $**$

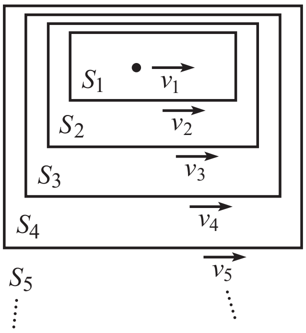

Figure 1.53

An object moves at speed $v_1/c \equiv \beta_1$ with respect to S_1 (we'll call the β's "speeds" here), which moves at speed β_2 with respect to S_2, which moves at speed β_3 with respect to S_3, and so on, until finally S_{N-1} moves at speed β_N with respect to S_N (see Fig. 1.53). Show by mathematical induction that the speed $\beta_{(N)}$ of the object with respect to S_N can be written as

$$\beta_{(N)} = \frac{P_N^+ - P_N^-}{P_N^+ + P_N^-}, \quad \text{where } P_N^+ \equiv \prod_{i=1}^{N}(1+\beta_i) \text{ and } P_N^- \equiv \prod_{i=1}^{N}(1-\beta_i). \quad (1.42)$$

1.19. Velocity addition from scratch $***$

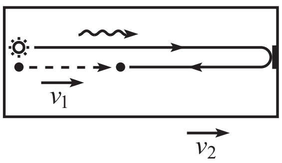

Figure 1.54

A ball moves at speed v_1 with respect to a train. The train moves at speed v_2 with respect to the ground. What is the speed of the ball with respect to the ground? Solve this problem (that is, derive the velocity-addition formula, $V = (v_1 + v_2)/(1 + v_1 v_2/c^2)$) in the following way. (Don't use time dilation, length contraction, etc. Use only the two postulates of relativity.)

Let the ball be thrown from the back of the train. At the same instant, a photon is released next to it; see Fig. 1.54. The photon heads to the front of the train, bounces off a mirror, heads back, and eventually runs into the ball. In both the train frame and the ground frame, calculate the fraction of the way along the train where the meeting occurs, and then equate these fractions.

1.20. Time dilation and Lv/c^2 $***$

A person walks very slowly at speed u from the back of a train of proper length L to the front. The time-dilation effect in the train frame can be made arbitrarily small by picking u to be sufficiently small, because the effect is second order in u, while the travel time is only first order in $1/u$. (See Problem 1.3.) Therefore, if the person's watch agrees with a clock at the back of the train when he starts, then it also (essentially) agrees with a clock at the front when he finishes.

Now consider this setup in the ground frame, where the train moves at speed v. The rear clock reads Lv/c^2 more than the front, so in view of the preceding paragraph, the time gained by the person's watch during the process must be Lv/c^2 less than the time gained by the front clock (because they agree in the end). By working in the ground frame, explain why this is the case.[9] Since we are assuming u is small, you may assume $u \ll c$.

1.21. Modified twin paradox $***$

Figure 1.55

Consider the following variation of the twin paradox. A, B, and C each have a clock. In A's reference frame, B flies past A with speed v to the right, as shown in Fig. 1.55. When B passes A, they both set their clocks to zero. Also, in A's reference frame, C starts far to the right and moves to the left with speed v. When B and C pass each other, C sets his clock to read the same as B's. Finally, when

[9] If you line up a collection of these train systems around the circumference of a circular rotating platform, then the present result implies the following fact. Let person A be at rest on the platform at a point on the circumference, and let person B start at A and walk arbitrarily slowly around the circumference. Then when B returns to A, B's clock will read less than A's. This is true because the above reasoning shows (as you will figure out) that an inertial observer sees B's clock running slower than A's. This result, that you can walk arbitrarily slowly in a particular reference frame and have your clock lose synchronization with other clocks, is a consequence of the fact that in some accelerating reference frames it is impossible to produce a consistent method (that is, one without a discontinuity) of clock synchronization. See Cranor *et al.* (2000) for more details.

C passes A, they compare the readings on their clocks. At this moment, let A's clock read T_A, and let C's clock read T_C.

(a) Working in A's frame, show that $T_C = T_A/\gamma$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

(b) Working in B's frame, show again that $T_C = T_A/\gamma$.

(c) Working in C's frame, show again that $T_C = T_A/\gamma$.

1.8 Exercises

Section 1.3: The fundamental effects

1.22. **Effectively speed c** $*$

A rocket flies between two planets that are one light-year apart. What should the rocket's speed be so that the time elapsed on the captain's watch is one year?

1.23. **A passing train** $*$

A train with length $15\,c$s moves at speed $3c/5$. (1 cs is one "light-second." It equals $(1)(3 \cdot 10^8\ \text{m/s})(1\ \text{s}) = 3 \cdot 10^8\ \text{m}$.) How much time does it take to pass a person standing on the ground, as measured by that person? Solve this by working in the frame of the person, and then again by working in the frame of the train.

1.24. **Simultaneous waves** $*$

Alice flies past Bob at speed v. Right when she passes, they both set their watches to zero. When Alice's watch shows a time T, she waves to Bob. Bob then waves to Alice simultaneously (as measured by him) with Alice's wave (so this is before he actually *sees* her wave). Alice then waves to Bob simultaneously (as measured by her) with Bob's wave. Bob then waves to Alice simultaneously (as measured by him) with Alice's second wave. And so on. What are the readings on Alice's watch for all the times she waves? And likewise for Bob?

1.25. **Overtaking a train** $**$

Train A has proper length L. Train B moves past A (on a parallel track, facing the same direction) with relative speed $4c/5$ (as measured by either train; so each one sees the other move at $4c/5$). The length of B is such that A says that the fronts of the trains coincide at exactly the same time as the backs coincide. What is the time difference between the fronts coinciding and the backs coinciding, as measured by B? Solve this in two ways: (a) by using length contraction, and (b) by using the rear-clock-ahead effect (among other things).

1.26. **Walking on a train** $**$

A train with proper length L and speed $3c/5$ approaches a tunnel with length L. At the moment the front of the train enters the tunnel, a person leaves from the front of the train and walks (briskly) toward the back. She arrives at the back of the train right when it (the back) leaves the tunnel.

(a) How much time does this take in the ground frame?

(b) What is the person's speed with respect to the ground?

(c) How much time elapses on the person's watch?

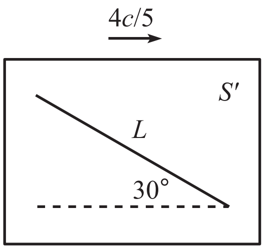

Figure 1.56

1.27. **Diagonal stick** ∗∗

In frame S' a stick with proper length L is at rest and is tilted at 30°. Frame S' moves to the right at speed $4c/5$ with respect to the ground. See Fig. 1.56.

(a) In the ground frame, what is the horizontal span of the stick?

(b) You stand far away from the stick (below it in the picture). When you see the stick at its closest point to you, what is the *apparent* horizontal span of the stick? That is, what would a photograph show, taken from a distant camera?

1.28. **Triplets** ∗∗

Triplet A stays on the earth. Triplet B travels at speed $4c/5$ to a planet (a distance L away) and back. Triplet C travels out to the planet at speed $3c/4$, and then returns at the necessary speed to arrive back exactly when B does. How much does each triplet age during this process? Who is youngest?

1.29. **Seeing the light** ∗∗

A and B leave from a common point (with their clocks both reading zero) and travel in opposite directions with relative speed v (that is, each sees the other move with speed v). When B's clock reads T, he sends out a light signal. When A receives the signal, what time does her clock read? Answer this by doing the calculation entirely in (a) A's frame, and then (b) B's frame. (This problem is basically a derivation of the longitudinal Doppler effect, discussed in Section 2.5.1. It is one of the few cases where we're actually concerned with the time is takes light to reach someone's eye.)

1.30. **Twin paradox and Lv/c^2** ∗∗

In the twin-paradox example near the end of Section 1.3.2, we noted that B (the traveler) is in different inertial frames for the outward and return trips. In B's frame during the outward trip (as the universe flies past B, from B's point of view), the star clock is the rear clock. But in B's (new) frame during the return trip, the earth clock is the rear clock. During B's turnaround, the earth clock therefore goes from being Lv/c^2 behind the star clock, to being Lv/c^2 ahead of it (where L is the earth-star proper distance). The earth clock must therefore quickly whip ahead by $2Lv/c^2$ during the turnaround, from B's (briefly noninertial) point of view. Explain quantitatively, by working in B's frame(s), how B puts everything together to conclude that at the end of the trip, the earth clock has advanced more than B's clock, by a factor γ.

1.31. **Backward photon** ∗∗

A train with proper length L has clocks at the front and back. A photon is fired from the front to the back. Working in the train frame, we can easily say that if the photon leaves the front of the train when a clock there reads zero, then it arrives at the back when a clock there reads L/c.

Now consider this setup in the ground frame, where the train travels by at speed v. Rederive the above frame-independent result (namely, if the photon leaves the front of the train when a clock there reads zero, then it arrives at the back when a clock there reads L/c) by working *only* in the ground frame.

1.32. **Pole's clocks in barn** ∗∗

Consider the pole-in-barn setup from Problem 1.6. In the solution to that problem, we stated that in the barn frame, the pole's right clock (when the right ends of the pole and barn coincide) reads *less* than the pole's left clock (when the left ends coincide), even though the right event happens *after* the left event. Demonstrate this by explicitly calculating the reading on the right clock when the right ends coincide. (Assume for simplicity that the left clock reads zero when the left ends coincide.) Do this by working in (a) the pole frame, and (b) the barn frame.

1.33. **Twice simultaneous** $**$

A train with proper length L moves at speed v with respect to the ground. When the front of the train passes a tree on the ground, a ball is simultaneously (as measured in the *ground frame*) thrown from the back of the train toward the front, with speed u with respect to the train. What should u be so that the ball hits the front simultaneously (as measured in the *train frame*) with the tree passing the back of the train? Show that in order for a solution for u to exist, we must have $v/c < (\sqrt{5} - 1)/2$, which happens to be the inverse of the golden ratio.

1.34. **People clapping** $**$

Two people stand a distance L apart along an east-west road. They clap their hands simultaneously in the ground frame. You are driving eastward along this road at speed $4c/5$. You notice that you are next to the western person at the same instant (as measured in your frame) that the eastern person claps. Later on, you notice that you are next to a tree at the same instant (as measured in your frame) that the western person claps. Where is the tree along the road? (Describe its location in the ground frame.)

