Chapter 5

Longitudinal waves

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In Chapter 4 we discussed transverse waves, in particular transverse waves on a string. We'll now move on to longitudinal waves. Each point in the medium (whatever it consists of) still oscillates back and forth around its equilibrium position, but now in the longitudinal instead of the transverse direction. Longitudinal waves are a bit harder to visualize than transverse waves, partly because everything is taking place along only one dimension, and partly because of the way the forces arise, as we'll see. Most of this chapter will be spent on sound waves, which are the prime example of longitudinal waves.

The outline of this chapter is as follows. As a warm up, in Section 5.1 we take another look at the longitudinal spring/mass system we originally studied in Section 2.4, where we considered at the continuum limit (the $N \to \infty$ limit). In Section 5.2 we study actual sound waves. We derive the wave equation (which takes the same form as all the other wave equations we've seen so far), and then look at the properties of the waves. In Section 5.3 we apply our knowledge of sound waves to musical instruments.

5.1 Springs and masses revisited

Recall that the wave equation for the continuous spring/mass system was given in Eq. (2.80) as

$$\frac{\partial^2 \psi(x,t)}{\partial t^2} = \frac{E}{\mu} \frac{\partial^2 \psi(x,t)}{\partial x^2} \,, \tag{1}$$

where ψ is the longitudinal position relative to equilibrium, μ is the mass density, and E is the elastic modulus. This wave equation is very similar to the one for transverse waves on a string, which was given in Eq. (4.4) as

$$\frac{\partial^2 \psi(x,t)}{\partial t^2} = \frac{T}{\mu} \frac{\partial^2 \psi(x,t)}{\partial x^2} \,, \tag{2}$$

where ψ is the transverse position relative to equilibrium, μ is the mass density, and T is the tension.

These equations take exactly the same form, so all of the same results hold. However, the fact that ψ is a longitudinal position in the former case, whereas it is a transverse position in the latter, makes the former case a little harder to visualize. For example, if we plot ψ for a sinusoidal traveling wave (either transverse or longitudinal), we have the picture shown in Fig. 1. The interpretation of this picture depends on what kind of wave we're talking about.
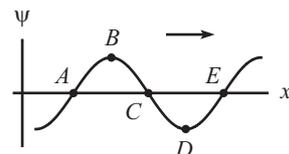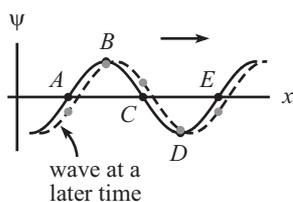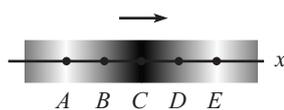

Figure 1

Figure 2

Figure 3

For a *transverse* wave, ψ is the transverse displacement, so Fig. 1 is what the string actually *looks* like from the side. The wave is therefore very easy to visualize – you just need to look at the figure. It's also fairly easy to see what the various points in Fig. 1 are doing as the wave travels to the right. (Imagine that these dots are painted on the string.) Points B and D are instantaneously at rest, points A and E are moving downward, and point C is moving upward. To verify these facts, just draw the wave at a slightly later time. The result is shown in Fig. 2, with the new positions of the dots being represented by gray dots. Remember that the points keep their same longitudinal position and simply move up or down (or not at all). They don't travel longitudinally along with the wave.

However, for a *longitudinal* wave, ψ is the longitudinal displacement, so although Fig. 1 is a perfectly valid plot of ψ, it does *not* indicate what the wave actually looks like. There is no transverse motion, so the system simply lies along a straight line. What changes is the *density* along the line. You could therefore draw the wave by shading it as in Fig. 3, but this is a bit harder to draw than Fig. 1. For a longitudinal wave, the statements in the preceding paragraph about the motion of the various points in Fig. 1 are still true, provided that "downward" is replaced with "leftward," and "upward" is replaced with "rightward. But what do things actually *look* like along the 1-D line? In particular, how does Fig. 3 follow from Fig. 1?

At points B and D in Fig. 3, the density of the masses equals the equilibrium density, because nearby points all have essentially the same displacement (see Fig. 1). But at points A and E, the density is a minimum, because points to the left of them have a negative displacement, while points to the right have a positive displacement (again see Fig. 1). The opposite is true for point C, so the density is maximum there. Various properties of the wave are indicated in Fig. 4. You should stare at this figure for a while and verify all of the stated properties. We'll talk more about the relation among the various quantities when we discuss Fig. 8 later on when we get to sound waves.
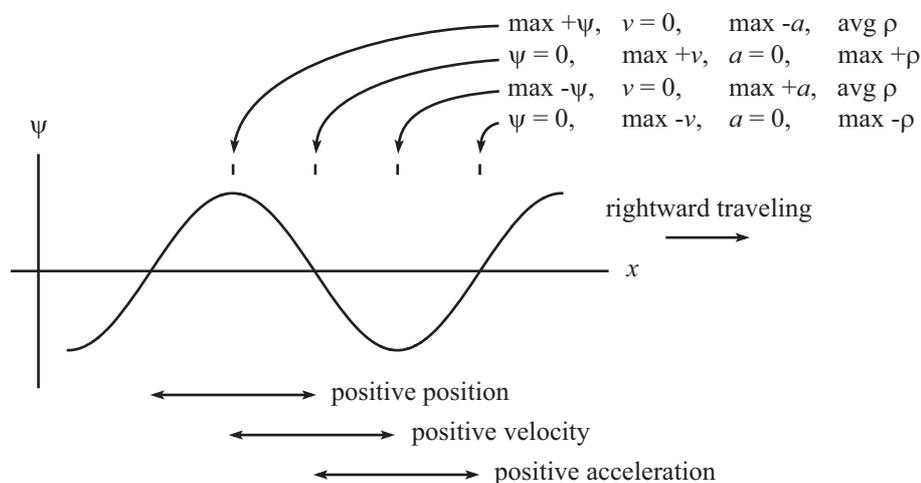
Figure 4

In the Fig. 4, the relation between ψ, v, and a is the same as always, namely, a is 90° ahead of v, and v is 90° ahead of ψ. But you should think about how these relate to the density ρ. For example, from the preceding paragraph, the (excess) ρ is proportional to the negative of the slope (see Problem [to be added] for a rigorous derivation of this fact). But we already know that v is proportional to the negative of the slope; see Eq. (4.44). Therefore, the (excess) ρ is proportional to v. For a *leftward* traveling wave, the same statement about

ρ is still true, but now v is proportional to the slope (with no negative sign). So the (excess) ρ is proportional to $-v$.

