A brief introduction to the work of Haruzo Hida

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Abstract

I want to thank the organizers for scheduling this hour as an introduction to the work of Haruzo Hida. What a pleasure it is to reflect on the span of Haruzo's mathematics! The extraordinary range of his great accomplishments include

- his seminal analytic formulas related to adjoint representations of automorphic forms, and his profound discoveries on both the analytic and arithmetic sides of this picture—
- his invention and development of *Hida families* of ordinary and nearly ordinary automorphic forms of reductive groups, which is so crucial to much current progress in our field—
- his work on the anti-cyclotomic main conjecture (joint with J. Tilouine)—
- his study of $\mathcal{L}$-invariants and exceptional zeroes—
- his contributions to the theory of Iwasawa μ-invariants.

What follows is a very brief introduction to two of these themes and their interconnection.

Contents

1 Plan

Since we only have an hour for this *introduction* and Hida's contributions are vast, we must focus things a bit, and we'll do that by beaming onto an important breakthrough of Hida's dating back to 1979—in an unpublished paper written jointly with Doi—a result which one can view as a *seed idea* whose significance foreshadows Hida's later approach to some of his great contributions, and which encompasses much current deep work in the subject as well.

The nature of this idea is neatly conveyed in a quote from the article [3] written by Hida two years later.

> We are going to establish a coincidence of the following two rational numbers; namely the discriminant of the skew-symmetric or symmetric $\mathbf{Q}$-bilinear form associated to the primitive cusp form f of weight $k \geq 2$, and the rational part of the special value at the integer k of the 'zeta function' associated with the same cusp form. *As an application of this fact, one can prove congruences between this cusp form and another (non-Galois-conjugate) cusp form.*

Here I should clarify that in those days the term 'zeta function of f' was sometimes used to signify the adjoint L-function, i.e., the L-function of the symmetric square of the automorphic form f; so—for example—if

$$f(q) = q^1 + a_2q^2 + a_3q^3 + \cdots$$

is the Fourier expansion of the cuspidal Hecke eigenform f of weight k on $\Gamma_0(N)$, then the 'zeta function,' $Z(f,s)$, referred to in the quotation above would be the meromorphic (often entire) function now called $L(\mathrm{symm}^2(f), s)$ that is the meromorphic continuation of the (appropriate) Dirichlet series whose Euler factor at a 'good' prime p has the form

$$L^{\{p\}}(\mathrm{symm}^2(f), s) := 1 - \alpha_p^2 p^{-s})^{-1}(1 - p^{k-1-s})^{-1}(1 - \beta_p^2 p^{-s})^{-1}$$

where α_p, β_p are the roots of the quadratic polynomial $X^2 - a_pX + p^{k-1}$.

Now *any* significant text—and especially in mathematics—is never in isolation; it can't be, for otherwise it would be unintelligible; it is in conversation with things that have been written or said in the past, and it is anticipating conversation with what is to be said or written in the future. My plan, then, is to discuss what is behind the mathematics referred to in the above quotation, touching on some of its forerunners, and explaining how the seed ideas in it have expanded subsequently in Hida's work.

Here again is the sentence in it that I italicized:

> *As an application of this fact, one can prove congruences between this cusp form and another (non-Galois-conjugate) cusp form.*

The essential theme here then is to give computable conditions that predict the *existence* of cuspforms that are congruent to a given one. With this in mind, we might begin our excursion talking about the most classical aspects of congruences of cuspforms.

2 Congruences and the 'continuous spectrum'

2.1 Elementary congruences

1. Congruences are everywhere, and even the most elementary of them, such as

$$(a+b)^{\ell} \equiv a^{\ell} + b^{\ell} \mod \ell$$

for ℓ a prime number, have had a profound effect on our field. To cite an utterly randomly chosen minor example of this, consider the infinite product expansion of the Fourier series for the classical cuspform of weight 12 of level 1:

$$\Delta(q) = q\prod_{m}(1-q^m)^{24}$$

whose Fourier coefficients $n \mapsto \tau(n)$ are given by Ramanujan's tau-function (which is related to a great number of important arithmetic questions) and compare it to that of the cuspform of weight 2 of level 11,

$$\omega(q) = q\prod_{m}(1-q^m)^2(1-q^{11m})^2 \; = \; \sum_{n} a_n q^n$$

whose prime Fourier coefficients $\{p \mapsto a_p\}$ give us $N_E(p) :=$ the number of rational points mod p on the elliptic curve

$$E : y^2 + y = x^3 - x^2$$

by the formula $N_E(p) = 1 + p - a_p$. Since

$$1 - q^{11m} \equiv (1-q^m)^{11} \mod 11$$

and since this switch from $1 - q^{11m}$ to $(1-q^m)^{11}$ converts one of these infinite products into the other, we get that

$$\tau(n) \equiv a_n \mod 11$$

telling us that—among all the other things it gives us— $\tau(p)$ also gives us the number, at least mod 11, of solutions mod p of the equation $y^2+y=x^3-x^2$. Also—using our standard convention of saying that two modular forms are *congruent modulo* m if their corresponding Fourier coefficients are—we can say:

$$\Delta \equiv \omega \mod 11.$$

2. One of the most prolific generators of congruences among modular forms is the elementary *Fermat's Little Theorem* which says that

$$a^k \equiv a^{p-1+k} \mod p$$

where p is any prime number, and k any integer, and $a \not\equiv 0 \mod p$ and, more generally, Euler's extension of it:

$$a^k \equiv a^{\phi(N)+k} \mod N$$

for $(a,N)=1$ and where $\phi(N)$ is Euler's Phi-function.

The power of *Fermat's Little Theorem* to generate congruences is illustrated by comparing the classical Eisenstein series of weight k (here $k \geq 2$, k is even; and keep $k \not\equiv 0 \mod (p-1)$ in this discussion).

$$G_k = -\frac{b_k}{2k} + \sum_{n=1}^{\infty}\{\sum_{d \mid n} d^{k-1}\}q^n$$

taken modulo p (i.e., its Fourier coefficients being taken modulo p) with the Eisenstein series of weight $k+p-1$

$$G_{k+p-1} = -\frac{b_{k+p-1}}{2(k+p-1)} + \sum_{n=1}^{\infty}\{\sum_{d \mid n} d^{k-1}d^{p-1}\}q^n$$

or, more generally, comparing it modulo p^r with the Eisenstein series of weight $k+\phi(p^r)$:

$$G_{k+\phi(p^r)} = -\frac{b_{k+\phi(p^r)}}{2(k+\phi(p^r))} + \sum_{n=1}^{\infty}\{\sum_{d \mid n} d^{k-1+\phi(p^r)}\}q^n.$$

Here it is Euler's Theorem that guarantees that the *nonconstant* coefficients of $G_{k+\phi(p^r)}$ are congruent to the corresponding *nonconstant* coefficients of G_k modulo p^r. And it is the classical *Kummer congruence* that guarantees the analogous result for the constant coefficients, giving us that

$$G_{k+\phi(p^r)} \equiv G_k \mod p^r.$$

In fact, an argument of Serre (see [16]; also [15]) and by Katz ([13]) allows you to *use* modularity of these Fourier series together with the congruences we've just discussed between nonconstant coefficients to *prove* the analogous congruence for the constant coefficients; i.e., to prove the classical *Kummer congruences* as a derivative of, in effect, Euler's Theorem.