1.35. **Photon, tree, and house** $**$

(a) A train with proper length L moves at speed v with respect to the ground. At the instant the back of the train passes a tree, someone at the back of the train shines a photon toward the front. The photon happens to hit the front of the train at the instant the front passes a house. As measured in the ground frame, how far apart are the tree and the house? Solve this by working in the ground frame.

(b) Now look at the setup from the point of view of the train frame. Using your result for the tree-house distance from part (a), verify that the house meets the front of the train at the same instant the photon meets it.

1.36. **Four clock readings** $**$

A train has proper length L. In the frame of the train, a photon is fired from the back of the train to the front. Assume that a clock at the back reads zero when the photon is fired. Then a clock at the front of course reads L/c when the photon arrives there.

Now consider the setup in the ground frame, where the train moves to the right at speed $3c/5$. In this frame, it is observed that the train enters a tunnel when the photon is fired, and the train leaves the tunnel when the photon arrives at the front, as shown in Fig. 1.57.

(a) What is the (proper) length of the tunnel?

(b) What are the four R_i readings on the train clocks at the two instants shown?

(c) As observed in the ground frame, verify that the time elapsed on a given train clock is related to the time elapsed on a ground clock by the appropriate γ factor. (So you will need to first determine the ground time via a method other than time dilation.)

(d) Now return to the train frame. Draw a reasonably accurate picture of what things look like at the instant the photon is fired. Label all distances necessary to describe the location of the tunnel.

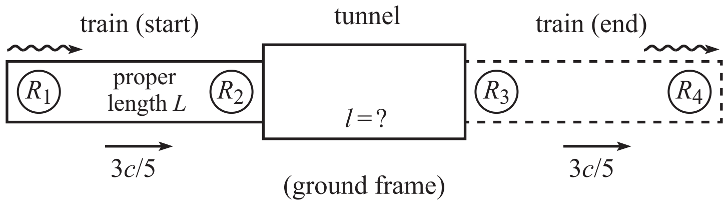

Figure 1.57

1.37. **Tunnel fraction** $**$

A person runs with speed v toward a tunnel of length L. A light source is located at the far end of the tunnel. At the instant the person enters the tunnel, the light source simultaneously (as measured in the tunnel frame) emits a photon that travels down the tunnel toward the person. When the person and the photon eventually meet, the person's location is a fraction f along the tunnel. What is f? Solve this by working in the tunnel frame, and then again by working in the person's frame.

1.38. **Through the hole?** $***$

A stick with proper length L moves at speed v in the direction of its length. It passes over a infinitesimally thin sheet that has a hole of diameter L cut in it. As the stick passes over the hole, the sheet is raised so that the stick passes through the hole and ends up underneath the sheet. Well, maybe . . .

In the lab frame, the stick's length is contracted to L/γ, so it appears to easily make it through the hole. But in the stick frame, the hole is contracted to L/γ, so it appears that the stick does *not* make it through the hole (or rather, the hole doesn't make it around the stick, since the hole is what is moving in the stick frame). So the question is: Does the stick end up on the other side of the sheet or not?

1.39. **Short train in a tunnel** $**$

Consider the scenario in Problem 1.7, with the only change being that the train now has length rL, where r is some numerical factor. What is the largest value of r, in terms of v, for which it is possible for the bomb not to explode? Verify that you obtain the same answer working in the train frame and in the tunnel frame.

1.40. **Charge density** $**$

Consider the setup in Problem 1.13. If the electrons' speed is $v_0 = 4c/5$ and the charge q's speed is $v = 3c/5$, what is the charge density of the wire in q's rest frame? Solve this from scratch; that is, don't just invoke the result in Eq. (1.59) (although you can of course check your answer with that result; likewise for any intermediate steps).

Section 1.5: Velocity addition

1.41. **γ's for relativistic addition** $*$

Show that the relativistic addition (or subtraction) of the velocities u and v has a γ factor given by $\gamma = \gamma_u \gamma_v (1 \pm uv)$, or $\gamma_u \gamma_v (1 \pm uv/c^2)$ with the c's.

1.42. **Equal speeds** $*$

A travels at speed $4c/5$ toward B, who is at rest. C is between A and B. How fast should C travel so that she sees both A and B approaching her at the same speed? (This problem is a special case of Problem 1.17. Solve this one from scratch, but feel free to check you answer with the one from Problem 1.17.)

1.43. **Running away** $*$

A and B both start at the origin and simultaneously head off in opposite directions, each with speed $3c/5$ with respect to the ground. A moves to the right, and B moves to the left. Consider a mark on the ground at $x = L$. As viewed in the ground frame, A and B are a distance $2L$ apart when A passes this mark. As viewed by A, how far away is B when A coincides with the mark?

1.44. **Again simultaneous** $**$

A train with proper length L moves at speed v with respect to the ground. When the front of the train passes a tree on the ground, a ball is simultaneously (as measured in the ground frame) thrown from the back of the train toward the front, with speed u with respect to the train. What should u be so that the ball hits the front simultaneously (as measured again in the ground frame) with the back of the train passing the tree? What is the maximum value of v for which a solution for u exists?

1.45. **Overlapping trains** $**$

An observer on the ground sees two trains, A and B, both with proper length L, move in opposite directions at speed v with respect to the ground. She notices that at the instant the trains overlap, clocks at the front of A and rear of B both read zero, as shown in Fig. 1.58. From the rear-clock-ahead effect, she therefore also notices that clocks at the rear of A and front of B read Lv/c^2 and $-Lv/c^2$, respectively. Now imagine riding along on A. When the rear of B passes the front of your train (A), clocks at both of these places read zero (a frame-independent statement). Explain, by working only in the frame of A, why clocks at the back of A and the front of B read Lv/c^2 and $-Lv/c^2$, respectively, when these points coincide.

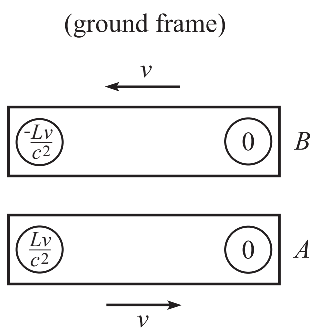

Figure 1.58

1.46. **Running on a train** $**$

A train with proper length L moves at speed v_1 with respect to the ground. A passenger runs from the back of the train to the front at speed v_2 with respect to the train. How much time does this take, as viewed by someone on the ground? Solve this in two different ways:

(a) Find the relative speed of the passenger and the train (as viewed by someone on the ground), and then find the time it takes for the passenger to erase the initial head start that the front of the train had.

(b) Find the time elapsed on the passenger's clock (by working in whatever frame you want), and then use time dilation to get the time elapsed on a ground clock.

1.47. **Velocity addition** $**$

The fact that the previous exercise can be solved in two different ways suggests a method of deriving the velocity-addition formula: A train with proper length L moves at speed v_1 with respect to the ground. A ball is thrown from the back of the train to the front at speed v_2 with respect to the train. Let the speed of the ball with respect to the ground be V. Calculate the time of the ball's journey, as measured by an observer on the ground, in the two different ways described in the previous exercise, and then equate the results to solve for V in terms of v_1 and v_2. (This gets rather messy. And yes, you have to solve a quadratic.)

1.48. **Velocity addition again** $**$

A train with proper length L moves at speed v with respect to the ground. A ball is thrown from the back of the train to the front at speed u with respect to the train.

(a) Find the time of the process in the ground frame by looking at how much a clock at rest in the train frame advances (for example, a clock at the front of the train), and then applying time dilation to this one clock.

(b) Find the time of the process in the ground frame by applying time dilation to the ball's clock. Your answer will contain the unknown speed V of the ball with respect to the ground.

(c) Equate your results from parts (a) and (b) to show that $\gamma_V = \gamma_u \gamma_v (1+uv/c^2)$. Then solve for V to produce the velocity-addition formula.

1.49. **Bullets on a train** $**$

A train moves at speed v. Bullets are successively fired at speed u (relative to the train) from the back of the train to the front. A new bullet is fired at the instant (as measured in the train frame) the previous bullet hits the front. In the frame of the ground, what fraction of the way along the train is a given bullet, at the instant (as measured in the ground frame) the next bullet is fired? What is the maximum number of bullets that are in flight at a given instant, in the ground frame?

1.9 Solutions

1.1. **Consistency with Lv/c^2**

In the setup that led to Eq. (1.6), the two events (light hitting rear and light hitting front) were simultaneous in the train frame, because the light source was located at the center of the train. The difference $t_R - t_L$ is the time difference between these events, as measured in the ground frame. From Eq. (1.6) we have

$$t_R - t_L = \frac{\ell'}{c-v} - \frac{\ell'}{c+v} = \frac{2\ell' v}{c^2 - v^2} = \frac{2\ell' v}{c^2(1 - v^2/c^2)} = \frac{2\gamma^2 \ell' v}{c^2} . \qquad (1.43)$$

Now, $2\ell'$ is the total length of the train, as measured in the ground frame. But due to length contraction, this length is shorter than the train's proper length L, by a factor γ. That is, $2\ell' = L/\gamma$. Substituting this into Eq. (1.43) gives

$$t_R - t_L = \frac{\gamma L v}{c^2} . \qquad (1.44)$$

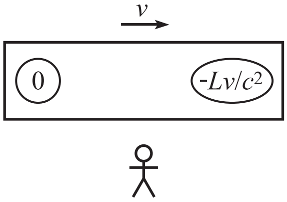

Figure 1.59

This is consistent with the rear-clock-ahead result of Lv/c^2, for the following reason. Since the events are simultaneous in the train frame, clocks at the front and rear of the train have the same reading when the photons hit. Assume for concreteness that this reading is zero. Then as viewed in the ground frame, at the instant the rear event occurs, the situation is shown in Fig. 1.59. The rear clock reads zero (a frame-independent statement), and the front clock reads $-Lv/c^2$ due to the rear-clock-ahead effect. The ground observer must then wait for the front clock to advance to zero, at which point the front event occurs (again a frame-independent statement). But the front clock (along with every other clock on the train) runs slow due to time dilation. So it takes a time (in the ground frame) of $\gamma Lv/c^2$ to advance its reading by Lv/c^2 to zero. This elapsed time of $\gamma Lv/c^2$ in the ground frame agrees with the $t_R - t_L$ result in Eq. (1.44).