We can double check that this result makes sense with the following reasoning. Since the (excess) ρ is proportional to v, we can take the derivative of this statement to say that $\partial \rho / \partial x \propto \partial v / \partial x$ (the word "excess" is now not needed). But since a traveling wave takes the form of $\psi(x, t) = f(x - ct)$, the velocity $v = \partial \psi / \partial t$ also takes the functional form of $g(x - ct)$. Therefore, we have $\partial v / \partial x = -(1/c) \partial v / \partial t$. The righthand side of this is just the acceleration a, so the $\partial \rho / \partial x \propto \partial v / \partial x$ statement becomes

$$\frac{\partial \rho}{\partial x} \propto -a \quad \implies \quad a \propto -\frac{\partial \rho}{\partial x}. \tag{3}$$

Does this make sense? It says, for example, that if the density is an increasing function of x at a given point, then the acceleration is negative there. This is indeed correct, because a larger density means that the springs are more compressed (or less stretched), which in turn means that they exert a larger repulsive force (or a smaller attractive force). So if the density is an increasing function of x (that is, if $\partial \rho / \partial x > 0$), then the springs to the right of a given region are pushing leftward more than the springs to the left of the region are pushing rightward. There is therefore a net negative force, which means that the acceleration a is negative, in agreement with Eq. (3).

5.2   Sound waves

5.2.1   Notation

Sound is a longitudinal wave, in both position and pressure/density, as we'll see. Sound can exist in solids, liquids, and gasses, but in this chapter we'll generally work with sound waves in air. In air, molecules push and (effectively, relative to equilibrium) pull on each other, so we have a sort of spring/mass system like the one we discussed above.

The main goal in this section is to derive the wave equation for sound waves in air. We'll find that we obtain exactly the same type of wave equation that we had in Eq. (1) for the spring/mass system. The elastic modulus E appears there, so part of our task below will be to find the analogous quantity for sound waves. We'll consider only one-dimensional waves here. That is, the waves depend only on x. Waves like this that are uniform in the transverse y and z directions are called "plane waves."

To emphasize the 1-D nature of the wave, let's consider a tube of air inside a cylindrical container, with cross-sectional area A. Fig. 5 shows a given section of air at equilibrium, and then also at a later time. Let the ends of this section be located at x and $x + \Delta x$ at equilibrium, and then at $x + \psi(x)$ and $x + \Delta x + \psi(x + \Delta x)$ at a later time, as shown. So the function ψ measures the displacement from equilibrium.

If we define $\Delta \psi$ by $\psi(x + \Delta x) \equiv \psi(x) + \Delta \psi$, then $\Delta \psi$ is how much more the right boundary of the region moves compared with the left boundary. The molecules of air are in thermal motion, of course, so it's not as if the molecules that form the boundary at position x in the first picture correspond to the molecules that form the boundary at position $x + \psi(x)$ in the second picture. But we'll ignore this fact and just pretend that it's the same molecules, for ease of discussion. It doesn't actually matter. Note that $\Delta \psi \approx (\partial \psi / \partial x) \Delta x$ for small Δx, by definition of the derivative. In actual sound waves in air, $\Delta \psi$ is much less than Δx. In other words, $\partial \psi / \partial x$ is very small.

(equilibrium)

x $x+\Delta x$

(later)

$\Delta x+\Delta\psi$

$x+\psi(x)$ $x+\Delta x+\psi(x+\Delta x)$

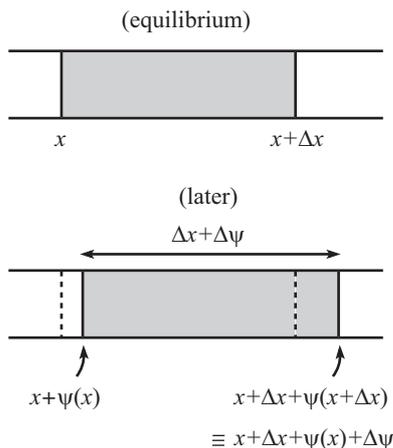$\equiv x+\Delta x+\psi(x)+\Delta\psi$

Figure 5

A note on terminology: We're taking x to be the position of a given molecule at equilibrium. So even after the molecule has moved to the position $x + \psi(x)$, it is still associated with the same value of x. So x is analogous to the index n we used in Section 2.3 and the beginning of Section 2.4. The movement of the particle didn't affect its label n there, and it doesn't affect its label x here.