2.2 p-Adic interpolation

We'll exclude the case $p=2$ from now on. Working with these congruences modulo p^r between modular forms of weights k and $k+\phi(p^r)=k+(p-1)p^{r-1}$, it is natural to pass to the (projective) limit of the sequence:

$$\cdots \to \mathbf{Z}/(p-1)p^{r-1}\mathbf{Z} \to \cdots \to \mathbf{Z}/(p-1)p\mathbf{Z} \to \mathbf{Z}/(p-1)\mathbf{Z}.$$

When the dust settles—i.e., as $r \to \infty$—we get a continuous one-parameter p-adic space (a commutative Lie group, in fact)

$$W := \lim_{r\to\infty} \mathbf{Z}/\phi(p^r)\mathbf{Z} = \lim_{r\to\infty} \mathbf{Z}/(p^{r-1}(p-1)\mathbf{Z}) = \lim_{r\to\infty} \mathbf{Z}/p^{r-1}\mathbf{Z} \ \times \ \mathbf{Z}/(p-1)\mathbf{Z}$$

which we will refer to as *p-adic weight space*. This isomorphism provides W with a canonical product decomposition

$$W = \mathbf{Z}_p \times \mathbf{Z}/(p-1)\mathbf{Z}$$

and we'll write $\kappa = (s, i)$ following this product decomposition, with $s \in \mathbf{Z}_p$, and $i \in \mathbf{Z}/(p-1)\mathbf{Z}$ being the image of κ under the projections to the factors. W can be viewed as a union of $p-1$ disjoint closed unit discs:

$$W = \sqcup_{i\in\mathbf{Z}/(p-1)\mathbf{Z}} W_i$$

where W_i is the inverse image of i under the natural map $W \to \mathbf{Z}/(p-1)\mathbf{Z}$. W contains the monoid of natural numbers

$$\mathbf{N} \subset W$$

the elements of which are called "classical weights." Since p is assumed odd, there is a natural further projection:

$$W \to \mathbf{Z}/(p-1)\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z},$$

allowing us to decompose W into *even* weights and *odd* weights; this will be useful below.

$$W = W_{\text{even}} \ \sqcup \ W_{\text{odd}}.$$

For each point $\kappa \in W$ and any integer d, we can define

$$d^{\{\kappa\}} := \lim_{r\to\infty} d^{w_r} \in \lim_{r\to\infty} \mathbf{Z}/p^r\mathbf{Z} \simeq \mathbf{Z}_p.$$

where the sequence $\{w_r\}_r$ are positive integers tending to infinity[1] such that $w_r \equiv \kappa \mod \phi(p^r)\mathbf{Z}$ for all r. Note that if d is divisible by p then $d^{\{\kappa\}} = 0$.

We have, thus, "p-adically interpolated" versions of the exponential function $k \mapsto d^k$ to give what are p-adic analytic functions of W. As a result we've also interpolated all the nonconstant Fourier coefficients of our family of Eisenstein series:

$$\kappa \mapsto \sum_{d \ | \ n} d^{\{\kappa\}-1}.$$

For $\kappa \in W_{\text{even}}$, the important construction of Kubota and Leopoldt—i.e., their "p-adic L-function"— performs, in effect, the analogous interpolation of the constant term of our family of classical Eisenstein series;

[1]Note that *even if $\kappa = w$ is an ordinary integer*, we will want the approximating w_r's to be positive numbers tending to infinity.

i.e., of interpolating the function $k \mapsto -\frac{b_k}{2k}$ for k ranging through positive integers, to obtain a p-adic meromorphic function ranging through $\kappa \in W_{\text{even}}$.

More specifically, first consider $\kappa \in W_{\text{even}} - W_0$ so that (by the first part of Kummer's Congruence)

$$-\frac{b_k}{2k} \in \mathbf{Z}_p.$$

Moreover, fixing a sequence of even positive integers $\{k_j\}_j$ which go to infinity (when viewed in $\mathbf{R}$) and which have the limit

$$\lim_{j\to\infty} k_j \;=\; \kappa \;\in\; W_{\text{even}} - W_0$$

(when viewed in W) we define

$$-b_\kappa/2\kappa := \lim_j -b_{k_j}/2k_j \;\in\; \mathbf{Z}_p.$$

Writing $\kappa = (s, i)$ as discussed above, one has

$$b_\kappa/2\kappa = L_p(1-s, \omega^i)$$

Here ω is the Teichmüller character, and "L_p" denotes the Kubota-Leopoldt p-adic L-function.

2.3 The continuous p-adic family of Eisenstein series

Putting all this together we have a construction of Serre (in [16]): giving a p-adic interpolation of the Fourier coefficients of the classical family of Eisenstein series. More exactly we may form, for every weight $\kappa \in W_{\text{even}}$ such that κ projects to the even number $i \neq 0$ modulo $p-1$, the p-adic continuous series of 'p-adic Hecke eigenforms on $\Gamma_0(p)$':

$$G_\kappa^{\{p\}}(q) := -\frac{1}{2}L_p(1-s;\omega^i) + \sum_n \big\{ \sum_{d \mid n} d^{\{\kappa\}-1} \big\} \cdot q^n \;\subset \mathbf{Z}_p[[q]].$$

The half-scare quotes around *'p-adic Hecke eigenforms on $\Gamma_0(p)$'* are just to record that one has to give a sense of what this means, specifically for nonclassical weights κ. But even when κ is the image of an even integer $k \geq 2$, and even though, in this case, $G_\kappa^{\{p\}}(q)$ is identifiable with a classical eigenform, there is a significant difference between $G_k(q)$ and $G_\kappa^{\{p\}}(q)$. For one thing, the coefficients of $G_\kappa^{\{p\}}(q)$ are just p-adic numbers; however they *are* in the image of

$$\mathbf{Q}(\mu_{p-1}) \hookrightarrow \mathbf{Q}_p$$

where we make an identification

$$\mu_{p-1} \;\overset{\sigma}{\simeq}\; \mathbf{F}_p^*$$

and then embed

$$\mathbf{F}_p^* \subset \mathbf{Z}_p^* \subset \mathbf{Q}_p^* \subset \mathbf{Q}_p$$

in the natural way. Using such an identification we may view $G_\kappa^{\{p\}}(q)$ as a genuine classical modular form with Fourier coefficients in $\mathbf{Q}(\mu_{p-1}) \hookrightarrow \mathbf{C}$; it has weight k and is on the group $\Gamma_0(p)$ with a nebentypus character that depends on σ and k.

Also, we don't just throw away the Eisenstein series with weights $2 \leq k \equiv 0 \mod p-1$, these having—by the von Staudt-Clausen Theorem—a constant term $-\frac{b_k}{2k}$ with negative ord_p. We have other plans for these Eisenstein series: We just divide by their constant terms to get a sequence

$$E_k(q) = 1 - \frac{2k}{b_k} \sum_{n=1}^{\infty} \{ \sum_{d \mid n} d^{k-1} \} q^n$$

which has the very useful property of being $\equiv 1 \mod p$, and which interpolates to give another family

$$E_\kappa^{\{p\}}(q)$$

for $\kappa \in W_0 \subset W_{\text{even}}$. The Fourier expansion of *any* member of this family is congruent to 1 mod p, and if $\kappa = 0 \in W_0$ then

$$E_\kappa^{\{p\}}(q) = E_0^{\{p\}}(q) \; = \; 1,$$

a fact that plays an important role in this story.

These families of eigenfunctions, varying p-adic analytically in their weights were first put forward by Serre, and can be viewed as the starting point of a significant amount of modern (p-adic) number theory.

It is natural to *try to* compare these families, for example, with the so-called "non-analytic" family of complex Eisenstein series—eigenfunctions of the Laplacian on the upper half-plane—that, at least, have Fourier coefficients varying as functions of a complex parameter s:

$$\mathrm{Eis}_s^{\{\infty\}}(z) := \frac{1}{2} \sum_{\gcd(m,n)=1} \frac{y^s}{|mz+n|^{2s}}$$

for $z = x + iy$ in the upper half plane $y > 0$ that had beginnings in work of Hecke and Reidemeister, and was developed by Maass (and, of course, later much more substantially developed by Langlands). These two families—the p-adic Eisenstein family and the nonanalytic complex Eisenstein family[2]—are parallel in a certain sense and can be thought of as being (part of) the 'continuous spectrum' of the eigenfunction decomposition of Banach and Hilbert spaces of modular functions. In the complex case the existence of such a continuous L_2-spectrum is related to the *noncompactness* of the fundamental domain due to the allowed behavior of these Eisenstein series at the cusps. In the p-adic case, one has the analogous relationship to the cusps; but as Hida has taught us, there are p-adic surprises coming from the so-called "discrete spectrum" here.

