Arithmetic in the Geometry of Symmetric Spaces

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Introduction. The interesting homology of the classical modular curves $X_0(N)$, $N \geq 1$, occurs in dimension one. Viewing $X_0(N)$ as the compactification of the quotient of the upper half plane H by the action of the subgroup $\Gamma_0(N) \subset SL_2(Z)$, one has available a 'natural family' of (relative) one-dimensional cycles -- paths on the Riemann surface whose endpoints lie in the set of cusps--- with which one may systematically understand the relative homology group $H_1(X_0(N), \text{cusps}; Z)$ and the action of the Hecke operators.

These cycles are, in effect, the modular symbols whose study was initiated by Birch, and concerning which a sizeable literature has developed . Their study was motivated by consideration of the special values of L functions arising as Mellin transforms of parabolic modular forms. They are, moreover, an indispensible tool in the p-adic analysis of these special values.

The family of cycles is readily defined: to each rational number r, consider the closed vertical line $I_r = \{r+iy \mid 0 \leq y \leq +\infty\}$ in the extended upper half plane $H^* = H \cup P^1(Q)$ ( ). The natural parametrization of $X_0(N)$ by H^* maps I_r to a path in $X_0(N)$ whose endpoints are cusps. Orienting I_r from ∞ to r, we obtain a relative cycle, whose class is denoted $\{r\} \in H_1(X_0(N), \text{cusps}; Z)$. This relative cycle depends only on r modulo one, and thus we may view the classical modular symbol as a mapping

$$\{\ \} : Q/Z \longrightarrow H_1(X_0(N), \text{cusps}; Z).$$

The object of this note is to draw attention to a generalization of the classical modular symbol to arbitrary reductive groups over $\mathbb{Q}$. The construction we shall outline produces 'natural families' of relative cycles in $\Gamma\backslash X$ the quotient of the symmetric space of a reductive algebraic group $G/_{\mathbb{Q}}$ by an arithmetic subgroup Γ. One such family is obtained for each admissible (cf. §) parabolic subgroup $P \subset G$ defined over $\mathbb{Q}$. The deep questions concerning these 'natural families' remain unanswered in any great generality:

(a) What portion of the compact homology of $\Gamma\backslash X$ is generated by each of the families constructed?

(b) What is the relation between these families and special values of L functions associated to automorphic representations of $G/_{\mathbb{Q}}$?

Nevertheless, some fragmentary reults already obtained impel us to focus on these questions in a broad context.

In § it will be shown that when $G = SL_n$ and P is an admissible Borel subgroup, the associated family of cycles 'interpolate p-adically' in a suitable sense. I understand that Michael Harris has generalized the construction of § to the case of G an arbitrary Chevalley group over Q ( ).

In § question (a) will be answered in what may be the only easy case: G a split group over Q, and P an admissible Borel subgroup.

In a joint paper with David Kazhdan, we will then answer (b) for certain L functions $L(\pi, \sigma, \chi, s)$ considered by Jacquet and Piatetski-Shapiro ( ) where π is a ("special" cf.) cuspidal automorphic representation of $GL_3/_{\mathbb{Q}}$, σ is a classical automorphic representation of $GL_2/_{\mathbb{Q}}$ (of weight 2), and χ is a finite Dirichlet character. ~~As a result we shall be able to p-adically interpolate these L functions under mild conditions.~~

1. <u>The modular symbol</u>.

Fix a reductive group G over $\mathbb{Q}$, and K, a maximal compact subgroup in its group of real points $G(\mathbb{R})$. Let $X = G(\mathbb{R})/K$. Then X is diffeomorphic to euclidean space. Let $d(G)$ denote the dimension of X. Fix $\Gamma \subset G(\mathbb{Q})$, an arithmetic subgroup. The group Γ acts properly on X. Let $Y = \Gamma\backslash X$.

If M is a reductive subgroup of G defined over $\mathbb{Q}$, we shall say that M is <u>admissible</u> (relative to the fixed data: K, Γ) if

(a) $M(\mathbb{R}) \cap K$ is a maximal compact subgroup of $M(\mathbb{R})$.

(b) $\Gamma \cap M(\mathbb{R})$ is torsionfree and operates as a group of orientation-preserving transformations on the euclidean space $M(\mathbb{R})/M(\mathbb{R}) \cap K$.