In obtaining the wave equation, we'll need to get a handle on the pressure at the two ends of the given section of air, and then we'll figure out how these pressures cause the section to move. Let the pressure in the tube at equilibrium be p_0. At sea level, the atmospheric pressure happens to be about 14.7 lbs. per square inch. The first picture in Fig. 6 shows the pressures at the two ends at equilibrium; it is simply p_0 at both ends (and everywhere else).
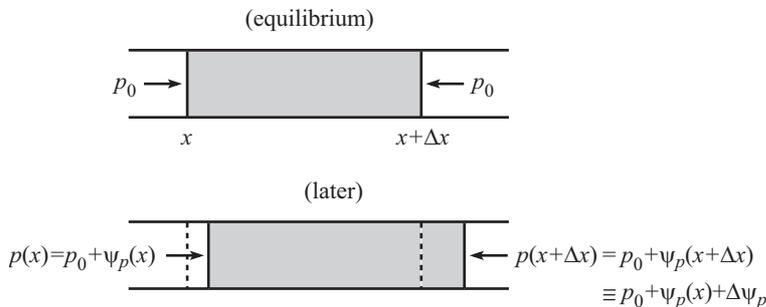
(equilibrium)

$p_0 \rightarrow$ $\leftarrow p_0$

x $x+\Delta x$

(later)

$p(x)=p_0+\psi_p(x) \rightarrow$ $\leftarrow p(x+\Delta x)=p_0+\psi_p(x+\Delta x)$

$\equiv p_0+\psi_p(x)+\Delta\psi_p$

Figure 6

What about at a later time? Let $\psi_p(x)$ be the *excess* pressure (above p_0) as a function of x. (Remember that x labels the equilibrium position of the molecules, not the present position.) The total pressure at the left boundary of the section is then $p_0+\psi_p(x)$. However, this total pressure won't be too important; the change, $\psi_p(x)$, is what we'll be concerned with. At the right boundary, the total pressure is, by definition, $p_0 + \psi_p(x + \Delta x)$. If we define $\Delta\psi_p$ by $\psi_p(x + \Delta x) \equiv \psi_p(x) + \Delta\psi_p$, then $\Delta\psi_p$ is how much the pressure at the right boundary exceeds the pressure at the left boundary. Note that $\Delta\psi_p \approx (\partial\psi_p/\partial x)\Delta x$ for small Δx. In practice, ψ_p is much smaller than p_0. And $\Delta\psi_p$ is infinitesimally small, assuming that we have picked Δx to be infinitesimally small. The pressures at a later time are summarized in the second picture in Fig. 6.

5.2.2 The wave equation

Having introduced the necessary notation, we can now derive the wave equation for sound waves. The derivation consists of four main steps, so let's go through them systematically. Our strategy will be to find the net force on a given volume of air, and then write down the $F = ma$ equation for that volume.

1. HOW THE VOLUME CHANGES: First, we need to determine how the volume of a gas changes when the pressure is changed. Qualitatively, if we increase the pressure on a given volume, the volume decreases. By how much? The decrease should certainly be proportional to the volume, because if we put two copies of a given volume next to each other, we will obtain twice the decrease. And the decrease should also be proportional to the pressure increase, provided that the increase is small. This is a reasonable claim, but not terribly obvious. We'll derive it in step 2 below. Assuming that it is true, we can write (recalling that ψ_p is defined to be the increase in pressure relative to equilibrium) the change in volume from equilibrium as $\Delta V \propto -V \psi_p$. This correctly incorporates the two proportionality facts above. The minus sign is due to the fact that an increase in pressure causes a decrease in volume. This equation is valid as long as ΔV is small compared with V, as we'll see below.

   Let's define κ to be the constant of proportionality in $\Delta V \propto -V \psi_p$. κ is known as the *compressibility*. The larger κ is, the more the volume is compressed (or expanded), for a given increase (or decrease) in pressure, ψ_p. In terms of κ, we have

$$\Delta V = -\kappa V \psi_p. \tag{4}$$

   But from the first picture in Fig. 5, we see that the volume of gas is $V = A\Delta x$, where A is the cross-sectional area. And the change in the volume between the two pictures shown is $\Delta V = A\Delta\psi$.[1] Eq. (4) therefore becomes

$$\frac{\Delta V}{V} = -\kappa\psi_p \quad\Longrightarrow\quad \frac{A\Delta\psi}{A\Delta x} = -\kappa\psi_p \quad\Longrightarrow\quad \boxed{\frac{\partial\psi}{\partial x} = -\kappa\psi_p} \tag{5}$$

   where we have taken the infinitesimal limit and changed the Δ's to differentials (partial ones, since ψ is a function of t also). The quantity $\partial\psi/\partial x$ indicates how the displacement from equilibrium grows as a function of x. Equivalently, $\partial\psi/\partial x$ is the "stretching fraction." If the displacement ψ grows by, say, 1 mm over the course of a distance of 10 cm, then the length (and hence the volume) of the region has increased by 1/100, and this equals $\partial\psi/\partial x$. Eq. (5) says that the stretching fraction is proportional to the change in pressure, which is quite reasonable.

2. CALCULATING THE COMPRESSIBILITY, κ: We'll now be rigorous about the above statement that the decrease in volume should be proportional to the pressure increase, provided that the increase is small. In the course of doing this, we'll find the value of κ.

   Let's first give a derivation that isn't quite correct. From the ideal gas law (which we'll accept here; we have to start somewhere), we have $pV = nRT$. If the temperature T is constant, then any changes in p and V must satisfy $(p + dp)(V + dV) = nRT$. Subtracting the original $pV = nRT$ equation from this one, and ignoring the second-order $dp\,dV$ term (this is where the assumption of small changes comes in), we obtain

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[1] We can alternatively say that the volume is $V = A(\Delta x + \Delta\psi)$, by looking at the second picture. But we are assuming $\Delta\psi \ll \Delta x$, so to leading order we can ignore the $A\Delta\psi$ term in the volume. However, we can't ignore it in the *change* in volume, because it's the entire change.

$p\,dV + dp\,V = 0 \implies dV = -(1/p)V\,dp$. But dp is what we've been calling ψ_p above, so if we compare this result with Eq. (4), we see that the compressibility κ equals $1/p$. Note that what we did here was basically take the differential of the $pV = C$ equation (where C is a constant) to obtain $p\,dV + dp\,V = 0$.