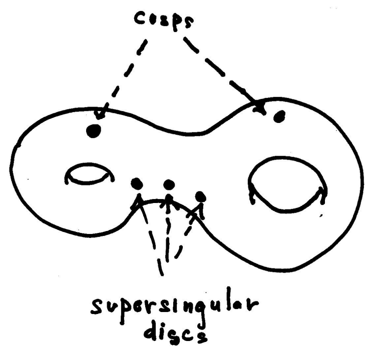
2.4 Noncompactness and continuous spectra

These p-adic modular eigenfunctions, and all the others we will be dealing with today all have the technical property of being *overconvergent* in the sense that

[2]One can also consider in the roster of Eisenstein-families, the more naive analytic family parametrized by the complex variable s with Fourier series:

$$\frac{1}{2}\zeta(1-s) \; + \; \sum_{n=1}^{\infty}\{\sum_{d \mid n} d^{s-1}\}q^n.$$

- they can be viewed ‘geometrically’ as sections of the appropriate line bundle on the appropriate p-adic modular curve,
- they are allowed to have essential singularities–but only of a specific controlled type—in small p-adic discs centered above the supersingular points in characteristic p. (To a p-adic modular form, every modular curve is noncompact!)



The Hecke operators and the U_p operator act as correspondences on the modular curves and induce operators on the corresponding spaces of overconvergent sections. By “eigenform” we will always mean an overconvergent p-adic modular form that is an eigenvector for the appropriate Hecke operators. Since these sections may have (allowed) singularities at the supersingular points, one is dealing with noncompactness—when working with any of these families—whether or not one requires regularity at the cusps. This is the underlying reason why there can be continuous families of p-adic cuspforms, i.e., despite the fact that the classical members of the family correspond to the *discrete series* in the (classical) harmonic analysis of these modular curves.

3 Hida’s early work on the L-function of the adjoint representation

3.1 f-Intrinsic versus f-extrinsic concepts

Let us return to the quotation from Hida’s 1981 article [3] that we started with this hour:

“ We are going to establish a coincidence of the following two rational numbers; namely:

$D(f) :=$ { the discriminant of the **Q**-bilinear form associated to the primitive cusp form f }

and

$L(f) :=$ { the rational part of the special value at the integer k of the 'zeta function' associated with f }

As an application of this fact, one can prove congruences between this cusp form and another (non-congruent) cusp form."

To connect this with the title of this section, the "$D(f)$" will be the *f-extrinsic concept* in that its definition involves consideration of the placement of f among all the eigenforms of its own weight and level; the "$L(f)$" is *f-intrinsic* insofar as its direct calculation involves only constructions related to f alone.

For this lecture, let's just concern ourselves with the rational prime divisors p of these numbers $D(f)$ and $L(f)$ and also only those p that are large compared to the weight $k \geq 2$, where—to be very specific—**large** means that $p > \max\{k-2, 3\}$.

3.2 The large prime divisors of $D(f)$

These are the (large) primes dividing the order of the fundamental module that describes—essentially tautologically—all possible congruences that f or its $\mathbf{Q}$-conjugates can have with other eigenforms of its weight and level. They are also what might be called *primes of fusion.* To be specific, for example, let f be a newform of weight $k \geq 2$ on $\Gamma_0(N)$ and let $S := S_k(N)^{\text{new}}$ denote the complex vector space generated by all cuspidal newforms of weight k on $\Gamma_0(N)$. Let $S_f = \mathbf{C} \cdot f \subset S$ be the complex line generated by f, and $S^f \subset S$ the orthogonal complement to S_f under the Peterson inner product. So,

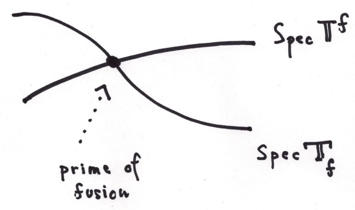
$$S = S_f \oplus S^f.$$

Now let $\mathbf{T} := \mathbf{T}_k(N)^{\text{new}}$ be the $\mathbf{Z}$-algebra acting faithfully on $S = S_k(N)^{\text{new}}$ generated over $\mathbf{Z}$ by the Hecke operators T_ℓ for ℓ not dividing N and by the Atkin-Lehner automorphisms and U_ℓ operators for $\ell \mid N$. Let $\mathbf{T}_f$ and $\mathbf{T}^f$ denote the quotient ($\mathbf{Z}$-)algebras of $\mathbf{T}$ that operate faithfully on S_f and S^f respectively. So $\mathbf{T}_f$ can be seen to be a sub-ring in the ring of integers of a number field[3] and more specifically, $\mathbf{T}_f$ is the subring generated by the Fourier coefficients of all $\mathbf{Q}$-conjugates of f. Denoting by $\pi_f : \mathbf{T} \to \mathbf{T}_f$ and $\pi^f : \mathbf{T} \to \mathbf{T}^f$ the natural surjections, we get an injection of $\mathbf{T}$-modules,

$$\mathbf{T} \quad \xrightarrow{\pi} \quad \mathbf{T}_f \bigoplus \mathbf{T}^f,$$

where $\pi := \pi_f \oplus \pi^f$. Viewing this geometrically on spectra, we have the picture

[3] that happens to be totally real, since we are dealing with Γ_0



and the primes of fusion associated to f measure the intersection of Spec($\mathbf{T}_f$) and Spec($\mathbf{T}^f$) viewed as closed subschemes of Spec($\mathbf{T}$). To phrase things more algebraically: since π_f and π^f are surjections, the cokernel of π is seen to be a finite cyclic $\mathbf{T}$-module, projecting isomorphically to (finite cyclic, of course) $\mathbf{T}_f$- and $\mathbf{T}^f$-modules under the homomorphisms π_f and π^f, respectively. The annihilator ideal of this cyclic $\mathbf{T}_f$-module is an ideal, $\mathcal{I}_f \subset \mathbf{T}_f$, that might be called the **congruence ideal** of f. Any prime p dividing the order of its index could be called a *prime of fusion* for f.

3.3 The large prime divisors of $L(f)$

Here, to be notationally as simple as possible but still retaining the essence of the idea, suppose that our eigenform f has rational integral coefficients; equivalently $\mathbf{T}_f = \mathbf{Z}$. So our congruence module $\mathcal{I}_f \subset \mathbf{T}_f$ is generated by an integer $c_f > 0$. The formula for $L(f)$ is then given by:

$$L(f) = \{\text{elementary term}\} \cdot \frac{\omega(f)}{\pi^{k+1}} \cdot Z(f,k)$$

where

- the "elementary term" is a truly elementary rational number: a power of 2 times a power of 3 times $(k-1)!N\phi(N)$,
- $\omega(f)$ is a certain period, and
- $Z(f,k)$ is the value at $s=k$ of the 'zeta function' we mentioned at the beginning of the hour[4].

[4]i.e., $Z(f,s) = L(\text{symm}^2(f),s)$ is the entire function extending the Dirichlet series $L(\text{symm}^2(f),s) = \prod_p L^{\{p\}}(\text{symm}^2(f),s)$ where $L^{\{p\}}(\text{symm}^2(f),s)$ is the appropriate Euler factor at p, which at good primes p is given by the formula

$$L^{\{p\}}(\text{symm}^2(f),s) = (1-\alpha_p^2 p^{-s})^{-1}(1-p^{k-1-s})^{-1}(1-\beta_p^2 p^{-s})^{-1}$$

where α_p, β_p are the roots of the quadratic polynomial $X^2 - a_pX + p^{k-1}$ and a_p is the p-th coefficient of the Fourier

By theorems of Sturm [18] and Shimura [17] this $L(f)$ is a rational number.

Now the reason for the label *f-intrinsic* for this type of information about f is that—as you can see—no data regarding modular eigenforms other than f itself has been invoked. And, yet(!) given Hida's theorem relating the large prime divisors of $L(f)$ to those of $D(f)$, any such prime divisor p of this f-intrinsic quantity *predicts* the existence of some other eigenform of the same weight and level as f, admitting a mod p congruence to f.

We'll discuss why this is so, in a while, but first, a comment about "predictions," and then a few randomly chosen examples.

3.4 Predictions

Hida's result, that the large prime divisors of $D(f)$ are **equal to** the large prime divisors of $L(f)$ gives us, then, two ways of obtaining those prime divisors: either by computing $D(f)$—which is perhaps best done via modular symbols technology—or $L(f)$, where the natural computations to make (of period and L-value) are of quite a different sort. It would be interesting to get good asymptotic bounds for the running times relative to weight and conductor of each of these kinds of algorithms to see which side "wins," and when. My guess is that, in general, the modular symbols methods are significantly faster, but that when restricting attention to certain types of eigenforms—for example, CM-forms—the $L(f)$-computation may very well win asymptotically[5]. I.e., so that one does get a serious *pre*-diction of new eigenforms congruent to the given f. See, for example, the illuminating computations of Hida on page 259 of [3], one of which we will make use of in subsection 3.5 below.