By an <u>admissible parabolic</u> P we shall mean a parabolic subgroup of G defined over $\mathbb{Q}$ (also denoted P) with a given ~~Levi~~ decomposition $P = M \ltimes U$ defined over $\mathbb{Q}$, whose reductive part M is admissible, and such that $M(\mathbb{R})/M(\mathbb{R}) \cap K$ is endowed with an orientation.

If $P = M \ltimes U$ is an admissible parabolic, and $u \in U(\mathbb{Q})$, let

$$Y(u,M) = \Gamma \cap u^{-1} \cdot \Gamma \cdot u \cap M(\mathbb{R}) \backslash M(\mathbb{R}) / M(\mathbb{R}) \cap K .$$

Then the spaces $Y(u,M)$ are connected oriented manifolds of dimension $m = d(M)$.

Let

$$F_u : Y(u,M) \longrightarrow Y$$

denote the natural map induced by

$$M(\mathbb{R}) \longrightarrow G(\mathbb{R}) ; \qquad x \longmapsto u \cdot x \qquad x \in M,$$

<u>Proposition</u> (Borel, Prasad): The mappings F_u are proper.

Proof of Lemma of Borel and Prasad.

Let $\mathbb{M} \subseteq G$ be reductive algebraic groups over $\mathbb{Q}$, and $\Gamma \subseteq G(\mathbb{Q})$ discrete subgroup. Then $M\cap\Gamma\backslash\mathbb{M} \longrightarrow \Gamma\backslash G$ is proper.

Proof: By Chevalley's theorem ( ) there is a finite-dimensional $\mathbb{Q}$-rational representation of G on a vector space $\underline{V}$, and a one-dimensional $\mathbb{Q}$-subspace $\underline{W}$ of $\underline{V}$ such that $\mathbb{M}$ is the stabilizer of $\underline{W}$ in $\underline{G}$. By complete reducibility, $\underline{V} = \underline{W} \oplus \underline{W}'$ (over $\mathbb{Q}$) as $\mathbb{M}$-module. If $\underline{V}^*$ is the dual of $\underline{V}$ and $\underline{W}^*$ the (1-dim'l) annihilator of $\underline{W}'$, then $\mathbb{M}$ fixes $\underline{W} \otimes \underline{W}^* \subseteq \underline{V} \otimes \underline{V}^*$ pointwise. So: $\mathbb{M}$ may be viewed as the isotropy subgroup of any nonzero element ω of $\underline{W} \otimes \underline{W}^*$. Clearly ~~[illegible]~~ $\Gamma\omega$ ~~[illegible]~~ is a discrete (hence closed) subset of $V \otimes V^* = (\underline{V} \otimes \underline{V}^*)(\mathbb{R})$. Therefore, the subset $E = \{g \mid g \in G, g\omega \in \Gamma\omega\}$ is a closed subset of G. But it is evident that $E = \Gamma \cdot M$. This proves that $\Gamma \cdot M$, hence also $M \cdot \Gamma$ is closed in G.

q.e.d.

Let $\underline{H}_i$ denote Borel-Moore homology with infinite supports ( ). Thus, if A is a finite polyhedron, and $B \subset A$ a finite closed sub-polyhedron, $\underline{H}_i(A;\ Z) \cong H_i(A;\ Z)$ and $\underline{H}_i(A-B;\ Z) \cong H_i(A,B;\ Z)$ where H_i denotes singular homology.

If $\overline{X}$ denotes the Borel-Serre completion of X ( ) then Γ acts properly on $\overline{X}$ and $\overline{Y} = \Gamma\backslash\overline{X}$ is compact (9.3); $\overline{Y}$ is a compactification of Y (the Borel-Serre compactification). If Γ is torsionfree, $\overline{Y}$ has, in a natural way, the structure of C^{∞} manifold with corners. One has an isomorphism:

$$\underline{H}_i(Y;\ Z) \cong H_i(\overline{Y}, \partial\overline{Y};\ Z)\ .$$

Since the spaces $Y(u,M)$ $(u \in U(Q))$ coming from an admissible parabolic P are oriented, connected m-dimensional manifolds, compactifiable in the Borel-Serre manner, we have canonical isomorphisms:

$$\underline{H}_m\,(Y(u,M)\ ;\ \ Z\ ) \xleftarrow[\cong]{\ \Theta\ } Z$$

Since the mapping F_u is proper, it induces a homomorphism on Borel-Moore homology ( ). Define $\phi(u,M) \in \underline{H}_m(Y\ ;\ Z\ ) \cong H_m(\ \overline{Y}, \partial\overline{Y};\ Z)$ to be the image of $1 \in Z$ under the composition:

$$Z \xrightarrow[\cong]{\ \Theta\ } \underline{H}_m(Y(u,M)\ ;\ Z) \xrightarrow{\ \ F_u\ \ } \underline{H}_m(\ Y\ ;\ Z)$$