1.2. **Here and there**

If the setup is to be possible, then in the train frame the person must run the length L of the train in a time Lv/c^2 (or slightly less). His speed with respect to the train must therefore be at least $L/(Lv/c^2) = c^2/v$. But $c^2/v = c(c/v) > c$, which is an impossible speed. So it is not possible for the person to perform the stated task. You will therefore *not* see him simultaneously at both the front and the back. This is good, because we could produce all sorts of paradoxes if someone were actually at two places at once in a given frame. Imagine a brick wall being constructed between the "two" people, and a bucket of paint being dropped on one of them.

1.3. **Synchronizing clocks**

In the frame of A and B, it takes a time of L/v for C to travel from A to B. During this time, C runs slow by a factor γ, so only $L/\gamma v$ elapses on C during the journey. Therefore, when C reaches B, the reading on C is $L/\gamma v$, and the reading on B is L/v. If v is small (more precisely, if $v \ll c$), we can use $\sqrt{1-\epsilon} \approx 1 - \epsilon/2$ to approximate C's reading as

$$\frac{L}{\gamma v} = \frac{L}{v}\sqrt{1-\frac{v^2}{c^2}} \approx \frac{L}{v}\left(1-\frac{v^2}{2c^2}\right) = \frac{L}{v} - \frac{Lv}{2c^2}. \tag{1.45}$$

The difference between C's and B's readings is therefore $Lv/2c^2$. This goes to zero as $v \to 0$, as desired. We see that even though the total time L/v in A's and B's frame goes to infinity as $v \to 0$, the $Lv/2c^2$ difference between C's and B's readings goes to zero. This is due to the fact that L/v has only one power of v in the denominator, whereas the γ factor depends quadratically on v. If the γ factor were instead equal to $1/\sqrt{1-v/c}$, then the difference in the readings would take on the fixed value of $L/2c$ in the $v \to 0$ limit.

1.4. **Rotated square**

Fig. 1.60 shows the square at the instant (in your frame) when it is closest to you. Its length is contracted along the direction of motion, so it takes the shape of a rectangle with sides L and L/γ. This is what the shape *is* in your frame (where *is*-ness is defined by where all the points of an object are at simultaneous times). But what does the square *look* like to you? That is, what is the nature of the photons hitting your eye at a given instant?

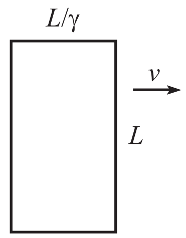

Figure 1.60

Photons from the far side of the square have to travel an extra distance L to get to your eye, compared with photons from the near side. This is true because we're assuming that you are far from the square, which means that the paths to you from the various points on the square are all essentially parallel. If you were instead close to the square, then we would have to use the Pythagorean theorem to obtain the distances, and things would be much more difficult and messy.

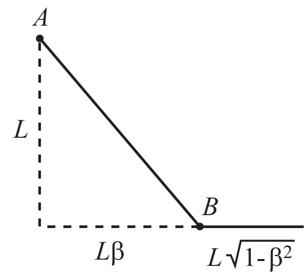

Figure 1.61

The photons from the far side need an extra time L/c of flight, compared with the photons from the near side. During this time L/c, the square moves a distance $v(L/c) \equiv L\beta$ sideways, where $\beta \equiv v/c$. Therefore, referring to Fig. 1.61, a photon emitted at point A reaches your eye at the same time as a photon emitted from point B, as do all the photons emitted from the near side, of course, and as do (as you can verify) all the photons emitted from the trailing (left) side, between A and B. This means that the trailing side of the square spans a distance $L\beta$ across your field of vision, while the near side spans a distance $L/\gamma = L\sqrt{1-\beta^2}$. But this is exactly what a rotated square of side L looks like, as shown in Fig. 1.62. From the figure, we see that the angle of rotation is given by $\sin\theta = \beta$, or equivalently $\cos\theta = \sqrt{1-\beta^2}$. So for $v \ll c$ the square is only slightly rotated, while for $v \to c$ the rotation angle approaches 90°. For the case of a circle instead of a square, see Hollenbach (1976).

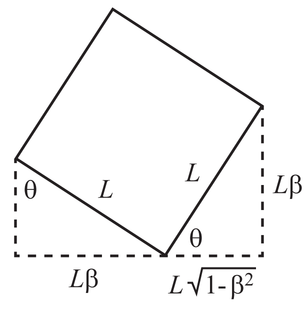

Figure 1.62

1.5. **Deriving length contraction**

In the ground frame, the given rule tells us that the length of the train is aL. (We'll drop the subscript v for convenience.) So the front of the train has a head start of aL over the photon. The photon closes this gap at a relative speed of $c - v$, as measured in the ground frame. The time of the process is therefore $t = aL/(c-v)$. During this time, the photon travels a distance $ct = caL/(c-v)$. This then is the tree-house distance in the ground frame.

Now consider the setup in the train frame. The starting and ending pictures are shown in Fig. 1.63.

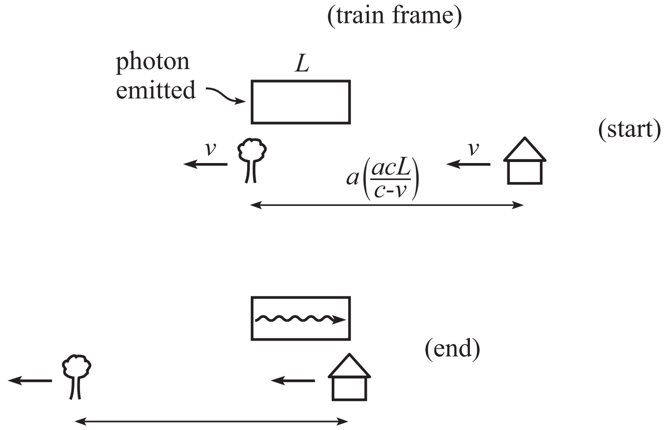

Figure 1.63

The photon is emitted when the tree coincides with the back of the train (a frame-independent statement). The train has length L (its proper length), and the tree-house distance is $a(acL/(c-v))$. This is true because our rule states that if an object is moving, the observed length equals a times the proper length, which we found above to be $acL/(c-v)$ for the tree-house separation. The initial distance between the house and the front of the train is therefore $a^2cL/(c-v) - L$ (subtracting off the train's length). The house covers this distance at speed v. The photon covers the length L of the train at speed c. Since the house and the photon arrive at the front of the train at the same time, we must therefore have

$$\frac{\frac{a^2cL}{c-v} - L}{v} = \frac{L}{c} \quad \Longrightarrow \quad \frac{a^2}{1-\frac{v}{c}} - 1 = \frac{v}{c} \quad \Longrightarrow \quad a^2 = 1 - \frac{v^2}{c^2}$$

$$\Longrightarrow \quad a = \sqrt{1 - \frac{v^2}{c^2}} \equiv \frac{1}{\gamma}, \tag{1.46}$$

as desired.

1.6. **Pole in barn**

The proper answer to the question is: The question cannot be answered without more information. More precisely, the question is frame dependent; it must be finished with the qualifier, "... in the barn frame," or "... in the pole frame." The question is indeed frame dependent, because in the barn frame the pole is length contracted down to L/γ. So the pole certainly fits inside the barn; see Fig. 1.64. (For the purpose of drawing the figure, we have chosen γ to be a slightly larger than 3.) But in the pole frame the barn is length contracted down to L/γ. So the pole certainly *doesn't* fit in the barn.

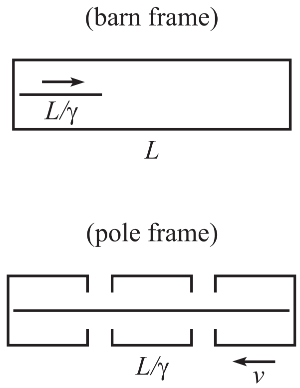

Figure 1.64

In retrospect, it is no surprise that the question is frame dependent, because the qualifier "with both doors closed" is shorthand for "with both doors closed *simultaneously*." And as soon as we start talking about simultaneity, we know that frame dependence will come into play, because events that are simultaneous in one frame are not simultaneous in another. We purposely didn't include the word "simultaneously" in the statement of the problem, in order to not give too much of a hint.

Remark: Fig. 1.64 makes it clear that the order of the two events, "left ends coinciding" and "right ends coinciding," is reversed in the two frames. In the barn frame the left ends coincide first, whereas in the pole frame the right ends coincide first. There is nothing wrong with the order of events in one frame being different from the order in another.

However, for this to be possible, the two events must be so-called *spacelike separated*. We'll discuss this term in Section 2.3.

Consider two clocks at the ends of the pole (synchronized in the pole frame). Assume that the left clock reads zero when it coincides with the left end of the barn. Then from the pole-frame view in Fig. 1.64, it is clear that the right clock has a *negative* reading when it coincides with the right end of the barn, because the right event happens *before* the left event (and because the pole clocks are synchronized in the pole frame; this fact is important.) These readings of zero (on the left clock when the left ends coincide) and a negative value (on the right clock when the right ends coincide) are frame independent, so they must be the same in the barn frame. In other words, in the barn frame the pole's right clock must read *less* than the left clock (at the two events at hand), even though the right event happens *after* the left event. The task of Exercise 1.32 is to explain explicitly why this is true, and to calculate the actual reading on the right clock. ♣

1.7. **Train in a tunnel**

Yes, the bomb explodes. This is clear in the frame of the train; see Fig. 1.65. In this frame, the train has length L, and the tunnel speeds past it. The tunnel is length contracted down to L/γ. Therefore, the far end of the tunnel passes the front of the train before the near end passes the back. So the bomb explodes.