However, this $\kappa = 1/p$ result isn't correct, because it is based on the assumption that T is constant in the relation $pV = nRT$. But T isn't constant in a sound wave (in either space or time). The compressions are actually adiabatic, meaning that heat can't flow quickly enough to redistribute itself and even things out. Basically, the time scale of the heat flow is large compared with the time scale of the wave oscillations. So a region that heats up due to high pressure stays hot, until the pressure decreases. The correct relation (which we'll just accept here) for adiabatic processes turns out to be $pV^\gamma = C$, where γ happens to be about $7/5$ for air (γ is $7/5$ for a diatomic gas, and air is 99% N_2 and O_2). Taking the differential of our new $pV^\gamma = C$ equation gives

$$p \cdot \gamma V^{\gamma-1}\,dV + dp\,V^\gamma = 0 \quad \implies \quad dV = -\left(\frac{1}{\gamma p}\right)V\,dp. \tag{6}$$

Therefore, the correct value of the compressibility κ is

$$\kappa = \frac{1}{\gamma p_0} \approx \frac{5}{7 p_0}\,. \tag{7}$$

The above incorrect result wasn't so bad; it was off by only a factor of $7/5$.

3. CALCULATING THE DIFFERENCE IN PRESSURE, $\Delta\psi_p$: Eq. (5) involves the quantity ψ_p. However, what we're actually concerned with is not ψ_p but the *change* in ψ_p from one end of the volume to the other, because in writing down the $F = ma$ equation for the volume, we're concerned with the *net* force on it, and this involves the difference in the pressures at the ends. So let's see how ψ_p changes with x. Differentiating Eq. (5) with respect to x gives

$$\frac{\partial\psi_p}{\partial x} = -\frac{1}{\kappa}\frac{\partial^2\psi}{\partial x^2} \quad \implies \quad \Delta\psi_p = -\frac{1}{\kappa}\frac{\partial^2\psi}{\partial x^2}\Delta x, \tag{8}$$

where we have multiplied both sides by ∂x and switched back to the Δ notation. Or equivalently, we have multiplied both sides by Δx, and then used the relation $\Delta\psi_p = (\partial\psi_p/\partial x)\Delta x$. Note that the second derivative $\partial^2\psi/\partial x^2$ appears here. From Eq. (5), a *constant* value of $\partial\psi/\partial x$ corresponds to a constant excess pressure $\psi_p(x)$; the tube of air just stretches uniformly if the pressure is the same everywhere. So to obtain a varying value of ψ_p, we need a varying value of $\partial\psi/\partial x$. That is, we need a nonzero value of $\partial^2\psi/\partial x^2$.

4. THE $F = ma$ EQUATION: We can now write down the $F = ma$ equation for a given volume of air. The net rightward force on the tube of air in the second picture in Fig. 6 is the cross-sectional area times the difference in the pressure at the ends. So we have, recalling the definition of $\Delta\psi_p$,

$$F_{\text{net}} = A\big(p(x) - p(x + \Delta x)\big) = A(-\Delta\psi_p). \tag{9}$$

Using the expression for $\Delta\psi_p$ we found in Eq. (8), the $F_{\text{net}} = ma$ equation on a given tube of air is (with ρ being the mass density)

$$
\begin{aligned}
-A\Delta\psi_p &= (\rho V)\frac{\partial^2\psi}{\partial t^2} \\
\implies -A\left(-\frac{1}{\kappa}\frac{\partial^2\psi}{\partial x^2}\Delta x\right) &= \rho(A\Delta x)\frac{\partial^2\psi}{\partial t^2}\,.
\end{aligned}
\tag{10}
$$

Canceling the common factor of $A\Delta x$ and using $\kappa = 1/\gamma p_0$ yields

$$\boxed{\frac{\partial^2 \psi}{\partial t^2} = \frac{\gamma p_0}{\rho} \cdot \frac{\partial^2 \psi}{\partial x^2}} \qquad (\textit{wave equation}) \qquad (11)$$

This is the desired wave equation for sound waves in air. We see that $\gamma p_0/\rho$ is the coefficient that replaces the E/μ coefficient for the longitudinal spring/mass system, or the T/μ coefficient for the transverse string.

The solutions to the wave equation are the usual exponentials,

$$\psi(x,t) = Ae^{i(\pm kx \pm \omega t)}, \qquad (12)$$

where ω and k satisfy $\omega/k = \sqrt{\gamma p_0/\rho}$. And since ω/k is the speed c of the wave, we have $c = \sqrt{\gamma p_0/\rho}$. As always, the speed is simply the square root of the factor on the righthand side. As a double check on the units, we have

$$\frac{\gamma p_0}{\rho} = \frac{(\text{unitless}) \cdot (\text{force/area})}{\text{mass/volume}} = \frac{(\text{kg\,m/s}^2)/\text{m}^2}{\text{kg/m}^3} = \frac{\text{m}^2}{\text{s}^2}, \qquad (13)$$

which has the correct units of velocity squared. What is the numerical value of the speed of sound waves? For air at sea level, the pressure is $p_0 \approx 10^5$ kg/ms^2 and the density is $\rho \approx 1.3$ kg/m^3. So we have

$$c = \sqrt{\frac{\gamma p_0}{\rho}} \approx \sqrt{\frac{(7/5)(10^5\,\text{kg/ms}^2)}{1.3\,\text{kg/m}^3}} \approx 330\,\text{m/s}. \qquad (14)$$

This decreases with ρ, which makes sense because the larger the density, the more inertia the air has, so the harder it is to accelerate it. The speed increases with p_0. This follows from the fact that if p_0 is large, then the compressibility κ is small (meaning the gas is not easily compressed). So for a given value of $\partial^2\psi/\partial x^2$, the force on the left side of Eq. (10) is large, which implies large accelerations.