Since $\phi(u,M)$ depends only on the right coset of u modulo $U(Q) \cap \Gamma$, we obtain the mapping

$$\phi\ :\ U(Q)/\ U(Q)\cap\Gamma \longrightarrow \underline{H}_m\,(Y\ ;\ Z) \cong H_m(\overline{Y}, \partial\overline{Y};\ \mathbb{Z})$$

which we shall call <u>the modular symbol</u> associated to the (admissible) parabolic subgroup P . A close analysis of special values of L series coming from adelic integrals leads one to study the geometry of modular symbols. Here are some questions that arise naturally, and have applications to the theory of L functions. If P is an admissible parabolic, let H(P) denote the sub- vector space of $H_m(\bar{Y},\partial\bar{Y};Q)$ generated by the cycles $\phi(u,M)$ for all $u \in U(Q)$. Determine a finite set of elements $u_1,\ldots,u_r \in U(Q)$ such that H(P) is generated by $\phi(u_j,M)$ $j=1,\ldots,r$. What is H(P) for the various admissible parabolics? Is $H_*(\bar{Y},\partial\bar{Y};\mathbb{Q})$ generated by all the H(P) ?

<u>Examples</u>:

1. $G = SL_{2/Q}$; $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$; $m = 1$; X = upper half-plane . If $u = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ for $r \in Q$, then $\phi(u,M)$ is the image of the vertical half-line $I_r = \{r + iy \mid \infty \geq y \geq 0\}$ in $H_1(\bar{Y},\partial\bar{Y}; Z)$. Thus, ϕ is the 'classical' modular symbol.

2. $G = Tr_{L/Q}\left(SL_{2/L}\right)$ where L is a number field, and $Tr_{L/Q}$ denotes the 'Weil trace' from L to Q. Let P be the 'Weil trace' of the parabolic $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ in $SL_{2/L}$. Here $m = r_1+r_2$ where r_1 = the number of real primes of L, and r_2 = the number of complex primes of L. The modular symbol is then a mapping

$$\phi : L/\mathfrak{a} \longrightarrow H_{r_1+r_2}(\bar{Y}, \partial\bar{Y}, Z)$$

where we have identified U(Q) with the additive group of the field L, and $\mathfrak{a}$ is the fractional ideal in L given as the image of $U(Q)\cap\Gamma$ under this identification.

3. $G = SL_3/\mathbb{Q}$. Then Y is a 5-dimensional topological space (a manifold if Γ is torsionfree). Its relative homology groups $H_i(\overline{Y}, \partial\overline{Y}; \mathbb{Q})$ can be nonzero only when $i = 0, 2, 3,$ or 5 . Thus the 'interesting' homology occurs in dimensions 2 and 3 . Taking P to be the Borel subgroup

$$P = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

we get m = 2, and the associated modular symbol is a mapping

$$\Phi : U(\mathbb{Q})/U(\mathbb{Q})\cap\Gamma \longrightarrow H_2(\overline{Y}, \partial\overline{Y}; \mathbb{Z})$$

whose image generates the vector space $H_2(\overline{Y}, \partial\overline{Y}; \mathbb{Q})$. (§)

Taking P to be one of the two maximal parabolics, e.g.

$$P = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$$

we get m=3. What is H(P) in this case?

§2. Hecke operators in symmetric spaces for SL_m :

Let V be an m-dimensional real inner product space, with $O(V) \subset GL(V)$ the associated orthogonal group. Let $RO(V)$ be the subgroup of $GL(V)$ generated by $O(V)$ together with real homotheties. By a lattice in V with full level structure we mean a pair (L,a) where $L \subset V$ is a lattice, and $a: L \to Z^m$ is a surjective (hence bijective) homomorphism. Equivalently, such a pair is given by a homomorphism $a': Q^m \longrightarrow V$ such that $a' \otimes \mathbb{R}$ is a bijection of real vector spaces. The correspondence $a' \longleftrightarrow (L,a)$ is given by setting $L = a'(Z^m)$ and $a = \left(a'^{-1} \mid L\right)$.