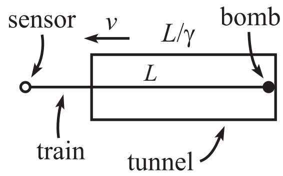

Figure 1.65

We can, however, also look at things in the frame of the tunnel; see Fig. 1.66. Here the tunnel has length L, and the train is length contracted down to L/γ. Therefore, the deactivation device gets triggered *before* the front of the train passes the far end of the tunnel. So you might think that the bomb does *not* explode. However, all observers must agree on whether or not the bomb explodes; the explosion (or lack thereof) is frame independent. So we appear to have a paradox.

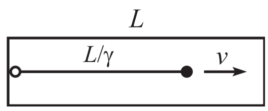

Figure 1.66

The resolution to this paradox is that the deactivation device cannot instantaneously tell the bomb to deactivate itself; the signal can't travel faster than the speed of light. (If signals could travel faster than c, we would be able to generate setups that violate causality. We'll talk about this in Section 2.3.) It therefore takes a nonzero time for the signal to travel the length of the train (or actually a longer distance, since the train is moving) from the sensor to the bomb. And it turns out that this transmission time makes it impossible for the deactivation signal to get to the bomb before the bomb gets to the far end of the tunnel, no matter how fast the train is moving. The bomb therefore explodes. Let's quantitatively demonstrate this.

The signal has the best chance of winning the "race" if it has speed c, so let's assume this is the case. The time it takes the signal to reach the bomb is $(L/\gamma)/(c-v)$, because the train has length L/γ in the ground frame, and because the relative speed of the light and the bomb is $c-v$ in the ground frame. The time it takes the bomb to get to the far end of the tunnel is $(L-L/\gamma)/v$, because the bomb is already a distance L/γ through the tunnel, and because it is moving at speed v. So if the bomb is *not* to explode, the former of these times must be less than the latter. With $\beta \equiv v/c$, this gives

$$\begin{aligned} \frac{L/\gamma}{c-v} < \frac{(L-L/\gamma)}{v} \quad &\Longleftrightarrow \quad \frac{1}{\gamma}\left(\frac{1}{1-\beta}+\frac{1}{\beta}\right) < \frac{1}{\beta} \\ \Longleftrightarrow \quad \sqrt{1-\beta^2}\cdot\frac{1}{(1-\beta)\beta} < \frac{1}{\beta} \quad &\Longleftrightarrow \quad \sqrt{1-\beta^2} < 1-\beta \\ &\Longleftrightarrow \quad \sqrt{1+\beta} < \sqrt{1-\beta}. \end{aligned} \tag{1.47}$$

This is never true. Therefore, the signal always arrives too late, and the bomb always explodes, consistent with the conclusion in the train frame.

1.8. **Bouncing stick**

Assume that a series of clocks are lined up along the stick, and assume that in the ground frame they all read zero when the stick bounces. In the frame of someone running leftward at speed v, the stick is moving rightward (and vertically). The rear (left) clock on the stick is ahead of all the other clocks, so it will reach zero and bounce off the ground first. (It is a frame-independent fact that a clock reads zero when it bounces off the ground.) Clocks along the stick will successively reach zero and the stick will bounce at those points, until finally the clock at the front (right) end reads zero and that end bounces. Snapshots of the stick therefore look like the ones shown in Fig. 1.67.

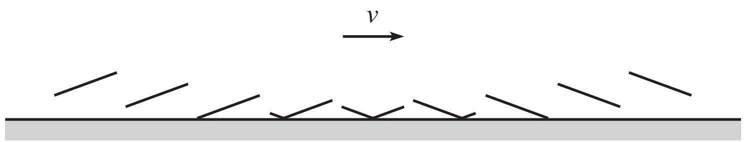

Figure 1.67

There is nothing wrong with the stick having a sharp bend in it. The stick doesn't break in the ground frame, so it doesn't break in the person's frame, either. The sharp bend doesn't imply any severe forces in the stick. The molecules in the stick think everything is perfectly normal; they have no clue that someone in running by to the left and that the stick is bent in this person's frame.

Remark: If we want to get quantitative, we can calculate the angle the stick makes with respect to the horizontal, in the person's frame. Although the above reasoning involved clocks that were at rest in the *stick* frame, it will be easier in the following reasoning to work with a set of clocks that are at rest in the *ground* frame.

Let's work in the ground frame for a moment. Assume that the two ends of the stick slide down two vertical rails. If the proper length of the stick is L, then the rails are a distance L apart, because there is no transverse length contraction. Imagine that a large number of clocks are attached to the two rails, to make two towers of clocks (at rest in the ground frame). Let all of the clocks in the towers read zero when the two ends of the stick (along with the rest of the stick) bounce off the ground simultaneously in the ground frame. For future reference, note that at a time Lv/c^2 (as measured in the ground frame) before the stick bounces (that is, when all of the clocks in the towers read $-Lv/c^2$), the stick is at a height $u(Lv/c^2)$ above the ground, where u is the stick's (vertical) speed in the ground frame. (Assume that this speed is essentially constant; ignore the vertical acceleration near the ground.)

Now go back to the person's frame. When the back end of the stick hits the ground, all of the clocks in the back vertical tower read zero (a frame-independent statement), and all of the clocks in the front vertical tower read $-Lv/c^2$, due to the rear-clock-ahead effect. But from the preceding paragraph, we know that the front of the stick is at a height $u(Lv/c^2)$ above the ground (in the ground frame and hence also in the person's frame, since there is no transverse length contraction) when it is next to a tower clock that reads $-Lv/c^2$. The horizontal distance between the ends is L/γ_v, because this is the length-contracted distance between the two rails. The angle that the stick makes with the horizontal in the person's frame is therefore given by

$$\tan\theta = \frac{Luv/c^2}{L/\gamma_v} = \frac{\gamma_v uv}{c^2}\,. \qquad ♣ \tag{1.48}$$

1.9. **Seeing behind the board**

First note that the reasonings can't both be correct, because the closest mark you can see has a frame-independent value. It can't depend on which frame we arbitrarily choose to do the calculation in.

The first reasoning is the correct one. You will be able to see a mark on the ruler that is less than L units from the wall. You will actually be able to see a mark even closer to the wall than L/γ, as we'll show below.

The error in the second reasoning (in the board's frame) is that the second picture in Fig. 1.48 is *not* what *you* see. This second picture shows where things are at simultaneous times in the *board's* frame, which are not simultaneous times in *your* frame. Alternatively, the error is the implicit assumption that signals travel instantaneously; but in fact the back (left) end of the board cannot know that the front (right) end has been hit by the wall until a nonzero time has passed. During this time, the ruler (and the wall and you) travels farther to the left, allowing you to see more of the ruler. Let's be quantitative about this and calculate (in both frames) the closest mark to the wall that you can see.

Your frame: In your reference frame, the board has length L/γ. Therefore, when the board hits the wall, you can see a mark a distance L/γ from the wall. You will, however, be able to see a mark even closer to the wall, because the back end of the board will keep moving forward, since it doesn't know yet that the front end has hit the wall. The stopping signal (shock wave, etc.) takes time to travel.

Let's assume that the stopping signal travels along the board at speed c. (We could instead work with a general speed u, but the speed c is simpler, and it yields an upper bound on the closest mark you can see.) Where will the signal reach the back end of the board? Starting from the time the board hits the wall, the signal travels backward from the wall at speed c, while the back end of the board travels forward at speed v (from a point L/γ away from the wall). The relative speed (as viewed by you) of the signal and the back end is $c + v$. This is the rate at which the initial gap of L/γ is closed. Therefore, the signal hits the back end after a time $(L/\gamma)/(c + v)$. During this time, the signal has traveled a distance $c \cdot (L/\gamma)/(c + v)$ from the wall. This is where the back end stops. The closest point to the wall that you can see on the ruler is therefore the mark with the value (with $\beta \equiv v/c$)

$$\frac{c(L/\gamma)}{c+v} = \frac{L}{\gamma(1+\beta)} = \frac{L\sqrt{1-\beta^2}}{1+\beta} = L\sqrt{\frac{1-\beta}{1+\beta}}\,. \tag{1.49}$$

Board frame: In the board's reference frame, the wall is initially moving leftward with speed v. After the wall hits the right end of the board, the signal moves to the left with speed c, while the wall keeps moving to the left with speed v (because the wall/earth is much more massive than the board). Where is the wall when the signal reaches the left end of the board (at which point the left end starts moving leftward along with the ruler)? The wall travels v/c as fast as the signal, so it travels a distance Lv/c in the time that the signal travels the distance L. This means that the wall is $L(1 - v/c)$ away from the left end of the board when the signal reaches the left end. This distance in the board's frame (or rather, in the board's original frame) corresponds to a distance $\gamma L(1 - v/c)$ on the ruler, because the (moving) ruler is length contracted. So the left end of the board is at the mark with the value

$$\gamma L(1 - v/c) = L\gamma(1-\beta) = \frac{L(1-\beta)}{\sqrt{1-\beta^2}} = L\sqrt{\frac{1-\beta}{1+\beta}}\,, \tag{1.50}$$

in agreement with Eq. (1.49).

1.10. **Cookie cutter**

Let the diameter of the cookie cutter be L, and consider the two following reasonings.

- In the lab frame, the dough is length contracted, so the cutter's diameter L corresponds to a distance larger than L (namely γL) in the dough frame. Therefore, when you buy a cookie, it is stretched by a factor γ in the direction of the belt.[10]
- In the dough frame, the cookie cutter is length contracted down to L/γ in the direction of motion. So in the frame of the dough, the cookies have a length of only L/γ. Therefore, when you buy a cookie, it is squashed by a factor γ in the direction of the belt.

Which reasoning is correct? The first one is. The cookies are stretched out. The fallacy in the second reasoning is that the various parts of the cookie cutter do *not* strike the dough

[10]The shape is an ellipse, since that's what a stretched circle is. The eccentricity of an ellipse is the focal length divided by the semi-major axis length. As an exercise, you can show that this equals $\beta \equiv v/c$ here.

simultaneously in the dough frame. What the dough sees is this: Assuming that the cutter moves to the left, the rightmost point on the cutter stamps the dough, then nearby points on the cutter stamp it, and so on, until finally the leftmost point stamps it. But by this time the front (that is, the left) of the cutter has moved farther to the left. So the cookie turns out to be longer than L. Let's now show (by working in the dough frame) that the length of the cookie is in fact γL, as the first of the above reasonings correctly states.