Given the current interest in algorithms, and given our capability of making large computational experiments, one can be motivated to view certain equations as saying—among whatever else they are saying— that the algorithm implied by the RHS has the same outcome as the algorithm implied by the LHS, and therefore as raising the subsidiary problem of determining the comparative asymptotic time-estimates for each of those

expansion

$$f(q) = q^1 + a_2q^2 + a_3q^3 + \cdots$$

[5] Commenting on an early draft of these notes, William Stein wrote:

> I agree that the L-functions method to compute this number will win – dramatically – if you're in *any* situation where you somehow know a lot of coefficients of the modular form. One example is CM forms, but there are others, e.g., anything that can be expressed in terms of CM forms and forms with known q-expansions like Eisenstein series (I have a recent paper `http://wstein.org/papers/nimft/` with Coates, Dokchitser, etc., where we compute certain special values for non-CM f using that we know lots of coefficients). Another situation is computing modular degrees, where Mark Watkins's approach via Flach/Shimura's formula is dramatically faster than using modular symbols, since one knows the a_p for an elliptic curve efficiently; until about 15 years ago, everybody computed these modular degrees using modular symbols, which was massively slower. Using Watkins's approach, one can predict congruences (by computing modular degrees) that are infeasible to ever actually see, e.g., the first (known) curve of rank 5 has modular degree $2^7 * 258659$. Thus there is a mod 258659 congruence between the newform f attached to the elliptic curve $y^2 + y = x^3 - 79x + 342$ and some newform g in $S_2(\Gamma_0(19047851))$; note that 19047851 is a prime. That newform f is *probably* defined over a number field of degree around $793660 = [(\dim S_2)/2]$, and it's not likely anybody could ever write down g. In short, we know—because of $L(f)$—that there is a congruence $f \equiv g \mod 258659$, but, beyond this, we know absolutely nothing explicit about g, not even the degree of the field it is really defined over (with certainty), and it isn't currently feasible as far as I can tell to compute anything about g either. This is a very concrete example of how $L(f)$ can, at times, be more approachable than $D(f)$.

algorithms; briefly, asking the question: under varying conditions of the computation, which side of the equation can be used more efficiently?

3.5 Two randomly chosen examples

1. Let $K = \mathbf{Q}(\zeta_3)$ where ζ_3 is a primitive third root of unity, and denote by $A := \mathbf{Z}(\zeta_3) \subset K$ the ring of integers in K. Form the CM eigenform on $\Gamma_1(3)$ of weight 13:

$$f(q) := \frac{1}{6} \sum_{a \in A} a^{12} q^{Na}$$

where $Na \geq 1$ is the norm of a. One can compute—by either the $D(f)$ or $L(f)$ route—that 13 is the unique 'large' prime dividing these common numbers—predicting the existence of some cusp form g in $S_{13}(\Gamma_1(3))$ (different from f) with its Fourier coefficients congruent 'modulo 13' to f. I'm thankful to Ben Lundell for communicating to me that

$$f(q) = q + 729q^3 + 4096q^4 - 153502q^7 + 531441q^9 + 2985984q^{12} - 9397582q^{13} +$$

$$+16777216q^{16} + 17886962q^{19} + O(q^{20}),$$

and that in the three dimensional space $S_{13}(\Gamma_1(3))$ the other two newforms in $S_{13}(\Gamma_1(3))$ are Galois conjugates (pick one and call it g), and have Fourier coefficients in the field $\mathbf{Q}(\sqrt{-26})$. So there is a unique prime P above $p = 13$ in the ring generated by the Fourier coefficients of g, and (as follows from Hida's Theorem) one indeed has that modulo P the Fourier coefficients of g are equal to those of f mod 13.

2. The smallest prime number N for which there are two non-Galois-conjugate newforms of weight 2 with "even parity," i.e., Atkin-Lehner eigenvalue -1 is $N = 67$. Letting f be the eigenform associated to the unique isogeny class of elliptic curves of conductor 67 with even parity, one computes that there is a prime of fusion (namely $p = 5$) and that $p = 5$ is the unique 'large prime' dividing $D(f)$ is 5, and indeed there is an eigenform g such that the field generated by its Fourier coefficients is $\mathbf{Q}(\sqrt{5})$ and modulo the unique prime above 5 in the ring generated by its Fourier coefficients it is congruent to f.

   Regarding this example, I should also say that I know of *no* Hecke algebra $\mathbf{T} = \mathbf{T}_k^{\text{new}}(\Gamma_0(N); \epsilon)$ associated to newforms of weight k on $\Gamma_0(N)$ for N squarefree and with prescribed signs ϵ for all Atkin-Lehner operators[6] with the property that $\text{Spec}(\mathbf{T})$ is *disconnected*; i.e., such that $\text{Spec}(\mathbf{T})$ has multiple components and yet there is no prime of fusion "connecting them."

   Question 1. *Is there an example of such a* $\text{Spec}(\mathbf{T})$ *that is disconnected?*

   For computations inspired by Maeda's Conjecture, that raise the very interesting question of *irreducibility* of such Hecke algebras $\mathbf{T}_k^{\text{new}}(\Gamma_0(N); \epsilon)$ for N squarefree and $k >> 0$, see [19].

3.6 The theme of deformations

weight constant Rather than attempting to outline a proof of Hida's theorem relating $D(f)$ to $L(f)$—which would take up this entire introductory hour—it makes sense to restrict the discussion to a few words about Hida's insight as seen from the vantage point of lots that have gone on in our subject since 1981; this viewpoint includes later results of Hida himself, of Flach, Wiles, Taylor, Kisin, and others. The aim here

[6] I.e., $\epsilon : \ell \mapsto \epsilon(\ell) \in \{\pm 1\}$ is the function on newforms $f \in \mathbf{T}_k^{\text{new}}(\Gamma_0(N); \epsilon)$ that give the eigenvalues of the Atkin-Lehner operators; i.e., we have $w_\ell f = \epsilon(\ell) \cdot f$ for all $\ell |\ N$.

is to hint about why it is not unnatural for the f-intrinsic number $L(f)$ to predict the existence of 'other' eigenforms.

Briefly, $L(f)$ can be understood as related to deformations. If V is the (ℓ-adic) Galois representation attached to the eigenform f and to its 'standard' L-function, which is the analytic continuation of a Dirichlet series whose Euler factor at any 'good' prime p is

$$(1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1}$$

then $\mathrm{End}^0(V)$, the trace zero endomorphism ring of V endowed with the adjoint Galois representation, is attached (after appropriate Tate twist) to the symmetric square automorphic form $\mathrm{symm}^2(f)$ and to the 'zeta function'

$$L(\mathrm{symm}^2(f), s) = Z(f, s)$$

which is the analytic continuation of a Dirichlet series whose Euler factor at any 'good' prime p is

$$(1-\alpha_p^2 p^{-s})^{-1}(1-p^{k-1-s})^{-1}(1-\beta_p^2 p^{-s})^{-1},$$

i.e., $Z(f,s)$ is the 'zeta function' we have been discussing, whose special value $Z(f,k)$ (times the appropriate period) gives us $L(f)$.

The gateway to Galois deformations of V (deformations that keep the weight constant and are required to satisfy various further features) is given by the appropriate cohomology of the adjoint representation $\mathrm{End}^0(V)$—indeed the cohomology subject to local conditions (i.e., Selmer group conditions) connected to the features desired for those deformations. An analogue of the Birch- Swinnerton-Dyer conjecture associated to the automorphic form $\mathrm{symm}^2(f)$ would then make it "not unnatural" to imagine the connection between $L(f)$ and $D(f)$. This connection was what Hida established directly, way back then.

Having introduced the theme, deformations, it is time to pass to one of Hida's grand theories having to do with p-adic deformations of cuspforms ('ordinary' at p) of varying weight.