The two partial derivatives in Eq. (11) come about in the usual way. The second time derivative comes from the "a" in $F = ma$, and the second space derivative comes from the fact that it is the difference in the first derivatives that gives the net force. Eq. (5) tells us that the first space derivative of the displacement gives a measure of the force at a given location (just as with the spring/mass system, the first space derivative told us how much the springs were stretched, which in turn gave the force). The difference in the force at the two ends is therefore proportional to the second space derivative (again as it was with the spring/mass system).

As with the other wave equations we have encountered thus far in this book, the speed of sound waves is independent of ω and k. (This won't be the case for the dispersion-ful waves we discuss in the following chapter.) So all frequencies travel at the same speed. This is fortunate, because if it weren't true, then a music concert would sound like a complete mess!

5.2.3 Pressure waves

Eq. (11) gives the wave equation for the displacement, ψ, from equilibrium for a molecule whose equilibrium position is x. However, it is rather difficult to follow the motion of a single molecule, so it would be nice to obtain the wave equation for the excess pressure ψ_p, because the pressure is much easier to measure. (It is a macroscopic property of the average of a

large number of molecules, as opposed to the microscopic position of a particular molecule.) In view of the relation between ψ_p and ψ in Eq. (5), we can generate the wave equation for ψ_p by taking the $\partial/\partial x$ derivative of the wave equation in Eq. (11). Using the fact that partial derivatives commute, we obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial t^2}\right) = \frac{\gamma p_0}{\rho}\cdot\frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2}\right) \quad\Longrightarrow\quad \frac{\partial^2}{\partial t^2}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\gamma p_0}{\rho}\cdot\frac{\partial^2}{\partial x^2}\left(\frac{\partial \psi}{\partial x}\right). \tag{15}$$

But from Eq. (5) we know that $\partial\psi/\partial x \propto \psi_p$, so we obtain

$$\boxed{\frac{\partial^2 \psi_p}{\partial t^2} = \frac{\gamma p_0}{\rho}\cdot\frac{\partial^2 \psi_p}{\partial x^2}} \qquad (wave\ equation\ for\ pressure) \tag{16}$$
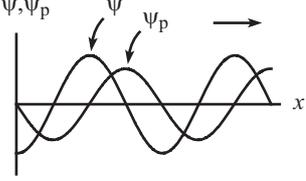
This is the *same* wave equation as the one for the displacement ψ in Eq. (11). So everything that is true about ψ is also true about ψ_p. The only difference is that since $\psi_p \propto -\partial\psi/\partial x$ (the minus sign is important here), the phase of ψ_p is 90° *behind* the phase of ψ. This is shown in Fig. 7. The pressure (and hence also the density) reaches its maximum value a quarter cycle after the displacement does. This is consistent with the values of ψ and ρ given in Fig. 4.

Figure 7

5.2.4 Impedance

What is the impedance of air? In other words, what is the force per velocity that a given region applies to an adjacent region, as a wave propagates? Remember that impedance is a property of the medium and not the wave, even though it is generally easiest to calculate it by considering the properties of a traveling wave. (However, when we discuss dispersion-ful systems in the next chapter, we will find that the impedance depends on the frequency of the wave.)

The velocity of a "sheet" of molecules whose equilibrium position is x is simply $v(x) = \partial\psi(x)/\partial t$. To find the force, consider a cross-sectional area A. We can use Eq. (5) to write the (excess) force that the sheet exerts on the region to its right as

$$F = A\psi_p = A\left(-\frac{1}{\kappa}\frac{\partial\psi}{\partial x}\right). \tag{17}$$

And since we are working with a traveling wave (no need for it to be sinusoidal), we have the usual relationship between $\partial\psi/\partial x$ and $\partial\psi/\partial t$, namely $\partial\psi/\partial x = \mp(1/c)(\partial\psi/\partial t)$ (the minus sign is associated with a rightward traveling wave). So Eq. (17) becomes

$$F = A\left(-\frac{1}{\kappa}\right)\left(\mp\frac{1}{c}\frac{\partial\psi}{\partial t}\right) = \pm\frac{A}{\kappa c}\cdot\frac{\partial\psi}{\partial t}. \tag{18}$$

The force that the sheet *feels* from the region on its right is the negative of this, but the sign isn't important when calculating the impedance Z, because Z is defined to be the magnitude of F/v. Using $v = \partial\psi/\partial t$, Eq. (18) gives the impedance as

$$Z \equiv \frac{F}{v} = \frac{A}{\kappa c}. \tag{19}$$

The impedance per unit area is the more natural thing to talk about, because Z/A is independent of the specific cross section chosen. The force F can be made arbitrarily large by making the area A arbitrarily large, so $Z = F/v$ isn't too meaningful. When people talk about the impedance of air, they usually mean "impedance per area," that is, force

per velocity per area. However, we'll stick with the $Z = F/v$ definition of impedance, in which case Eq. (19) tells us that the impedance per unit area is $Z/A = 1/\kappa c$. But $c = 1/\sqrt{\kappa \rho} \implies \kappa = 1/\rho c^2$ (this follows from writing the coefficient in Eq. (11) in terms of κ). So we have $Z/A = \rho c$. Using $c = \sqrt{\gamma p_0/\rho}$ from Eq. (14), we can write this alternatively as

$$\boxed{\frac{Z}{A} = \rho c = \sqrt{\gamma \rho p_0}} \tag{20}$$

5.2.5 Energy, Power

Energy

What is the energy density of a (longitudinal) sound wave? The kinetic energy density (per unit volume) is simply $(1/2)\rho(\partial \psi/\partial t)^2$, because the speed of the molecules is $\partial \psi/\partial t$. If we want to consider instead the kinetic energy density *per unit length* along a tube with cross-sectional area A, then this is $(1/2)(A\rho)(\partial \psi/\partial t)^2$, where $A\rho \equiv \mu$ is the mass density per unit length. We are assuming here that ρ and μ are independent of position. This is essentially true for actual sound waves, because $\partial \psi/\partial x$ is small, so the fractional change in the density from its equilibrium value is small.