The group $GL_m(Q)$ operates on the set of lattices with full level structure by the rule $a' \longmapsto a' \cdot \gamma^{-1}$ for $\gamma \in GL_m(Q)$ while $GL(V)$ operates by $a' \longmapsto g \cdot a'$ for $g \in GL(V)$. Two lattices with full level structure a' and b' are equivalent if $b' = g \cdot a'$ for some $g \in RO(V)$. The inner product structure of V determines a canonical equivalence class of lattices with full level structure. Indeed a' is a member of the canonical equivalence class if and only if a' applied to the standard basis of $Z^m \subset Q^m$ is an orthonormal basis of V. Thus, for such an a', $a' \otimes \mathbb{R}: Z^m \otimes \mathbb{R} \longrightarrow V$ is an isomorphism of inner product spaces. A choice of a member a' of the canonical equivalence class determines a Z-structure on the inner product space V via the above isomorphism. Fix such a 'lattice with full level structure':

$$\underline{a}' \longleftrightarrow (\underline{L},\underline{a}) \qquad \text{(the \underline{base} lattice)}$$

and hence a $\mathbb{Z}$-structure on V.

If $X = RO(V) \backslash GL(V)$ is the symmetric space, we may make the identification:

$$X \longleftrightarrow \text{equivalence classes of lattices in } V \text{ with full level structure}$$

by the rule: $\quad RO(V)\cdot g \longleftrightarrow RO(V)\cdot a'$

where $g = (a'\otimes R)\cdot(\underline{a}'\otimes R)^{-1}$ $\quad$ (later we shall merely say: $g = a'\cdot\underline{a}'^{-1}$)

Our choice of base lattice also determines an injection $i: GL_m(Q) \hookrightarrow GL(V)$ by the rule $\underline{a}'\cdot\gamma^{-1} = i(\gamma)\cdot\underline{a}'$.

Viewing $GL(V)$ as the Lie group of real-valued points of $GL_{m/Q}$ (here we have made use of the fixed Z- (and hence Q-) structure on V), we may complete X to obtain $\overline{X}$, a manifold with corners, in the Borel-Serre manner. The action of $GL_m(Q)$ on X

$$(\gamma, x) \longmapsto x.i(\gamma)^{-1}$$

extends to an action on $\overline{X}$.

The discrete subgroups of interest to us will be:

(a) For N a positive integer $\Gamma(N)$, the principal subgroup of $SL_m(Z)$ of level N, is defined to be the subgroup of $SL_m(Z)$ consisting in all matrices congruent to 1 modulo N.

(b) For p a prime number and r an integer $0 \leqslant r \leqslant m$ $\Gamma(p,r)$ is the subgroup of $SL_m(Z)$ consisting in matrices which stabilize the kernel of the projection $Z^m \twoheadrightarrow (Z/pZ)^r$ (projection to the ~~first~~ last. r coordinates modulo p). [over]

(c) For N a positive integer, p a prime number not dividing N and r an integer $o \leqslant r \leqslant m$ set $\Gamma(N;p,r) = \Gamma(N)\cap\Gamma(p,r)$.

Also, for p,r as above, define $W_{p,r}$ to be the ~~diagonal~~ matrix

$$W_{p,r} \quad \begin{array}{cc} & \begin{array}{cc} m-r & r \end{array} \\ \begin{array}{c} m-r \\ r \end{array} & \left(\begin{array}{c|c} 0 & -1 \\ \hline p & 0 \end{array}\right) \end{array} \in GL_m(Q)$$

Note that if (L,a) is a lattice with full level structure, then $W_{p,r}^{-1}(L,a)$ is a lattice with full level structure $(\tilde{L},\tilde{a})$ where

$$\tilde{L} \subset L \subset p^{-1}\cdot\tilde{L} \subset V$$

and $L/\tilde{L}$ is an F_p-vector space of dimension r.

Since $w_{p,r}\cdot w_{p,m-r} = w_{p,m-r}\cdot w_{p,r} = -p$ the operators $w_{p,r}$ and $w_{p,m-r}$ on $\bar{X}$ are two-sided inverses of one another. [over]

Set $\bar{Y}(N) = \bar{X}/\Gamma(N)$, $Y(N) = X/\Gamma(N)$; $\bar{Y}(N;p,r) = \bar{X}/\Gamma(N;p,r)$ and $Y(N;p,r) = X/\Gamma(N;p,r)$.