4 Hida's p-ordinary cuspidal families

4.1 Revisiting the noncupidal families

We have already discusses p-adically varying families of objects parametrized by p-adic weight space; namely the p-adic Eisenstein families of Serre for $\kappa \in W_{\mathrm{even}}$:

$$G_\kappa^{\{p\}}(q), \quad E_\kappa^{\{p\}}(q).$$

Each $G_\kappa^{\{p\}}(q)$ is an eigenvector for the Hecke operators T_ℓ for $\ell \nmid p$:

$$T_\ell G_\kappa^{\{p\}} = (1+\ell^{\{\kappa\}})G_\kappa^{\{p\}}$$

and is fixed by the operator U_p:

$$U_p G_\kappa^{\{p\}} = G_\kappa^{\{p\}},$$

and similarly with $E_\kappa^{\{p\}}(q)$.

We can consider these noncuspidal families as setting up the *prototype* for Hida's vastly interesting class of p-adically varying continuous families (not merely of Eisenstein series, but) of p-ordinary p-adic cuspforms constructed by Hida.

4.2 The basic vocabulary of Hida's theory

Definition 1. *1.* **(The U_p-operator):** *If*

$$f(q) = \sum_n a(n)q^n$$

is a power series let

$$U_p f(q) := \sum_n a(pn)q^n.$$

(The classical Hecke operator U_p acting on modular forms on $\Gamma_0(N)$—if $p \mid N$—acts on Fourier series at the cusp ∞ via this formula.)

2. **(The p-ordinary condition):** *If $f(q) \in \mathbf{Z}_p[[q]]$ (or, more generally, in $A[[q]]$ where A is a discrete valuation ring which is a finite extension of $\mathbf{Z}_p$) say that $f(q)$ is a p-**ordinary** eigenvector for U_p if it is an eigenvector,*

$$U_p f = u_p \cdot f,$$

and its eigenvalue u_p is a unit in $\mathbf{Z}_p$ (or in the DVR A).

3. **(The p-ordinary projection operator):** *If $f(q) \in \mathbf{Z}_p[[q]]$ consider the following limit :*

$$f^{\mathrm{ord}} := \lim_{t\to\infty} U_p^{\phi(p^t)} f.$$

which (if it exists) will be called the **ordinary projection**[7] **of** f.

The ordinary projection operator is key to this theory; it is somewhat analogous to 'harmonic projection' in classical analysis. Note, for example, if f is a finite sum of U_p-eigenfunctions then f^{ord} will be the sum of all the p-ordinary eigenfunctions among them.

The key theorem that controls the rest of this extraordinary theory is the following. Let $S_k(\Gamma_1(N); \mathbf{Z}_p)$ be the $\mathbf{Z}_p$-module of "classical" cuspidal modular forms of weight k on $\Gamma_1(N)$ with Fourier coefficients in the ring[8] $\mathbf{Z}_p$.

Theorem 1. (Hida's theorem of constancy of p-ordinary rank) *Let $N = p \cdot N_0$ with N_0 not divisible by p. Then the rank of the p-ordinary subspace*

$$S_k^{\mathrm{ord}}(\Gamma_1(N); \mathbf{Z}_p) \subset S_k(\Gamma_1(N); \mathbf{Z}_p)$$

is independent of k if $k > 2$.

[7] More general, if the coefficients of f lie in a finite DVR extension A of $\mathbf{Z}_p$ with uniformizer π define:

$$f^{\mathrm{ord}} := \lim_{t\to\infty} U_p^{|(A/\pi^t A)^*|} f.$$

[8] You might wonder how you can get "classical" modular forms with coefficients in rings other than in $\mathbf{C}$, but there is a straightforward natural way of doing this by passing from $\mathbf{Z} \subset \mathbf{C}$ to $\mathbf{Z} \subset \mathbf{Z}_p$.

Put

$$S_k^{\text{ord}}(\Gamma_1(N); \mathbf{Q}_p) := S_k^{\text{ord}}(\Gamma_1(N); \mathbf{Z}_p) \otimes \mathbf{Q}_p.$$

Colloquially, one can say that the number of 'distinct' p-ordinary (cuspidal) eigenforms in $S_k(\Gamma_1(N); \mathbf{Q}_p)$ is constant ($k > 2$).

If you are interested in the number of distinct p-ordinary (cuspidal) eigenforms of weight k on $\Gamma_0(N)$ with *fixed character* ψ, this is periodic in the weight (> 2) with period $p - 1$. More precisely, the dimension of the $\mathbf{Q}_p$-vector space

$$S_k^{\text{ord}}(\Gamma_0(N), \psi\omega^{-k}; \mathbf{Q}_p).$$

is constant. For a discussion of this, with more related information specifically about weight $k = 2$ and p-ordinary forms on $\Gamma_1(Np)$, see sections 1 and 2 of [11].

A large-scale numerical investigation of the statistics of ordinary rank[9] might be an interesting project; e.g., what can one say about the distribution of the arithmetic function

$$r^{\text{ord}}(p, i) := \dim S_k^{\text{ord}}(\Gamma_0(p); \mathbf{Q}_p)$$

for $k \equiv i \mod p$?

This boils down to a computation of the characteristic polygon (mod p) of T_p acting on weight k cuspforms of level 1.

I asked William Stein about this, and he very quickly produced some data, and on reviewing it we soon realized that the essential data is best displayed if one has the following definition: For a given prime p and for an even integer $2 \leq k \leq (p+3)$ by the **p-ordinary defect of** k let us mean the difference of dimensions:

$$\delta(p, k) := \dim S_k(\Gamma_0(p); \mathbf{Q}_p) - \dim S_k^{\text{ord}}(\Gamma_0(p); \mathbf{Q}_p).$$

If the **p-ordinary defect of** k is zero, then every eigenform of low weight is p-ordinary. The following symmetry relationship holds for all data that he has computed so far (i.e. $p < 389$) :

$$\delta(p, k) = \delta(p, p + 3 - k).$$

Therefore the table below lists only pairs (p, k) for even integers $2 \leq k \leq (p+3)/2$ with the understanding that the full range $k \leq (p+3)$ can then be found, using the above symmetry [10].

[9]Kevin Buzzard had done such an investigation some years ago, and it might be good to extend the range of it.

[10]We see no reason for this symmetry to persist for all higher primes; see the appendix for the reason why —in any event—one might expect a close relationship between (p, k) and $(p, p + 3 - k)$.

(p,k)	defect
(59,16)	1
(79,38)	1
(107,28)	1
(131,40)	1
(139,36)	1
(151,60)	1
(173,24)	1
(193,72)	1
(223,72)	1
(229,116)	2
(257,50)	1
(257,100)	1
(257,130)	2
(263,98)	1
(269,78)	1
(277,92)	1
(283,72)	2
(307,78)	1
(313,114)	1
(331,84)	2
(353,76)	2
(379,56)	1

We have plans for a much larger project here.

4.3 A 'first example' of a Hida family

Let p be any prime in the range

$$11 \leq p \leq 7 \text{ billion}$$

except for $p = 2411$. (Visit the section "Conjecture on tau(n)" at wikipedia[11] which lists $p = 7758337633$ as the next prime number that should be avoided.)

For these primes p we have that the p-th Fourier coefficient, $\tau(p)$, of the classical newform Δ (of weight 12 and level 1) is a p-adic unit, and so we can factor

$$X^2 - \tau(p)X + p^{11} = (X - \alpha_p)(X - \beta_p) \in \mathbf{Z}_p[X]$$

where one of these roots, say α_p is a p-adic unit while $\mathrm{ord}_p(\beta_p) = 11$.

Form

$$\Delta^{\{p\}}(z) := \quad \Delta(z) - \beta_p \Delta(pz) \quad \in \quad S_{12}(\Gamma_0(p); \mathbf{Z}_p).$$

The newform Δ is "p-ordinary" in the sense that $\Delta^{\{p\}}$, its 'lifting' to $\Gamma_0(p)$, is a p-ordinary eigenform for (the Hecke operator T_ℓ for all primes $\ell \neq p$ and also for) U_p, where the eigenvalue of U_p is a p-adic unit:

$$U_p \Delta^{\{p\}} = \alpha_p \cdot \Delta^{\{p\}}.$$

Definition 2. *A modular form that is an eigenvector for U_p is said to be p-**ordinary** if its U_p-eigenvalue is a p-adic unit.*

[11] http://en.wikipedia.org/wiki/Tau-function#Conjectures_on

So $\Delta^{\{p\}}(z)$ is indeed p-ordinary. Since Δ is a generator of $S_{12}^{\rm ord}(\Gamma_0(1))$ and since any newform in $S_{12}^{\rm ord}(\Gamma_0(p))$ has slope[12] $5 = (12-2)/2$ it follows that $\Delta^{\{p\}}(z)$ alone generates $S_{12}^{\rm ord}(\Gamma_0(p); \mathbf{Q}_p)$.