Assume that all points on the cutter have little clocks associated with them. Since all points on the cutter strike the dough simultaneously in the lab frame (since the cutter is horizontal), all of the clocks have the same reading when they strike the dough. Let's assume that this value is zero.

Now let's look at what happens in the dough frame. Consider the moment when the rightmost (rear) point on the cutter strikes the dough. The clock there reads zero (a frame-independent reading). The clock at the leftmost (front) point on the cutter therefore reads $-Lv/c^2$, due to the rear-clock-ahead effect. This leftmost clock must then advance by Lv/c^2 by the time it strikes the dough when it reads zero (again a frame-independent reading). However, due to time dilation, this takes a time $\gamma(Lv/c^2)$ in the dough frame. During this time, the cutter travels a distance $v(\gamma Lv/c^2)$. But the front of the cutter was initially a distance L/γ (due to length contraction) ahead of the back. The total length of the cookie in the dough frame is this initial L/γ distance plus the extra $v(\gamma Lv/c^2)$ distance traveled by the front. The total length is therefore

$$\ell = \frac{L}{\gamma} + \frac{\gamma L v^2}{c^2} = \gamma L\left(\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right) = \gamma L\left(\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\right) = \gamma L, \tag{1.51}$$

as we wanted to show. (This is the same calculation as in Eq. (1.23).) If the dough is then slowly decelerated, the shape of the cookies won't change. So this is the shape you see in the store.

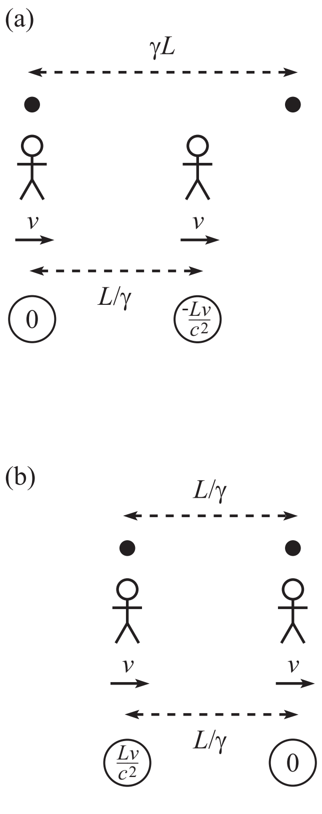

Figure 1.68

1.11. **Getting shorter**

(a) In the frame in which the balls are initially at rest, the people move rightward with speed v. The beginning picture is shown in Fig. 1.68(a). The left person catches the left ball when his clock reads zero (a frame-independent fact). The right person's clock therefore reads $-Lv/c^2$, due to the rear-clock-ahead effect. The balls are a distance γL apart, and the people's separation is length contracted down to L/γ.

The ending picture is shown in Fig. 1.68(b). The right person catches the right ball when his clock reads zero (again a frame-independent fact). The left person's clock is ahead, so it reads Lv/c^2. By the time the right person catches the ball, the left person has moved to the right while holding the left ball. Both distances are now L/γ.

(b) By looking at the two distances in Fig. 1.68(a), we see that the people travel a distance $\gamma L - L/\gamma$ between the beginning and ending moments.

Let's now use the clock readings to obtain the distance the people travel. The total time of the process is $\gamma(Lv/c^2)$ because each person's clock advances by Lv/c^2, but these clocks run slow in the frame we're working in. Since the speed of the people is v, the distance they travel is $v(\gamma Lv/c^2)$. This had better be equal to $\gamma L - L/\gamma$. And it is, because

$$\gamma L - \frac{L}{\gamma} = \gamma L\left(1 - \frac{1}{\gamma^2}\right) = \gamma L\left(1 - \left(1 - \frac{v^2}{c^2}\right)\right) = \frac{\gamma L v^2}{c^2}. \tag{1.52}$$

If we then shift to the people's frame where which everything is at rest, we see that the proper distance between the balls is L, because this is what gets length contracted down to the L/γ in Fig. 1.68(b).

(c) To sum up, the proper distance between the balls decreases because in the frame in which the balls are initially at rest, the left person catches the left ball first and then drags it closer to the right ball by the time the right person catches that ball. So it all comes down to the loss of simultaneity.

1.12. **Transforming the length**

The proper length of the stick is $\gamma_{3/5}L = 5L/4$, because this is the length that is contracted down to L in the ground frame. Using the velocity-addition (or rather, subtraction) formula, the speed at which you see the stick move rightward is

$$\frac{\frac{3c}{5} - \frac{c}{2}}{1 - \frac{3}{5} \cdot \frac{1}{2}} = \frac{\frac{c}{10}}{\frac{7}{10}} = \frac{c}{7}. \tag{1.53}$$

The length that you observe is obtained by contracting the proper length by the γ factor associated with this $c/7$ speed, which is $\gamma_{1/7} = 7/\sqrt{48}$. So the length you observe is

$$\frac{L_{\text{proper}}}{\gamma_{1/7}} = \frac{\gamma_{3/5}L}{\gamma_{1/7}} = \frac{5L}{4} \cdot \frac{\sqrt{48}}{7} = \frac{5\sqrt{3}}{7}L \approx (1.24)L. \tag{1.54}$$

This is larger than the length L in the ground frame, because the stick is moving slower in your frame (at $c/7$) than in the ground frame (at $3c/5$). So it is contracted less in your frame, from its proper length $5L/4$.

REMARK: Note that it is *not* correct to say, "Since the length of the stick in the ground frame is L, and since you are moving with speed $c/2$ with respect to the ground, you observe the length of the stick to be $L/\gamma_{1/2} = \sqrt{3}L/2$." This is incorrect because the standard length-contraction result applies only to the *proper* length. Length contraction says, "If a stick is moving at speed v with respect to you, then in your frame it is short (relative to its proper length) by a factor γ_v." This is why we had to first find the proper length in the above solution. You can't contract a non-proper length. (Equivalently, if you are using length contraction to go from one frame to another, the stick must be at rest in one of the frames.) This is clear in a special case: If in the above problem you are moving rightward also with speed $3c/5$, then the stick is at rest with respect to you, so you must observe a length that is *longer* than L (namely the proper length, $\gamma_{3/5}L = 5L/4$). The incorrect naive application of length contraction (contracting the ground frame's length L) would yield a length that is *shorter* than L (namely $L/\gamma_{3/5} = 4L/5$). ♣

1.13. **Magnetic force**

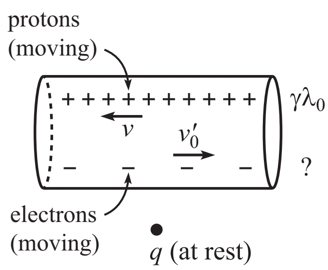

Figure 1.69

In q's rest frame, the situation is shown in Fig. 1.69. The protons are moving leftward with speed v (because they were at rest in the original frame). The distance between them is contracted by the factor $\gamma_v \equiv \gamma$, so the density is increased by the factor γ. The protons' charge density in q's frame is therefore

$$\lambda_{\text{protons}} = \gamma\lambda_0. \tag{1.55}$$

To determine the electrons' charge density in q's frame, we need to find the electrons' velocity (call it v_0') in q's frame. This is obtained via the velocity-addition (or subtraction) formula, which gives

$$v_0' = \frac{v_0 - v}{1 - v_0 v/c^2}. \tag{1.56}$$

If this is negative, then the electrons are actually moving leftward. Note that there are three different velocities that appear in this solution:

v: velocity of charge q in lab frame,
v_0: velocity of electrons in lab frame,
v_0': velocity of electrons in q's frame.

v is also the leftward speed of the protons in q's frame. The γ factors associated with each of the above three velocities will appear at various places in this solution.

Before getting quantitative, let's give a qualitative argument for why the net charge density of the wire is nonzero in q's frame. For the sake of drawing Fig. 1.69, we have assumed that $v < v_0$. The electrons are therefore still moving rightward, but with a speed (the v_0' in

Eq. (1.56)) that is smaller than v_0. Since this speed is smaller than the electrons' speed v_0 in the lab frame, the electrons are farther apart in q's frame than in the lab frame (because the proper distance between them isn't contracted by as large a factor). The electrons' charge density in q's frame is therefore *smaller* (in magnitude) than the density in the lab frame, which was λ_0. Since we found in Eq. (1.55) that the protons' charge density in q's frame is *larger* than λ_0, we see that the wire has a net positive charge. That is, it is *not* neutral in q's frame. Let's now be quantitative about what the charge density actually is.

To determine the separation between the electrons (and thereby their density) in q's frame, we can't simply contract the separation in the original frame. (See Problem 1.12 for a discussion of this erroneous method.) Instead, we must first find the proper separation (that is, in the rest frame of the electrons) and then length contract this distance by the appropriate γ factor. The proper separation between the electrons is $\gamma_{v_0} \equiv \gamma_0$ times the separation in the lab frame (because it is then contracted down to the distance in the lab frame shown in Fig. 1.50). The proper density is therefore smaller than λ_0 by a factor γ_0. So it equals $-\lambda_0/\gamma_0$, with the negative sign due to the fact that electrons are negatively charged.