What about the potential energy density? The task of Problem [to be added] is to show that the potential energy density (or rather, the excess over the equilibrium value) per unit length is $(1/2)A\gamma p_0(\partial \psi/\partial x)^2$. The total energy density per unit length (kinetic plus potential) is therefore

$$
\begin{aligned}
\mathcal{E}(x,t) &= \frac{1}{2}A\rho\left(\frac{\partial \psi}{\partial t}\right)^2 + \frac{1}{2}A\gamma p_0\left(\frac{\partial \psi}{\partial x}\right)^2 \\
&= \frac{1}{2}A\rho\left[\left(\frac{\partial \psi}{\partial t}\right)^2 + \frac{\gamma p_0}{\rho}\left(\frac{\partial \psi}{\partial x}\right)^2\right] \\
&= \frac{1}{2}A\rho\left[\left(\frac{\partial \psi}{\partial t}\right)^2 + c^2\left(\frac{\partial \psi}{\partial x}\right)^2\right].
\end{aligned}
\tag{21}
$$

This is the same as the result we found in Eq. (4.49) for transverse waves, since $A\rho$ is the mass per unit length, μ. As with Eq. (4.49), the present expression for $\mathcal{E}(x,t)$ is valid for an arbitrary wave. But if we consider the special case of a single traveling wave, then we have the usual relation, $\partial \psi/\partial t = \pm c\,\partial \psi/\partial x$. So the two terms in the expression for $\mathcal{E}(x,t)$ are equal at a given point and at a given time. We can therefore write the energy density per unit length as

$$\boxed{\mathcal{E}(x,t) = A\rho\left(\frac{\partial \psi}{\partial t}\right)^2} \qquad \text{(for traveling waves)} \tag{22}$$

The energy density per unit volume is then $\mathcal{E}/A = \rho(\partial \psi/\partial t)^2$.

Power

Consider a cross-sectional "sheet" of molecules. At what rate does the air on the left of the sheet do work on the sheet? (This is the same type of question that we asked in Section 4.4 for a transverse wave: At what rate does the string to the left of a dot do work on the dot?) In a small amount of the time, the work done by the air is $dW = F\,d\psi = (pA)\,d\psi$.

The power on the sheet whose equilibrium position is x is therefore

$$P = \frac{\partial W}{\partial t} = (p_0 + \psi_p) A \frac{\partial \psi}{\partial t} \,. \tag{23}$$

The $A p_0 (\partial \psi / \partial t)$ part of this averages out to zero over time, so we'll ignore it. The $A \psi_p (\partial \psi / \partial t)$ term, however, always has the same sign, for the following reason. From Eq. (5), we have $\psi_p = -(1/\kappa)(\partial \psi / \partial x)$. But as usual, $\partial \psi / \partial x = \mp (1/c)(\partial \psi / \partial t)$ (the minus sign is associated with a rightward traveling wave). So the power is

$$P = A \psi_p \frac{\partial \psi}{\partial t} = A \left(\pm \frac{1}{\kappa c} \cdot \frac{\partial \psi}{\partial t} \right) \frac{\partial \psi}{\partial t} = \pm \frac{A}{\kappa c} \left(\frac{\partial \psi}{\partial t} \right)^2 . \tag{24}$$

Using $c = 1/\sqrt{\kappa \rho} \Longrightarrow \kappa = 1/\rho c^2$, we have

$$\boxed{ P = \pm A \rho c \left(\frac{\partial \psi}{\partial t} \right)^2 } \tag{25}$$

Since $Z = A \rho c$ from Eq. (20), we can also write the power as

$$P = \pm Z \left(\frac{\partial \psi}{\partial t} \right)^2 , \tag{26}$$

which takes exactly the same form as the result in Eq. (4.53) for transverse waves.

If we compare Eqs. (22) and (25), we see that $P = \pm c \mathcal{E}$. This makes sense, because as with transverse waves, the power must equal the product of the wave velocity and the energy density, because the $\mathcal{E}$ curve moves right along with the wave.

If we want to write P in terms of the pressure ψ_p, we can do this in the following way. Using Eq. (5), Eq. (25) becomes

$$P = \pm A \rho c \left(\mp c \frac{\partial \psi}{\partial x} \right)^2 = \pm A \rho c^3 (-\kappa \psi_p)^2 = \pm A \rho c^3 \left(\frac{1}{\rho c^2} \right)^2 \psi_p^2 = \boxed{ \pm \frac{A}{\rho c} \psi_p^2 } \tag{27}$$

We see that when written in terms of ψ_p, the power decreases with ρ and c. But when written in terms of ψ (or rather $\partial \psi / \partial t$) in Eq. (25), it grows with ρ and c. The latter is fairly clear. For example, a larger ρ means that more matter is moving, so the energy density is larger. But the dependence on ψ_p isn't as obvious. It arises from the fact that there are factors of ρ and c hidden in ψ_p. So, for example, if ρ is increased (for a given function ψ), then ψ_p^2 grows faster than ρ, so the righthand side of Eq. (27) still increases with ρ. However, if ρ is increased for a given function ψ_p, then the power decreases, because the displacement ψ has to decrease to keep ψ_p the same, and this effect wins out over the increase in ρ, thereby decreasing P.