Hida's constant rank theorem then gives us that:

Corollary 2. *The dimension of*

$$S_k^{\rm ord}(\Gamma_0(p), \omega^{12-i}; \mathbf{Q}_p)$$

is equal to 1 *for all* $k \geq 2$.

Now multiply $\Delta^{\{p\}}$ by the Eisenstein family $E_\kappa^{\{p\}}$—for

$$\kappa \in W_0 := \{\kappa \in W \mid \kappa \equiv 0 \mod (p-1)\}.$$

Note two things: Since the Fourier expansion of $E_\kappa^{\{p\}}$ is congruent to $1 = 1 + 0 \cdot q + 0 \cdot q^2 + \cdots$ modulo p, it follows that for any $\kappa \in W_0$, the Fourier expansion of the product, $\Delta^{\{p\}} \cdot E_\kappa^{\{p\}}$, is congruent modulo p to the Fourier expansion of $\Delta^{\{p\}}$. Moreover, since $E_0^{\{p\}} = 1$ the product, $\Delta^{\{p\}} \cdot E_\kappa^{\{p\}}$ in weight $\kappa = 12$, is just $\Delta^{\{p\}}$.

Now apply the p-ordinary projection operator (which we can show converges). We get a family

$$\mathcal{F}_{12+\kappa}^{\{p\}} := \{\Delta^{\{p\}} \cdot E_\kappa^{\{p\}}\}^{\rm ord}$$

with extraordinary properties:

1. $\mathcal{F}_{12}^{\{p\}} := \Delta^{\{p\}}$.
2. For any $\kappa \in W_0$, the Fourier expansion of $\mathcal{F}_\kappa^{\{p\}}$, is congruent modulo p to the Fourier expansion of $\Delta^{\{p\}}$.
3. If $\kappa \in W_0$ is the image of the integer $k \geq 2$, then $\mathcal{F}_{\kappa+12}^{\{p\}}$ is the unique generator of $S_{k+12}^{\rm ord}(\Gamma_0(p); \mathbf{Q}_p)$. In particular[13] it is an eigenvector for the Hecke operator T_ℓ for all primes $\ell \neq p$ and also for U_p.
4. This family
$$\kappa \mapsto \mathcal{F}_\kappa^{\{p\}}$$
defined for $\kappa \in W_0$ extends[14] to a family $\kappa \mapsto \mathcal{F}_\kappa^{\{p\}}$ defined for all $\kappa \in W$ with essentially the same properties, except that at any classical weight $2 \leq k = \kappa \in W_i$ the p-adic modular form $\mathcal{F}_\kappa^{\{p\}}$ corresponds to a classical modular (cuspidal p-ordinary) eigenform[15] on $\Gamma_0(p)$ with nebentypus ω^{-i}.
5. The extended family $\kappa \mapsto \mathcal{F}_\kappa^{\{p\}}$ has a p-adic integral structure. To prepare to explain this, form
$$\Lambda := \mathbf{Z}_p[[\mathbf{Z}_p^*]],$$

[12]The **slope** of a U_p-eigenform is the p-adic ord of its U_p-eigenvalue.

[13]since it is the unique generator, and the Hecke operators preserve $S_{k+12}^{\rm ord}(\Gamma_0(p); \mathbf{Q}_p)$,

[14]To get the extension of this family to W_i for any $i \in \mathbf{Z}/(p-1)\mathbf{Z}$ perform the same construction starting not with Δ but rather with a generator of $S_k^{\rm ord}(\Gamma_0(p), \omega^{12-i}; \mathbf{Q}_p)$.

[15]It follows by continuity that for every $\kappa \in W$, $\mathcal{F}_\kappa^{\{p\}}$ is an eigenvector for the Hecke operator T_ℓ for all primes $\ell \neq p$ and also for U_p.

the **Iwasawa ring**. There is a natural homomomorphism $W \to \mathrm{End}(\mathbf{Z}_p^*)$ and any endomorphism of $\mathbf{Z}_p^*$ extends to a ring homomorphism

$$\Lambda := \mathbf{Z}_p[[\mathbf{Z}_p^*]] \to \mathbf{Z}_p.$$

In this manner, every $\kappa \in W$ gives us, in a natural way, a ring homomorphism which we'll denote by the same letter:

$$\kappa : \Lambda \to \mathbf{Z}_p.$$

For the family $\kappa \mapsto \mathcal{F}_\kappa^{\{p\}}$ (with $\kappa \in W$) that we have been discussing we have the following theorem:

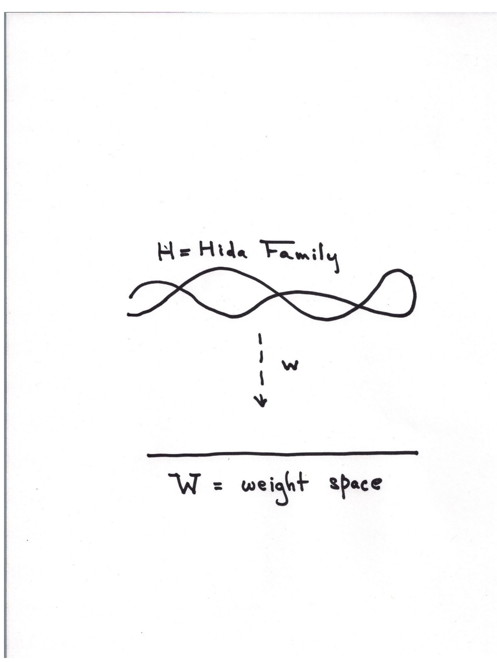
Theorem 3. *For any prime $\ell \neq p$ there is an element $t_\ell \in \Lambda$ such that for any $\kappa \in W$ the eigenvalue of T_ℓ acting on $\mathcal{F}_\kappa^{\{p\}}$ is equal to $\kappa(t_\ell) \in \mathbf{Z}_p$. Also, there is an element $u_p \in \Lambda$ such that for any $\kappa \in W$ the eigenvalue of U_p acting on $\mathcal{F}_\kappa^{\{p\}}$ is equal to $\kappa(u_p) \in \mathbf{Z}_p$.*

4.4 Hida Families for GL_2 over $\mathbf{Q}$

We have given an example, in the previous subsection, where Hida's theorem gave us *constant p-ordinary rank* 1. We constructed a family of p-ordinary p-adic cuspidal eigenforms—a p-adic analytic curve—that projected isomorphically onto weight space (a union of $p-1$ discs. The construction, though is vastly more general. For any $N = p \cdot N_o$ with $(p, N_o) = 1$, if

$$\dim_{\mathbf{Q}_p} S_k^{\mathrm{ord}}(\Gamma_0(N), \psi; \mathbf{Q}_p) \;=\; r$$

one obtains a p-adic analytic family of p-ordinary p-adic cuspidal eigenforms f forming a p-adic analytic space $H \to W$ projecting by a finite flat mapping of degree r to the corresponding weight space:



These are the *Hida Families*. Slightly more technically, we may view Hida families as p-adic rigid analytic spaces $w : H \to W$ and consequently we talk of their points *rational over* $\mathbf{Q}_p$ which is, in effect, all that we have done so far; or we can also go further and consider their rational points over extension fields of $\mathbf{Q}_p$ such as Tate's "p-adic complex numbers," $\mathbf{C}_p :=$ the p-adic completion of the algebraic closure of $\mathbf{Q}_p$. The Hecke operators T_ℓ for primes ℓ not dividing the level N, or the U_ℓ for $\ell \mid N$ act in a rigid p-adic analytic manner on the points f of H and their eigenvalues as functions of f then give us rigid p-adic analytic functions $T_\ell \mapsto \mathcal{T}_\ell$ and $U_\ell \mapsto \mathcal{U}_\ell$ on the p-adic analytic space H.