The electrons' separation in q's frame is then obtained by dividing the proper separation by the γ factor associated with the speed v'_0 in Eq. (1.56); let's call this γ'_0. Equivalently, the electrons' density in q's frame is obtained by multiplying the proper density (which is $-\lambda_0/\gamma_0$) by γ'_0. So the electrons' density in q's frame equals $\gamma'_0(-\lambda_0/\gamma_0)$. We must therefore determine γ'_0. A straightforward but slightly tedious calculation gives (switching to the β notation to that we can avoid writing all the c's)

$$
\begin{aligned}
\gamma'_0 &= \frac{1}{\sqrt{1-\beta_0'^2}} = \frac{1}{\sqrt{1-\left(\dfrac{\beta_0-\beta}{1-\beta_0\beta}\right)^2}} = \frac{1-\beta_0\beta}{\sqrt{(1-\beta_0\beta)^2-(\beta_0-\beta)^2}} \\
&= \frac{1-\beta_0\beta}{\sqrt{1+\beta_0^2\beta^2-\beta_0^2-\beta^2}} = \frac{1-\beta_0\beta}{\sqrt{1-\beta_0^2}\sqrt{1-\beta^2}} \\
&= \gamma_0\gamma(1-\beta_0\beta).
\end{aligned}
\tag{1.57}
$$

The electrons' density in q's frame is therefore

$$
\lambda_{\text{electrons}} = \gamma'_0 \cdot \frac{-\lambda_0}{\gamma_0} = \gamma_0\gamma(1-\beta_0\beta)\cdot\frac{-\lambda_0}{\gamma_0} = -\gamma\lambda_0(1-\beta_0\beta). \tag{1.58}
$$

Recalling Eq. (1.55), the *total* charge density of the positive protons and the negative electrons is then

$$
\begin{aligned}
\lambda_{\text{total}} &= \lambda_{\text{protons}} + \lambda_{\text{electrons}} \\
&= \gamma\lambda_0 - \gamma\lambda_0(1-\beta_0\beta) \\
&= \gamma\beta\beta_0\lambda_0.
\end{aligned}
\tag{1.59}
$$

This is nonzero, as we noted above. Assuming that β_0 is positive, λ_{total} is positive if β is positive (that is, q is moving rightward), but negative if β is negative (that is, q is moving leftward).

Remark: We have solved the stated problem, but let's now see how the result in Eq. (1.59) leads to the magnetic force. We'll mostly just work with proportionalities instead of equalities here, lest we get bogged down with various constants and definitions that belong more in a book on electromagnetism. We'll invoke a few facts from electromagnetism in the following discussion.

In q's frame, the force on q equals the product of q and the electric field E from the charged wire. We'll just accept here (quite reasonably) that the electric field from a wire is proportional to the charge density. So the force on q in q's frame is (using Eq. (1.59))

$$
F_{\text{in } q \text{ frame}} = qE \propto q\lambda_{\text{total}} = q\gamma\beta\beta_0\lambda_0. \tag{1.60}
$$

If this is positive (that is, if q and λ_{total} have the same sign), then the force is directed away from the wire. If it is negative, the force is directed toward the wire. In Chapter 3 we'll show that the transverse force on a particle is larger in the particle's frame than in any other frame (see Eq. (3.71)). The force in the original lab frame is then

$$F_{\text{in lab}} = \frac{F_{\text{in } q \text{ frame}}}{\gamma} = q\beta\beta_0\lambda_0 \propto qvv_0\lambda_0. \tag{1.61}$$

The current I in the wire in the lab frame is proportional (actually equal) to $v_0\lambda_0$. This makes sense; the larger that v_0 or λ_0 is, the more charge that passes by a given point on the wire. Using $v_0\lambda_0 \propto I$ in Eq. (1.61) yields

$$F_{\text{in lab}} \propto qvI. \tag{1.62}$$

Finally, the magnitude of the magnetic field B due to the current in the wire in the lab frame is proportional to the current I (we'll just accept this). So we finally have

$$F_{\text{in lab}} \propto qvB. \tag{1.63}$$

Although we dealt only with proportionalities in the above reasoning, we got a bit lucky; it turns out that qvB is either exactly the correct force, or off by only a factor of c, depending on which of the two most common systems of units you use.

In the present case where q is moving parallel to the wire, the force is directed either toward or away from the wire. For other directions of q's motion, the direction of the force in the lab frame is given by the general cross-product expression: $\mathbf{F} \propto q\mathbf{v} \times \mathbf{B}$. It can be shown that the magnetic field $\mathbf{B}$ (which is a vector) points in the tangential direction around the wire (that is, the field lines form circles around the wire). In the present setup, the velocity $\mathbf{v}$ and the magnetic field $\mathbf{B}$ are perpendicular, so the magnitude of $q\mathbf{v} \times \mathbf{B}$ is simply qvB, as we found above. Note that if $\mathbf{v}$ is parallel to $\mathbf{B}$, the magnetic force is zero. ♣

1.14. **Pythagorean triples**

The relativistic addition or subtraction of the two given β's has a β value of

$$\frac{\dfrac{a}{h} \pm \dfrac{b}{h}}{1 \pm \dfrac{ab}{h^2}} = \frac{(a \pm b)h}{h^2 \pm ab}. \tag{1.64}$$

The numerator and denominator here are two lengths in a Pythagorean triple, because

$$(h^2 \pm ab)^2 - ((a \pm b)h)^2 = h^4 + a^2b^2 - (a^2 + b^2)h^2 = a^2b^2, \tag{1.65}$$

where we have used the given information that $a^2 + b^2 = h^2$. The other leg is therefore ab, for both the addition and subtraction cases. So the full triple is

$$\big((a \pm b)h,\ ab,\ h^2 \pm ab\big). \tag{1.66}$$

The γ factor associated with the speed in Eq. (1.64) is

$$\gamma = \frac{1}{\sqrt{1 - \left(\dfrac{(a \pm b)h}{h^2 \pm ab}\right)^2}} = \frac{h^2 \pm ab}{\sqrt{(h^2 \pm ab)^2 - ((a \pm b)h)^2}} = \frac{h^2 \pm ab}{ab}, \tag{1.67}$$

which is the hypotenuse divided by the second leg. (You can show that this γ is consistent with the result from Exercise 1.41.) As an example, the initial triple $(3, 4, 5)$ (or $(4, 3, 5)$ if we take a to be the longer leg) gives the addition triple $(35, 12, 37)$ with $\gamma = 37/12$, and the subtraction triple $(5, 12, 13)$ with $\gamma = 13/12$.

1.15. **Fizeau experiment**

Since the light moves at speed c/n with respect to the water, and the water moves at speed v with respect to the ground, the velocity-addition formula gives the speed of the light with respect to the ground as

$$V = \frac{c/n + v}{1 + \dfrac{(c/n)v}{c^2}} = \frac{c/n + v}{1 + v/nc}. \tag{1.68}$$

To produce an approximate form of this answer when $v \ll c$, we can multiply the numerator and denominator each by $1 - v/nc$ and keep terms only up to order v/c. With $O(v^2/c^2)$ denoting terms of order v^2/c^2, we obtain

$$\begin{aligned} V = \frac{c(1/n + v/c)}{1 + v/nc} \cdot \frac{1 - v/nc}{1 - v/nc} &= c \cdot \frac{1/n + (v/c)(1 - 1/n^2) - O(v^2/c^2)}{1 - O(v^2/c^2)} \\ \approx c \cdot \left(\frac{1}{n} + \frac{v}{c}\left(1 - \frac{1}{n^2}\right)\right) &= \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right). \end{aligned} \tag{1.69}$$

The desired value of A is therefore $1 - 1/n^2$. We see that the speed of light in moving water increases with the velocity v of the water, but not as fast as the naive answer of $c/n + v$ would imply. Instead of adding v to c/n, we add only $(1 - 1/n^2)v$.

We can check the result in Eq. (1.69) in a few special cases. If $n = 1$, which means that we have vacuum instead of water, we obtain a speed of $c/1 + v(1 - 1) = c$. This is correct because we know that light always moves with speed c in vacuum. If n is very large, we obtain a speed of $c/n + v(1 - 0) = c/n + v$. This is correct because it is the naive addition of the speeds, which we know works perfectly fine when both speeds are much less than c.

In 1851, which was well before Einstein's velocity-addition formula was known, Fizeau performed an experiment to measure the speed (with respect to the ground) of light in moving water. His setup involved an interferometer similar to the one Michelson and Morley used in their experiment. He obtained a result consistent with our approximate formula in Eq. (1.69), so he conjectured that the formula held (exactly) in general. Many people then made unsuccessful attempts (involving frame dragging of the "ether," for example) to explain why the parameter A took on the value of $1 - 1/n^2$ instead of the naive value of 1. In retrospect, of course, failure was the likely result of their (commendable) efforts to generate an exact theory from an approximate result. It was more than half a century until Einstein produced the theory of special relativity in 1905, from which the correct explanation of A's value followed via the velocity-addition formula (along with the approximations we made in Eq. (1.69)). Conversely, the result of Fizeau's experiment was highly influential in Einstein's formulation of special relativity.

1.16. **Equal speeds**

First solution: Let C move at speed v with respect to the ground, and let the relative speed of C and both A and B be u (as viewed by C). Then two different expressions for u are the relativistic subtraction of v from $4c/5$, and the relativistic subtraction of $3c/5$ from v. Therefore,

$$\frac{\frac{4}{5} - v}{1 - \frac{4}{5}v} = \frac{v - \frac{3}{5}}{1 - \frac{3}{5}v}, \tag{1.70}$$

where we have temporarily ignored the c's, or equivalently used v to stand for $\beta \equiv v/c$, or equivalently pretended that c equals 1. (We'll do this in all three solutions here, since it keeps things from getting too messy.) After some algebra, you can show that Eq. (1.70) reduces to $0 = 35v^2 - 74v + 35 = (5v - 7)(7v - 5)$. Since the $v = 7/5$ root represents a speed larger than c, we want the other root:

$$v = \frac{5}{7}c, \tag{1.71}$$

where we have brought the c back in. This is the speed of C with respect to the ground. Plugging this back into either expression for u in Eq. (1.70) gives $u = c/5$. This is how fast C sees both A and B approaching her. Note that C's speed with respect to the ground *cannot* be obtained by simply taking the average of A's and B's speeds, which would give $7c/10$. Taking the average works for nonrelativistic speeds, but not for relativistic ones.

Second solution: With u and v defined as above, two different expressions for v are the relativistic subtraction of u from $4c/5$, and the relativistic addition of u to $3c/5$. Therefore,

$$\frac{\frac{4}{5} - u}{1 - \frac{4}{5}u} = \frac{\frac{3}{5} + u}{1 + \frac{3}{5}u}. \tag{1.72}$$

After some algebra, you can show that this reduces to $0 = 5u^2 - 26u + 5 = (5u - 1)(u - 5)$. Since the $u = 5$ root represents a speed larger than c, we want the other root:

$$u = \frac{c}{5}. \tag{1.73}$$

Plugging this back into either expression for v in Eq. (1.72) gives $v = 5c/7$.