5.2.6   Qualitative description

Let's take a look at what a given molecule in the air is doing at a few different times, as a rightward-traveling wave passes by. A number of snapshots with phase differences of $\pi/4$ are shown in Fig. 8. The darker regions indicate a higher pressure (and density), and the lighter regions indicate a lower pressure (and density). The vertical line, which represents the equilibrium position of the molecule, is drawn for clarity.

The three plots at the top of the figure give the values of the various parameters at $t = 0$, as functions of x. So these plots correspond to the first of the shaded snapshots. The plots

for the eight other snapshots are obtained by simply shifting the three plots to the right, by the same amount as the snapshots shift. The three plots at the left of the figure give the values of the various parameters at $x = 0$, as functions of t. So these plots correspond to the molecule in question (the little circle). The leftmost plot (the one for $\psi(0,t)$) is simply a copy of the circles as they appear in the snapshots.[2]

Let's discuss what's happening in each of the snapshots. In doing this, a helpful thing to remember is that for a rightward-traveling wave, the density (and hence pressure) is always in phase with the velocity (as we discussed in Section 5.1). And both the pressure and the velocity are 90° ahead of the position, as functions of time. (For a leftward-traveling wave, the pressure is out of phase with the velocity, which is, as always, 90° ahead of the position.) Again, the displacements are exaggerated in the figures. In reality, the displacements are much smaller than the wavelength. The commentary on the snapshots is as follows.

Figure 8

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[2]The ψ values are exaggerated for emphasis. In reality, they are much smaller than the wavelength of the wave. But if we drew them to scale, all of the circles would essentially lie on the $x = 0$ line, and we wouldn't be able to tell that they were actually moving.

1. In the first snapshot, the molecule is located at its equilibrium position and is moving to the right with maximum speed.[3] The pressure (and density) is also maximum. The pressure is the same on both sides of the molecule, so there is zero net force, consistent with the fact that it has maximum speed and hence zero acceleration.

2. The molecule is still moving to the right, but it is decelerating ($a < 0$) because there is higher pressure (which goes hand-in-hand with higher density) on its right than on its left.

3. It has now reached its maximum value of ψ and is instantaneously at rest. It has the maximum negative acceleration, because the pressure gradient is largest here; the pressure is changing most rapidly (as a function of x) halfway between the maximum and minimum pressures. The difference between the forces on either side of the molecule is therefore largest here, so the molecule experiences the largest acceleration.

4. It has now started moving leftward and is picking up speed due to the higher pressure on the right.

5. It passes through equilibrium again, but now with the maximum negative velocity. This ends the period of negative acceleration. Up to this time, there was always higher pressure on the molecule's right side. For the next half cycle, the higher pressure will be on the left side, so there will be positive acceleration; see the "$a(0,t)$" plot in the left part of the figure.

6. It is moving to the left but is slowing down due to the higher pressure on the left.

7. It has now reached its maximum negative value of ψ and is instantaneously at rest. As in the third snapshot, the pressure gradient is largest here.

8. It has started moving rightward and is picking up speed due to the higher pressure on the left.

9. We are back to the beginning of the cycle. The molecule is in the equilibrium position and is moving to the right with maximum speed.

5.3   Musical instruments

Musical instruments (at least the wind ones) behave roughly like pipes of various sorts, so let's start our discussion of instruments by considering the simple case of a standing wave in a pipe.

  Consider first the case where the pipe is closed at one end, taken to be at $x = 0$. The air molecules at the closed end can't move into the "wall" at the end. And they can't move away from it either, because there would then be a vacuum at the wall which would immediately suck the molecules back to the wall. This boundary condition at $x = 0$ tells us that the wall must be a node of the $\psi(x,t)$ wave. The standing wave for the displacement must therefore be of the form,

$$\psi(x,t) = A \sin kx \cos(\omega t + \phi). \tag{28}$$

What does the pressure wave look like? Since $\psi_p = -(1/\kappa)(\partial\psi/\partial x)$ from Eq. (5), we have

$$\psi_p(x,t) = -\frac{Ak}{\kappa} \cos kx \cos(\omega t + \phi). \tag{29}$$

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[3]Or, it is moving to the left if we have a leftward-traveling wave. After reading through this commentary, you should verify that everything works out for a leftward-traveling wave.

Looking at the x dependence of this function, we see that nodes of ψ correspond to antinodes of ψ_p, and vice-versa.

If we instead have an open end at $x = 0$, then the boundary condition isn't as obvious. You might claim that now the air molecules move a maximum amount at the open end, which means that instead of a node in ψ, we have an antinode. This is indeed correct, but it isn't terribly obvious. So let's consider the pressure wave instead. If we have a standing wave inside the pipe, then there is essentially no wave outside the pipe. (Well, there must of course be *some* wave outside, given that there are sound waves hitting your ear.) So the pressure outside must be (essentially) the atmospheric pressure p_0. In other words, $\psi_p = 0$ outside the pipe. And since the pressure must be continuous, the boundary condition at the open end at $x = 0$ is $\psi_p = 0$. So the pressure has a *node* there. The pressure can therefore be written as

$$\psi_p(x, t) = B \sin kx \cos(\omega t + \phi). \tag{30}$$

The $\psi(x, t)$ function that satisfies $\psi_p = -(1/\kappa)(\partial\psi/\partial x)$ is then

$$\psi(x, t) = \frac{B\kappa}{k} \cos kx \cos(\omega t + \phi). \tag{31}$$

(A nonzero constant of integration would just give a redefinition of the equilibrium position.) So ψ does indeed have an antinode at $x = 0$, as we suspected.