Even more striking is that Hida families all have analogous very tight p-adic integral structures as in Theorem 3 above[16]. That is, any Hida family is a component or union of components of the rigid analytic space attached to some "Hida-Hecke algebra" $\mathbf{T}$.

These Hida-Hecke algebras $\mathbf{T}$ contain elements identified with Hecke operators and have the property that for any continuous homomorphism $h : \mathbf{T} \to \mathbf{C}_p$ there is an overconvergent p-ordinary cuspidal eigenform f_h such that for all primes ℓ the image of $T_\ell \in \mathbf{T}$ (resp., $U_\ell \in \mathbf{T}$) under h gives the eigenvalue of the action of T_ℓ (resp., U_ℓ) on f_h. Moreover, *every* classical[17] p-ordinary cuspidal eigenform is a member of some Hida family.

4.5 The corresponding Galois representations

Over every Hida family of level N viewed as p-adic rigid analytic space $w : H \to W$, there is a canonical two-dimensional vector bundle[18] V with continuous action of the absolute Galois group $G_{\mathbf{Q}} := \mathrm{Gal}(\bar{\mathbf{Q}})$ on (the p-adic rigid analytic vector bundle) V over H,

$$\rho : G_{\mathbf{Q}} \to \mathrm{Aut}_H(V),$$

with the property that

- ρ is unramified outside the level $N = N_o p$ and
- the trace of the action of the Frobenius element(s) attached to any prime $\ell \nmid N$ on points f of H—when viewed as function on H—is equal to the function $\mathcal{T}_\ell$ defined in the previous subsection[19].

(It is also true that the determinant, $\det(\rho) : H \to \mathbf{G}_m$, is essentially–after mild renormalization, equal to the projection to weight space, $w : H \to W$.)

This associates to any Hida family $H \to W$ of level $H = N_o p$ a corresponding rigid p-analytic family of $G_{\{\mathbf{Q},N\}}$- representations parametrized by H. Here $G_{\{\mathbf{Q},N\}}$ is the Galois group of a maximal subextension, Galois over $\mathbf{Q}$, of an algebraic closure of $\mathbf{Q}$ that is unramified for all primes not dividing N. So, if we let

[16]This p-adic integrality feature of Hida families—plus the fact that the mapping to weight space is of finite degree—is where, at least at present–Hida families, which correspond to the "slope zero" part of the eigencurve, distinguish themselves from the other components of the eigencurve that parametrize p-adic overconvergent eigenforms of finite positive slope.

[17]or more generally overconvergent

[18]This vector bundle with Galois action is elegantly constructed by applying the *p-ordinary projection operator* to an appropriate limit of p-adic cohomology of a sequence of modular curves where p-power of the level of these curves tends to infinity.

[19]i.e., is equal to the eigenvalue of T_ℓ on f for f varying through the points of H.

denote the parameter of $\mathbf{C}_p$-valued points of H by $f \in H(\mathbf{C}_p)$, we have that f parametrizes a family of Galois representations

$$f \mapsto \{\rho_f : G_{\{\mathbf{Q},N\}} \longrightarrow \mathrm{GL}_2(\mathbf{C}_p)\},$$

these representations varying p-adic rigid analytically in the variable f.

These structures too have a p-adic integral property. That is, they are induced from continuous representations

$$G_{\{\mathbf{Q},N\}} \longrightarrow \mathrm{GL}_2(\mathbf{T})$$

where $\mathbf{T}$ is the appropriate Hida-Hecke algebra.

4.6 Universality

When studying a continuous representation of the absolute Galois group of a number field K into $\mathrm{GL}_n(R)$ for R a local noetherian ring it is standard, nowadays, to seek 'universality features.' That is, to formulate a certain collection of constraints, $\mathcal{C}$, such as specifying:

- the residual representation; i.e., the representation to the quotient $\mathrm{GL}_n(R) \to \mathrm{GL}_n(k)$ where k is the residue field of R,
- the determinant (or not specifying it),
- the local behavior, or some aspect of the local behavior, of the representation when restricted to the decomposition group at each place, nonarchimedean or archimedean,

and then proving that any representation of the absolute Galois group of K into $\mathrm{GL}_n(A)$ for any Artin local ring with residue field k that satisfies all the constraints $\mathcal{C}$ is induced from a unique homomorphism $R \to A$ that is compatible with the identity when one passes to residue fields.

The great work on modular lifting (due to Wiles, Taylor-Wiles, and others) have, as an application, that a large quantity of $G_{\mathbf{Q},N}$-representations attached to Hida families enjoy a *universal feature* (where one of the constraints in $\mathcal{C}$ is that the restriction of this Galois representation to a decomposition group at p has the p-ordinary Galois condition). The technique of Galois deformations has played an important role in some of Hida's other work, principally his establishment—jointly with Jacques Tilouine—of the anti-cyclotomic main conjecture for CM fields ([9]), thereby extending—but by a very different method—the earlier result of Karl Rubin who proved the conjecture for quadratic imaginary fields ([14]).

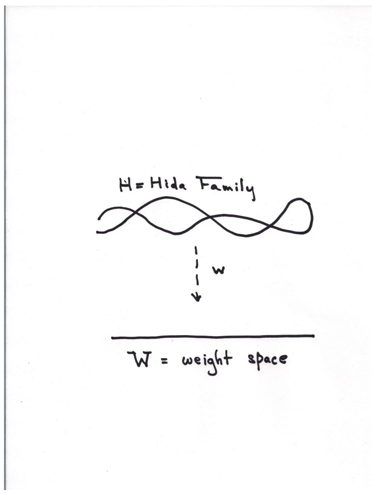
4.7 Hida's theory over more general reductive groups

This, of course, is merely the beginning of Hida's grand project which has now developed to encompass 'nearly ordinary' p-adic families for GL_n, (general n) and over general totally real fields. In the bibliography below we have selected some of Hida's early works connected—largely—to the aspects of his project concerning classical eigenforms, but for the full range, see the extensive bibliography of his books and articles in `http://www.math.ucla.edu/~hida/`

5 Returning to adjoint representations and the geometry of Hida families

5.1 The branch locus

In the figure of the previous section:



a salient feature is the discriminant locus of the projection $w : H \longrightarrow W$ of the Hida family to weight space. (Here if $\eta_\kappa \in H$ is an eigenform of weight κ, $w(\eta) = \kappa$.)

Since the L-function of the adjoint representation of a cuspform f of weight k has the property that $L(f)$, the algebraic normalization—i.e., the rational part—of its special value at $s = k$, detects congruences between f and other (non-Galois-conjugate) cuspforms of the same weight and level (see section 3) it is natural to guess that a more intimate geometric relationship between L-functions of adjoint representations and the discriminant locus of π could be established, by constructing (and concentrating on) the p-adic L-functions of the symmetric squares of the p-adic eigenforms in Hida families—or more generally, on the eigencurve. Without getting into any specifics, the general expectation is that if $L_p(\mathrm{symm}^2(f), \kappa = w(f))$ is the value of the p-adic L-function of $\mathrm{symm}^2(f)$ at the character $\kappa = w(f)$, and if

$$f \mapsto L_p(\mathrm{symm}^2(f), \kappa)$$

is viewed as a p-adic analytic function on the p-adic analytic space H—or more generally, on the eigencurve—then its divisor of zeroes should, if all goes well, give us the branch locus of the mapping w. Such a result was indeed proved by Walter Kim in his PhD thesis, and will be elaborated on in a forthcoming book by Joel Bellaïche on the eigencurve[20].

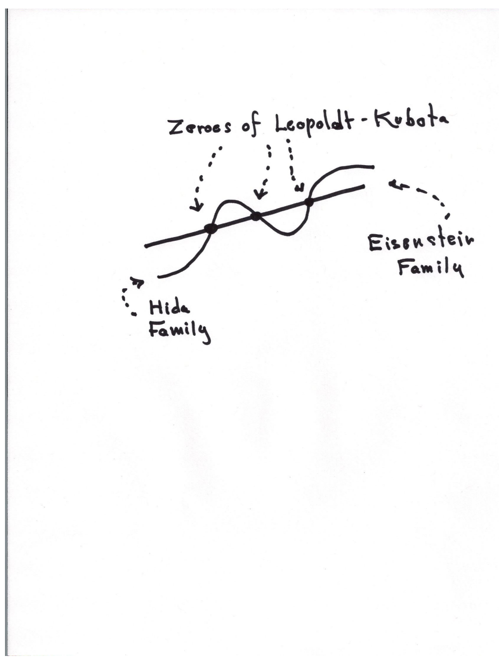
[20] In passing from classical to p-adic L-functions one sometimes picks up extra zeroes at certain critical points; the analysis of this phenomenon is therefore extremely important and related to Hida's contribution to the study of $\mathcal{L}$-invariants and exceptional zeroes.