Third solution: The relative speed of A and B (as viewed by either A or B) is

$$\frac{\frac{4}{5} - \frac{3}{5}}{1 - \frac{4}{5} \cdot \frac{3}{5}} = \frac{5}{13}. \tag{1.74}$$

In C's frame, A approaches with speed u from one side, and B approaches with speed u from the other. The relative speed of A and B (as viewed by either A or B) is therefore obtained by relativistically adding u with another u. But we just found that this relative speed is $5/13$. Therefore,

$$\frac{u + u}{1 + u^2} = \frac{5}{13} \quad \Longrightarrow \quad 5u^2 - 26u + 5 = 0, \tag{1.75}$$

as in the second solution.

1.17. **More equal speeds**

Let u be the speed at which C sees A and B approaching her. Then u is the desired speed of C with respect to B (that is, the ground). From C's point of view, the given speed v is the result of relativistically adding u with another u. Therefore,

$$v = \frac{2u}{1 + u^2/c^2} \quad \Longrightarrow \quad \left(\frac{v}{c^2}\right)u^2 - 2u + v = 0. \tag{1.76}$$

Solving this quadratic equation for u gives

$$u = \frac{c^2\left(1 - \sqrt{1 - v^2/c^2}\,\right)}{v} = \frac{c^2(1 - 1/\gamma)}{v}. \tag{1.77}$$

The quadratic equation also has a solution with a plus sign in front of the square root, but this solution cannot be correct, because it is greater than c, as you can verify (and in fact goes to infinity as v goes to zero). The above solution for u has the correct limit as v goes to zero, namely $u \to v/2$ (the expected nonrelativistic result); this can be obtained by using the Taylor approximation, $\sqrt{1 - \epsilon} \approx 1 - \epsilon/2$.

The ratio of the distances CB and AC in the ground frame is the same as the ratio of the differences in velocities as measured in the ground frame (because both A and C arrive at B at the same time, so you could imagine running the scenario backward in time). Therefore,

$$\begin{aligned}\frac{CB}{AC} = \frac{V_C - V_B}{V_A - V_C} &= \frac{\frac{c^2(1 - 1/\gamma)}{v} - 0}{v - \frac{c^2(1 - 1/\gamma)}{v}} = \frac{1 - 1/\gamma}{v^2/c^2 - 1 + 1/\gamma} \\ &= \frac{1 - 1/\gamma}{1/\gamma - (1 - v^2/c^2)} = \frac{1 - 1/\gamma}{1/\gamma - 1/\gamma^2} = \gamma. \end{aligned} \tag{1.78}$$

We see that C is γ times as far from B as she is from A, as measured in the ground frame. Note that for nonrelativistic speeds, we have $\gamma \approx 1$, so C is midway between A and B, as expected.

An intuitive reason for the simple factor of γ is the following. Imagine that A and B are carrying identical jousting sticks as they run toward C (in C's frame). In C's frame, the tips of the sticks reach C simultaneously, because in C's frame A and B are always the same distance from C. This is true because we are told that all three people eventually coincide at some instant. But since the sticks reach C simultaneously in C's frame, they do also in B's frame (the ground frame). This is true because since we're talking about the ends of sticks reaching C, everything happens right at C. The L in the Lv/c^2 rear-clock-ahead result is therefore zero, so we don't have to worry about any loss of simultaneity. Consider then the instant in B's frame (the ground frame) when both sticks reach C. B's stick is at rest, so it is uncontracted. But A's stick is moving with speed v, so it is length contracted by a factor γ. Therefore, in the ground frame, A is closer to C than B is, by a factor γ.

1.18. **Many velocity additions**

Let's first check the formula for $N = 1$ and $N = 2$. When $N = 1$, it gives

$$\beta_{(1)} = \frac{P_1^+ - P_1^-}{P_1^+ + P_1^-} = \frac{(1+\beta_1) - (1-\beta_1)}{(1+\beta_1) + (1-\beta_1)} = \beta_1, \tag{1.79}$$

as it should. And when $N = 2$, it gives

$$\beta_{(2)} = \frac{P_2^+ - P_2^-}{P_2^+ + P_2^-} = \frac{(1+\beta_1)(1+\beta_2) - (1-\beta_1)(1-\beta_2)}{(1+\beta_1)(1+\beta_2) + (1-\beta_1)(1-\beta_2)} = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}, \tag{1.80}$$

in agreement with the velocity-addition formula. You can check that the factors of c work out correctly when the β's are swapped for v's.

Let's now prove the formula for a general N. We will use induction. That is, we will assume that the result holds for a given N and then show that it also holds for $N + 1$. To find the speed, $\beta_{(N+1)}$, of the object with respect to S_{N+1}, we can relativistically add the speed of the object with respect to S_N (which is $\beta_{(N)}$) with the speed of S_N with respect to S_{N+1} (which is β_{N+1}). This gives

$$\beta_{(N+1)} = \frac{\beta_{N+1} + \beta_{(N)}}{1 + \beta_{N+1}\beta_{(N)}}. \tag{1.81}$$

Under the assumption that our formula holds for N, this becomes

$$\begin{aligned}
\beta_{(N+1)} = \frac{\beta_{N+1} + \dfrac{P_N^+ - P_N^-}{P_N^+ + P_N^-}}{1 + \beta_{N+1}\dfrac{P_N^+ - P_N^-}{P_N^+ + P_N^-}} &= \frac{\beta_{N+1}(P_N^+ + P_N^-) + (P_N^+ - P_N^-)}{(P_N^+ + P_N^-) + \beta_{N+1}(P_N^+ - P_N^-)} \\
&= \frac{P_N^+(1+\beta_{N+1}) - P_N^-(1-\beta_{N+1})}{P_N^+(1+\beta_{N+1}) + P_N^-(1-\beta_{N+1})} \\
&\equiv \frac{P_{N+1}^+ - P_{N+1}^-}{P_{N+1}^+ + P_{N+1}^-},
\end{aligned} \tag{1.82}$$

as we wanted to show. We have therefore shown that if the result holds for N, then it also holds for $N + 1$. Since we know that the result does indeed hold for $N = 1$, it therefore holds for all N.

The expression for $\beta_{(N)}$ has some expected properties. It is symmetric in the β_i. And if the given object is a photon with $\beta_1 = 1$, then $P_N^- = 0$, which yields $\beta_{(N)} = 1$ as it should. And if the given object is a photon with $\beta_1 = -1$, then $P_N^+ = 0$, which yields $\beta_{(N)} = -1$ as it should. Likewise, if any one of the β_i's equals 1 (or −1), then $P_N^- = 0$ (or $P_N^+ = 0$), which correctly yields $\beta_{(N)} = 1$ (or $\beta_{(N)} = -1$).

1.19. **Velocity addition from scratch**

As stated in the problem, we will use the fact that the meeting of the photon and the ball occurs at the same fraction of the way along the train, independent of the frame. This is true because, although distances may change depending on the frame, fractions remain the same, since length contraction doesn't depend on position. We'll compute the desired fraction in the train frame S', and then in the ground frame S.

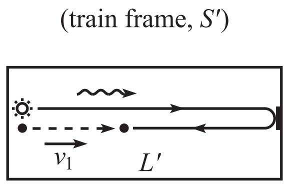

Figure 1.70

Train frame: Let the train have length L' in the train frame, S'. Let's first find the time at which the photon meets the ball. From Fig. 1.70, we see that the sum of the distances traveled by the ball and the photon, which is $v_1t' + ct'$, must equal twice the length of the train, which is $2L'$. The time of the meeting is therefore

$$t' = \frac{2L'}{c + v_1}. \tag{1.83}$$

The distance the ball has traveled is then $d' = v_1t' = 2v_1L'/(c+v_1)$, so the desired fraction F' is

$$F' = \frac{d'}{L'} = \frac{2v_1}{c + v_1}. \tag{1.84}$$

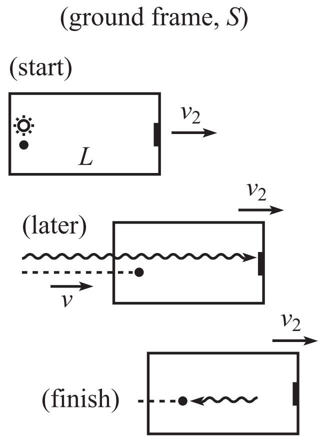

Figure 1.71

Ground frame: Let the speed of the ball with respect to the ground be v, and let the train have length L in the ground frame. (L equals L'/γ, but we're not going to use this.) Again, let's first find the time at which the photon meets the ball. From Fig. 1.71, we see that the photon takes a time $L/(c - v_2)$ to reach the mirror, because the initial gap of L is closed at a rate $c - v_2$ in the ground frame. At this time, the photon has traveled a distance $cL/(c - v_2)$. From the figure, we see that we can use the same reasoning we used in the train frame, but with the sum of the distances traveled by the ball and the photon, which is $vt + ct$, now equal to $2[cL/(c - v_2)]$. The time of the meeting in the ground frame is therefore

$$t = \frac{2cL/(c - v_2)}{(c + v)}. \tag{1.85}$$

The relative speed of the ball and the back of the train (as viewed in the ground frame) is $v - v_2$. This is the rate at which the gap between them is increasing. So the distance between the ball and the back of the train at the time of the meeting is $d = (v - v_2)t = (v - v_2) \cdot 2cL/[(c - v_2)(c + v)]$. The desired fraction F is therefore

$$F = \frac{d}{L} = \frac{2(v - v_2)c}{(c - v_2)(c + v)}. \tag{1.86}$$

We can now equate the above expressions for F' and F. For convenience, define $\beta \equiv v/c$, $\beta_1 \equiv v_1/c$, and $\beta_2 \equiv v_2/c$. Then $F' = F$ yields

$$\frac{\beta_1}{1 + \beta_1} = \frac{\beta - \beta_2}{(1 - \beta_2)(1 + \beta)}. \tag{1.87}$$

Solving for β in terms of β_1 and β_2 gives, after some algebra,

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}, \tag{1.88}$$

as desired. This problem is solved in Mermin (1983).