The above results hold for any open or closed end, independent of where it is located. It doesn't have to be located at the arbitrarily-chosen position of $x = 0$, of course. So Fig. 9 shows the lowest-frequency (longest-wavelength) modes for the three possible cases of combinations of end types: closed/open, closed/closed, and open/open. In practice, the pressure node is slightly outside the open end, because the air just outside the pipe vibrates a little bit.
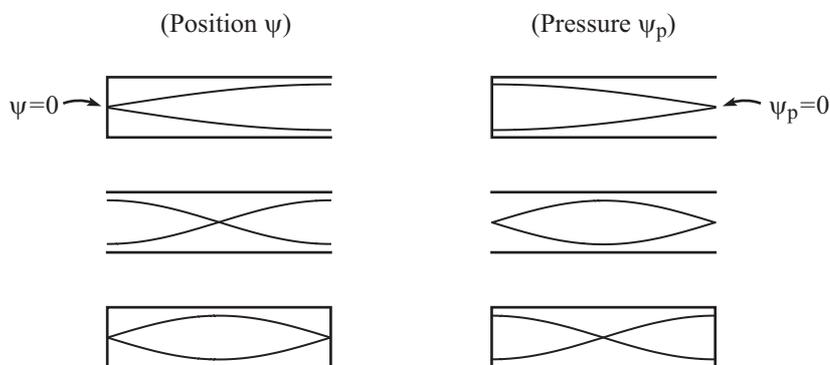

(Position ψ)   (Pressure ψ_p)

Figure 9

An instrument like a flute is essentially open at both ends (with one end being the mouthpiece). But most other instruments (reeds, brass, etc.) are open at one end and essentially closed at the mouthpiece end. This is due to the fact that the vibrating reed (or the vibrating lips in the mouthpiece) doesn't move much (so $\psi \approx 0$), but it is what is driving the pressure wave (so ψ_p is maximum there). A clarinet therefore corresponds to the first case (closed/open) in Fig. 9, while a flute corresponds to the second case (open/open). In view of this, you can see why a clarinet can play about an octave lower (which means half the frequency) than a flute, even though they have about the same length. The longest

(Pressure ψ_p)

L

$\lambda=4L$

$\lambda=(4/3)L$

$\lambda=(4/5)L$

$\lambda=(4/7)L$
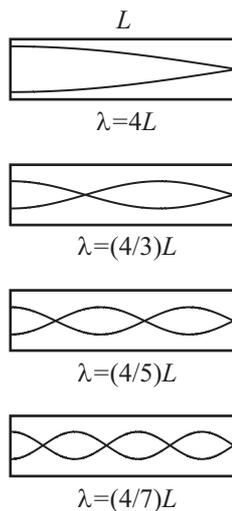
Figure 10

wavelength for a clarinet (which is four times the length of the pipe) is twice a long as the longest wavelength for a flute (which is two times the length of the pipe).[4]

The lowest four standing waves for a clarinet are shown in Fig. 10 (described by the pressure waves, which is customary). The wavelengths are $4L$, $4L/3$, $4L/5$, $4L/7$, etc. The frequencies are inversely proportional to the wavelengths, because $\nu\lambda = c \implies \nu \propto 1/\lambda$. So the frequencies are in the ratio of $1 : 3 : 5 : 7 : \cdots$. These notes are very far apart. For an open/open pipe like a flute, the wavelengths are $2L$, $2L/2$, $2L/3$, $2L/4$, etc., which means that the frequencies are in the ratio of $1 : 2 : 3 : 4 : \cdots$. These notes are also very far apart. So if a clarinet or a flute didn't have any keys, you wouldn't be able to play anywhere near all of the notes in a standard scale.

Keys remedy this problem in the following way. If all the keys are closed, then we simply have a pipe. But if a given key is open, then this forces the pressure wave to have a node at that point, because the pressure must match up with the atmospheric pressure there. So we have essentially shortened the pipe by creating an effectively open end at the location of the open key. With many keys, this allows for many different effective pipe lengths, and hence many different notes. And also many different ways to play a given note. If we have a particular standing wave in the instrument, and if we then open a key at the location of a (pressure) node, then this doesn't change anything, so we get the same note.

What about a trumpet, which has only three valves? It's a bit complicated, but the conical shape (at least near the end) has the effect of making the frequencies be closer together (and also higher). And the mouthpiece helps too. The end result (if done properly) is that the frequencies are in the ratio $2 : 3 : 4 : 5 : 6 : \cdots$ (for some reason, the 1 is missing) instead of the $1 : 3 : 5 : 7 : \cdots$ ratios for the closed/open case in Fig. 10. This is indeed the ratio of the frequencies of the notes (C,G,C,E,G,...) that can be be played on a trumpet without pressing down any valves. The valves then change the length of the pipe in a straightforward manner.

The flared bell of a trumpet has the effect (compared with a cylinder of the same length) of raising the low notes more than the high notes, because the long wavelengths (low notes) can't follow the bell as easily, so they're reflected sooner than the short wavelengths.[5] The longer wavelengths therefore effectively see a shorter pipe. However, another effect of the bell is that because the short wavelengths follow it so easily (right out to the outside atmospheric pressure), there isn't much reflection for these waves, so it's harder to get a standing wave. The high notes are therefore less well defined, and thus blend together (which is quite evident if you've ever heard a trumpet player screeching away in the high register).

Everything you ever wanted to know about the physics of musical instruments can be found on this website: http://www.phys.unsw.edu.au/music

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[4]The third option in Fig. 9, the closed/closed pipe, isn't too conducive to making music. Such an instrument couldn't have any keys, because they provide openings to the outside world. And furthermore, you couldn't blow into one like in reed or brass instrument, because there would be no place for the air to come out. And you can't blow across an opening like in a flute, because that's an open end.

[5]The fact that the shorter wavelengths (high notes) can follow the bell is the same effect as in the "Gradually changing string density" impedance-matching example we discussed in Section 4.3.2.