5.2 The collisions between the Eisenstein locus and Hida families

Returning to the family of p-adic overconvergent eigenforms given by Eisenstein series

$$\kappa \mapsto G_{\kappa}^{\{p\}}(q) \;=\; -\frac{1}{2}L_p(1-s;\omega^i) + \sum_{n}\Big\{\sum_{d\,\mid\,n} d^{\{\kappa\}-1}\Big\}\cdot q^n \;\subset \mathbf{Z}_p[[q]]$$

for $\kappa \in W_{\rm even}$ one immediately notices that for every zero of the Kubota-Leopoldt L-function (i.e., for the κ's such that $L_p(1-s;\omega^i)=0$) the corresponding p-adic eigenform $G_{\kappa}^{\{p\}}(q)$ is, in effect, cuspidal. Consequently—by the claim at the end of subsection 4.4—is a member of some Hida family of *cuspidal* eigenforms. There are, of course, mysteries here.



5.3 The Archimedean question

Since, for any prime number p, any p-adic overconvergent p-ordinary eigenform (Eisenstein or cuspidal) 'fits into' a natural one-parameter family of such eigenforms varying p-adic analytically over weight space, it is natural to wonder whether there is some archimedean version of this theory that somehow embraces classical cuspforms, despite the fact that they correspond to the *discrete* part of the spectrum. And if not, why not?

Are there such families[21] other than the continuous families of complex Eisenstein series described in sub-

[21] together with a well-working "U_∞-operator"

section 2.2 above; either the non-analytic version

$$\mathrm{Eis}_s^{\{\infty\}}(z) := \frac{1}{2} \sum_{\gcd(m,n)=1} \frac{y^s}{|mz+n|^{2s}}$$

or the analytic one alluded to in the footnote in that subsection[22]?

6 Appendix: a 'near-symmetry' of p-defect

Definition 3. *If $p > 2$ is a prime number and k an even number in the range:*

$$1 < k < p + 3$$

say that (p,k) is **admissible**.

Definition 4. *Say that an odd representation*

$$\rho : G_{\mathbf{Q},\{p,\infty\}} \to \mathrm{GL}(\bar{\mathbf{F}}_p)$$

contributes to an admissible pair (p,k) *if there is a newform f of level 1 and weight k whose associated $G_{\mathbf{Q}}$-representation mod π is equivalent to ρ. Here π is some prime ideal in $\mathcal{O}_f$ (the ring of Fourier coefficients of f) with residue field $F := \mathcal{O}_f/\pi$ of characteristic p.*

6.1 The case where the mod p $G_{\mathbf{Q}_p}$-representation is irreducible

Definition 5. *Say that an admissible (p,k) is* **exceptional** *if there is a newform f of level 1 and weight k whose associated $G_{\mathbf{Q}_p}$-representation mod π is irreducible.*

One can extract the following theorem from the literature.

Theorem 4. *The pair (p,k) is exceptional if and only if $(p, p+3-k)$ is exceptional.*

Proof: We use the discussion directly after the statement of the Theorem on section 4.5 in [2]; compare Lemma 3.17 of [1]. The general set-up and notation follows Serre's classical article [15].

As mentioned in section 4.5 of [2], the content of Theorem 4 is essentially a consequence of Prop. 3.3 (of loc.cit.). Namely, consider an odd representation

$$\rho : G_{\mathbf{Q},\{p,\infty\}} \to \mathrm{GL}(\bar{\mathbf{F}}_p).$$

[22]That is,

$$\mathcal{E}_s(q) := \frac{1}{2}\zeta(1-s) \;+\; \sum_{n=1}^{\infty}\{\sum_{d\,\mid\,n} d^{s-1}\}q^n,$$

whose Fourier coefficients lie in a ring, Λ_∞, of complex analytic functions expressible as Dirichlet series in appropriate half-planes, and that extend to meromorphic functions on the entire plane, with certain growth conditions.

If the $G_{\mathbf{Q}_p}$-representation induced from ρ is irreducible, and if the two (diagonal) characters of niveau 2 associated to this representation are ψ^a, ψ'^b where $\psi, \psi' = \psi^p$ are the two fundamental characters of niveau 2, and $0 \leq a < b \leq p-1$, then we have the following facts regarding modularity of ρ. To be sure, ρ *is* modular, thanks to Khare and Wintenberger. Moreover ρ arises from some–in fact many—newforms of level one (i.e., arises as the mod π-representation associated to a modular newform f of level 1 where $\pi \subset \mathcal{O}_f$ is a prime ideal of residual characteristic p in $\mathcal{O}_f$, the ring generated by the Fourier coefficients of the newform f). The question, then, is to find the list of weights $k \leq p+3$ that correspond to such newforms f. Reading the list given in section 4.5 of loc. cit., we see that the realizable k's (for $k \leq p+3$) are: $k_1 = 1+b-a$ and $k_2 = p+2+a-b = p+3-k_1$. So, this gives a stronger result than Theorem 4:

Theorem 5. *If an odd representation*

$$\rho : G_{\mathbf{Q},\{p,\infty\}} \to \mathrm{GL}(\bar{\mathbf{F}}_p)$$

whose associated $G_{\mathbf{Q}_p}$-representation is irreducible contributes to (p,k) then a twist of it contributes to (p,k); moreover no other twist contributes to an admissible pair.

6.2 The case where the mod p $G_{\mathbf{Q}_p}$-representation is indecomposable but not irreducible

Now suppose have an odd representation

$$\rho : G_{\mathbf{Q},\{p,\infty\}} \to \mathrm{GL}(\bar{\mathbf{F}}_p)$$

whose restriction to $G_{\mathbf{Q}_p}$ is *reducible*. Let the two (diagonal) characters of niveau 1 be ω^b, ω^a where ω is the Teichmüller character, a, b are in the range $[0, p-1]$, and if the splitting field of ρ is wildly ramified, i.e., if the representation restricted to $G_{\mathbf{Q}_p}$ is given as

$\begin{bmatrix} \omega^b & * \\ 0 & \omega^a \end{bmatrix}$ and "$* \neq 0$," then the two diagonal characters occur in the indicated order, and we've normalized as in Serre's original article; i.e., $0 \leq a < p-1$ and $0 < b \leq p-1$.

Suppose first that "$* \neq 0$." Reading section 4.5 of loc. cit., we see that there is only one twist of ρ by a power of the Teichmüller character that comes from a newform f of weight k in the range $2 \leq k < p+3$, namely $\rho \otimes \omega^{-a}$. In this case, f is p-ordinary. (So, we're happy.) What is left is the reducible split case, i.e., where ρ is the direct sum of two characters.

6.3 The case where the mod p $G_{\mathbf{Q}_p}$-representation is semisimple, and not irreducible

Here is where it gets interesting. The pertinent invariants for such a representation is the pair (a, b) as in the previous subsection.

Definition 6. *Say that an admissible (p,k)* **contains** *an (a,b) (we may as well normalize so that $a \leq b$ and both lie in the range $[0, p-2]$) if there is a ρ whose restriction to $G_{\mathbf{Q}_p}$ is $\begin{bmatrix} \omega^b & 0 \\ 0 & \omega^a \end{bmatrix}$ and such that ρ is associated to a newform of level 1 and weight k.*

Using the discussion of section 4.5 in Eidixhoven, and the general literature (including results of Gross, Coleman & Voloch, and Faltings & Jordan) as reviewed in [1], it seems that the "'symmetry'" enjoyed by the exceptional (p,k) in the range currently calculated may not quite be enjoyed by the admissible (p,k)'s that contain (a,b)'s. Some of these (a,b)'s contained in admissible (p,k)'s are ordinary, but as mentioned already, William Stein and I have a project to examine the data regarding these issues more extensively.

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