1.20. **Time dilation and Lv/c^2**

The velocity-addition formula gives the person's speed in the ground frame as $(u+v)/(1+uv)$, where we have dropped the c's. So in the ground frame, the person must close the initial gap of L/γ_v that the front of the train had, at a relative speed of $(u+v)/(1+uv) - v$. The time in the ground frame is therefore

$$t_g = \frac{L/\gamma_v}{\dfrac{u + v}{1 + uv} - v} = \frac{L(1 + uv)}{u\sqrt{1 - v^2}}. \tag{1.89}$$

Compared with this ground-frame time, the front clock on the train runs slow by the factor γ_v, and the person's watch runs slow by the γ factor associated with the speed

$(u+v)/(1+uv)$, which you can show equals $\gamma_u\gamma_v(1+uv)$; see Exercise 1.41. The difference in the elapsed times on the front clock and the person's watch is therefore

$$\begin{aligned}\Delta T_{\text{front}} - \Delta T_{\text{person}} &= \frac{L(1+uv)}{u\sqrt{1-v^2}}\left(\frac{1}{\gamma_v} - \frac{1}{\gamma_u\gamma_v(1+uv)}\right)\\ &= \frac{L}{u}\left(1+uv-\frac{1}{\gamma_u}\right)\\ &= \frac{Lv}{c^2}+\frac{L}{u}\left(1-\frac{1}{\gamma_u}\right),\end{aligned} \tag{1.90}$$

where we have put the c's back in to make the units correct. The second term here is negligible for the following reason. For small u, we can use the Taylor series $\sqrt{1-\epsilon}\approx 1-\epsilon/2$ to write $1/\gamma_u = \sqrt{1-u^2/c^2}\approx 1-u^2/2c^2$. The $(L/u)(1-1/\gamma_u)$ term then becomes $(L/u)(u^2/2c^2) = Lu/2c^2$. Since u is assumed to be small (more precisely, $u \ll c$), this term is negligible. So Eq. (1.90) becomes $\Delta T_{\text{front}} - \Delta T_{\text{person}} \approx Lv/c^2$. The front clock therefore gains essentially Lv/c^2 more time than the person's watch, as we wanted to show.

Since the front clock started Lv/c^2 behind the person's watch, we conclude that they end up showing the same time when the watch reaches the front, as we already knew from working in the train frame. The point here is that no matter how small u is, the result for $\Delta T_{\text{front}} - \Delta T_{\text{person}}$ is nonzero (namely Lv/c^2) because u appears at *first* order in the γ factor, $\gamma_u\gamma_v(1+uv)$, associated with $(u+v)/(1+uv)$, while it appears only at *second* order in γ_u. The difference between the γ factors is therefore first order in u, and this difference combines with the $1/u$ factor in the time to yield a nonzero result.

The result in Eq. (1.90) holds perfectly well for non-small u too, so it implies that the final readings on the front clock and the person's watch differ by $(L/u)(1-1/\gamma_u)$, for any u. In retrospect, this is clear from the train-frame calculation which gives the difference as $(L/u)-(L/u)/\gamma_u$, due to the time dilation of the watch.

1.21. **Modified twin paradox**

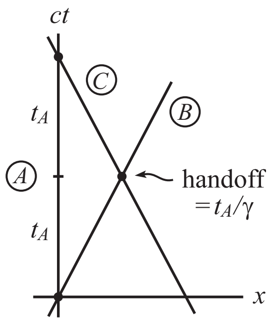

Figure 1.72

(a) To help visualize the setup in each frame, we'll draw the positions of the three people as functions of time. The resulting lines (or more generally, curves) are known as *worldlines*. In relativity, it is customary to put time on the vertical axis and space on the horizontal axis (the opposite of what is normally done). It is also customary to plot the value of ct, instead of t. This leads to the nice fact that light is represented by a lines with slope $\pm 45°$. The worldline of any (massive) object will always have a slope that is larger than $45°$, because $v < c$.

In A's frame, the worldlines of A, B, and C are shown in Fig. 1.72. A is at rest, so his worldline is vertical. B moves to the right at speed v, and C moves to the left at speed v. In A's frame, B's clock runs slow by a factor $1/\gamma$. Therefore, if A's clock reads t_A when B meets C, then B's clock reads only t_A/γ when he meets C. So the time he hands off to C is t_A/γ.

In A's frame, the time between the B-meets-C event and the C-meets-A event is again t_A, because B and C travel at the same speed. And A sees C's clock run slow by a factor $1/\gamma$, so A sees C's clock increase by only t_A/γ. Therefore, when A and C meet, A's clock reads $2t_A$, and C's clock reads $2t_A/\gamma$. In other words, $T_C = T_A/\gamma$.

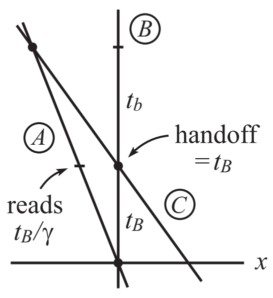

Figure 1.73

(b) Let's now look at things in B's frame. The worldlines of A, B, and C are shown in Fig. 1.73. A moves to the left at speed v, and C moves to the left at speed $2v/(1+v^2)$. This is the velocity addition of v with itself; we have ignored the c's, to keep things from getting cluttered.

From B's point of view, there are two competing effects that lead to the relation $T_C = T_A/\gamma$. The first is that B sees A's clock run slow, so the time that B hands off to C is *larger* than the time on A's clock at that moment. So C's clock reads more than A's at the handoff moment. The second effect is that from this point on, B sees C's clock run *slower* than A's (because the relative speed of C and B is greater than the relative speed of A and B). It turns out that this slowness wins out over the head start that C's clock had over A's. So in the end, C's clock reads less than A's. Let's be quantitative about this.

Let B's clock read t_B when C meets him. (t_B is the same as the t_A/γ in part (a), but we won't use that since we're doing things from B's point of view here.) Then when B hands off this time to C, A's clock reads only t_B/γ, because B sees A's clock run slow. We must determine how much additional time elapses on A's clock and on C's clock, by the time they meet. We'll find all times below in terms of t_B.

At time t_B (when C passes B) A is a distance vt_B from B. Let t_b be the additional time on B's clock between C passing him and C catching up with A. We can find t_b by noting that C closes the initial head start of vt_B that A had, at a relative speed of $2v/(1+v^2) - v$, as viewed by B. So

$$t_b = \frac{vt_B}{\dfrac{2v}{1+v^2} - v} \quad \Longrightarrow \quad t_b = t_B\left(\frac{1+v^2}{1-v^2}\right). \tag{1.91}$$

During the time t_b, B sees A's and C's clocks increase by t_b divided by the relevant time-dilation factor. For A this factor is $\gamma = 1/\sqrt{1-v^2}$, and for C it is

$$\gamma_C = \frac{1}{\sqrt{1-\left(\dfrac{2v}{1+v^2}\right)^2}} = \frac{1+v^2}{1-v^2}, \tag{1.92}$$

as you can verify. Therefore, the total time shown on A's clock when A and C meet is

$$\begin{aligned} T_A &= \frac{t_B}{\gamma} + \frac{t_b}{\gamma} = t_B\sqrt{1-v^2} + t_B\left(\frac{1+v^2}{1-v^2}\right)\cdot\sqrt{1-v^2} \\ &= \frac{2t_B}{\sqrt{1-v^2}}\,. \end{aligned} \tag{1.93}$$

And the total time shown on C's clock when A and C meet is the handoff time of t_B plus the time elapsed on C, so

$$T_C = t_B + \frac{t_b}{\gamma_C} = t_B + t_B\left(\frac{1+v^2}{1-v^2}\right)\cdot\left(\frac{1-v^2}{1+v^2}\right) = 2t_B. \tag{1.94}$$

Therefore, $T_C = T_A\sqrt{1-v^2} = T_A/\gamma$, as desired.

(c) Let's now work in C's frame. The worldlines of A, B, and C are shown Fig. 1.74. A moves to the right at speed v, and B moves to the right at speed $2v/(1+v^2)$. As in part (b), the time-dilation factor between B and C is $\gamma_B = (1+v^2)/(1-v^2)$. Also, as in part (b), let B and C meet when B's clock reads t_B. So this is the time that B hands off to C. We'll find all times below in terms of t_B.

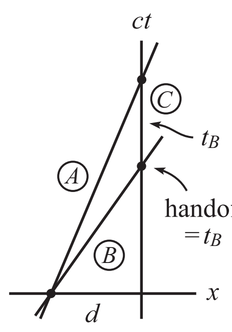

Figure 1.74

C sees B's clock running slow, so in C's frame it takes a time of $\gamma_B t_B$ for B's clock to advance by t_B, since when he met A. B therefore travels for a time of

$$t_{B\,\text{reach}\,C} = \gamma_B t_B = t_B\left(\frac{1+v^2}{1-v^2}\right) \tag{1.95}$$

between meeting A and meeting C. During this time, B covers a distance in C's frame equal to

$$d = t_B\left(\frac{1+v^2}{1-v^2}\right)\cdot\frac{2v}{1+v^2} = \frac{2vt_B}{1-v^2}\,. \tag{1.96}$$

A must travel this same distance (from where B passed him) to meet up with C. This allows us to find T_A. The time (as viewed by C) that it takes A to travel the distance d to reach C is

$$t_{A\,\text{reach}\,C} = \frac{d}{v} = \frac{2t_B}{1-v^2}\,. \tag{1.97}$$

But since C sees A's clock running slow by a factor $\sqrt{1-v^2}$, A's clock reads only

$$T_A = \frac{2t_B}{\sqrt{1-v^2}} \tag{1.98}$$

when he meets C. This agrees with Eq. (1.93), as it must, because the reading is frame-independent.

Now let's find T_C. To find T_C, we must take the handoff time of t_B and add to it the extra time it takes A to reach C, compared with the time it takes B to reach C. From Eqs. (1.95) and (1.97), this extra time is

$$t_{A\,\text{reach}\,C} - t_{B\,\text{reach}\,C} = \frac{2t_B}{1-v^2} - \frac{t_B(1+v^2)}{1-v^2} = t_B. \tag{1.99}$$

(This simple result is clear in A's frame, but not so clear in C's frame.) Therefore, C's clock reads

$$T_C = t_B + t_B = 2t_B, \tag{1.100}$$

which agrees with Eq. (1.94), as it must, because the reading is frame-independent. Hence, $T_C = T_A\sqrt{1-v^2} \equiv T_A/\gamma$, as desired.