There is a natural one-one correspondence between the points of $Y(N;p,r)$ and triples $(\tilde{L},L\ a_N)$ up to $RO(V)$ -equivalence, where,

(i) L is a lattice in V

(ii) $\tilde{L}$ is a sub-lattice such that

$$\tilde{L} \subset L \subset p^{-1}L \qquad \text{and } L/\tilde{L} \text{ is an } F_p\text{-vector space of dimension } r$$

($\tilde{L}$ will be called a (p,r)-<u>modification</u> of L)

(iii) a_N is a level N structure on L. That is, a_N is a surjective homomorphism from L to $(Z/NZ)^m$.

Indeed if a' represents a point of X, then its image in $Y(N;p,r)$ is represented by the triple $(\tilde{L},L,a_N)$ where $L = a'(Z^m)$, $\tilde{L} = (w_{p,r}^{-1}.a')(Z^m)$, and a_N = a modulo N.

The mapping $w_{p,r}: \bar{X} \longrightarrow \bar{X}$

induces an isomorphism (of manifolds with corners)

$$w^{-1}_{p,r}: \bar{Y}(N;p,r) \longrightarrow \bar{Y}(N;p,m-r)$$

The inclusion $\Gamma(N;p,r) \subset \Gamma(N)$ induces ~~/~~ proper morphisms

$$i: \bar{Y}(N;p,r) \longrightarrow \bar{Y}(N)$$
$$Y(N;p,r) \longrightarrow Y(N)$$

Define the Hecke operator $T_p(r)$ to be the endomorphism of $H^*(\bar{Y}(N), \partial\bar{Y}(N); A) = H^*_c(Y(N); A)$ given by the composition

$$H^*_c(Y(N);A) \xrightarrow[i^*]{} H^*_c(Y(N;p,r);A) \xrightarrow[w_{p,r}]{} H^*_c(Y(N;p,m-r); A) \xrightarrow[i_*]{} H^*_c(Y(N); A)$$

for an arbitrary coefficient ring A. In this manner we have defined operators $T_p(r)$ for all $p \nmid N$ and all integers r $0 \leqslant r \leqslant m$. These commute with each other. Moreover, $T_p(r)$ and $T_p(m-r)$ are adjoint with respect to cup-product in the sense that

$$T_p(r)\cdot x \cup y = x \cup T_p(m-r)\cdot y$$

§2 The modular symbol associated to admissible Borel subgroups.

Let $G = SL_{n+1}/\mathbb{Q}$ and let $B \subset G$ denote the (Borel) subgroup of upper triangular matrices. Fix Γ a torsionfree subgroup of $SL_{n+1}(\mathbb{Z})$ (of finite index). The arguments of this section will in fact work in a somewhat broader context (G a Q-split semisimple group, and B an admissible Borel subgroup). We retain the notational conventions of § 1. In particular, $\overline{X}$ denotes the Borel-Serre completion of X, and $\overline{Y} = \overline{X}/\Gamma$ is the (Borel-Serre) compactification of Y. Since Γ is assumed torsionfree, Γ acts freely on $\overline{X}$.

LEMMA 1. The natural map

$$i: \quad H^n(\overline{Y}, \partial\overline{Y}; A) \longrightarrow H^n(\overline{X}, \partial\overline{X}; A)^{\Gamma}$$

is an isomorphism, where A is any coefficient group.

Proof: Cohomology will mean with coefficients in A, unless explicitly stated to the contrary. Lemma 1 is standard if $n=1$; suppose $n \geq 2$. ~~The above~~ lemma follows more or less directly from the results proved in Borel-Serre. Note that n is the Q-rank of G. Since Γ acts freely on $\overline{X}$ and on $\partial\overline{X}$ we have the Hochschild-Serre spectral sequences for both actions. Using that $\overline{X}$ is contractible and $\partial\overline{X}$ is a wedge of n-1 spheres we evaluate these spectral sequences to produce the following commutative diagram:

$$\begin{tikzcd}
H^{p-n}(\Gamma, H^{n-1}(\partial\bar{X})) \arrow[r] & H^p(\Gamma, H^0(\partial\bar{X})) \arrow[r] & H^p(\partial\bar{Y}) \arrow[r] & H^{p-n+1}(\Gamma, H^{n-1}(\partial\bar{X})) \\
 & H^p(\Gamma, H^0(\bar{X})) \arrow[u, "\cong"'] \arrow[r, "\cong"'] & H^p(\bar{Y}) \arrow[u] & \\
 & & H^p(\bar{Y}, \partial\bar{Y}) \arrow[u] \arrow[r, "i"] & H^p(\bar{X}, \partial\bar{X})^{\Gamma} \\
H^{p-n-1}(\Gamma, H^{n-1}(\partial\bar{X})) \arrow[r] & H^{p-1}(\Gamma, H^0(\partial\bar{X})) \arrow[r] & H^{p-1}(\partial\bar{Y}) \arrow[u] \arrow[r] & H^{p-n}(\Gamma, H^{n-1}(\partial\bar{X})) \arrow[u, dashed] \\
 & H^{p-1}(\Gamma, H^0(\bar{X})) \arrow[u, "\cong"'] \arrow[r, "\cong"'] & H^{p-1}(\bar{Y}) \arrow[u] &
\end{tikzcd}$$

In the above diagram the vertical "column" is the cohomology sequence of the pair $(\bar{Y}, \partial\bar{Y})$. The vertical broken arrow represents a morphism which exists only when p=n. In this case it is merely the map induced from the coboundary isomorphism $H^{n-1}(\partial\bar{X}) \longrightarrow H^n(\bar{X}, \partial\bar{X})$.

When the broken arrow exists, the square that it forms is immediately seen to be commutative.

If we make use of the asserted isomorphisms, the above diagram "collapses" to the following diagram when p=n:

$$\begin{tikzcd}
 & & & H^n(\bar{Y}, \partial\bar{Y}) \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & H^{n-1}(\bar{Y}) \arrow[r] & H^{n-1}(\partial\bar{Y}) \arrow[ur] \arrow[dr] & & H^n(\bar{Y}) \arrow[r] & H^n(\partial\bar{Y}) \\
 & & & H^n(\bar{X}, \partial\bar{X})^{\Gamma} \arrow[ur] & &
\end{tikzcd}$$

where both horizontalish routes are exact. The lemma follows from the five-lemma.

Now note that in our case, M is the Cartan subgroup of diagonal matrices in SL_{n+1}, and $\Gamma \cap M$ is trivial. We may identify $M(R) \cap K$ with diagonal matrices in SL_{n+1} whose entries are ± 1. Thus $Y(u,M)$ may be identified with $M(R)^+$, the subgroup of diagonal matrices in $SL_{n+1}(R)$ whose entries are all positive real numbers.

The boundary:

Using the results of Borel-Serre (§8) we have isomorphisms $(m > 1)$

$$H^m(\bar{X}, \partial \bar{X}; A) \cong H^{m-1}(\partial \bar{X}; A) = H^{m-1}(|T|; A)$$

where T is the Tits building of $SL_{m+1/Q}$. As in , if we make the choice of s an m-1 dimensional simplex in T (we [redacted] choose the simplex s corresponding to the Borel subgroup B) then $|T|$ is homotopic (by a canonical mapping) to a wedge of m-1 spheres, each m-1 sphere being the geometric realization of an "apartment" of T containing s. The set of such apartments Σ can be identified with the set of split tori in (defined over Q) in B(Q). Since B(Q) acts transitively on this set (by conjugation) we may, in turn, identify Σ [redacted] with elements in B(Q)/M(Q).

In this way we get an isomorphism

$$H^m(X, \partial X; A) \longrightarrow \text{Functs}(B(Q)/M(Q); A) = \text{Functs}(U(Q); A)$$

which is compatible with the action of Γ, where Γ acts on the spaces of functions via its ~~entire~~ natural action on the Tits building Γ. Of course, to the geometric realization of the apartment associated to the torus M' in the Tits building;

The identification between the homotopy types of $\overline{X}$ and a wedge of (m−1)-spheres indexed by Σ may be described more precisely as follows: (§§) $\overline{X}$ has a locally finite cover by contractible spaces $\overline{X} = \coprod_P e(P)$ where P ranges through all parabolic subgroups of G defined over $\mathbb{Q}$. Letting M' range through all split tori (over $\mathbb{Q}$) in B, and $S_{M'} = \coprod_{G \neq P \supset M'} e(P)$ the indicated closed subspace of $\partial\overline{X}$, we have a natural mapping $\bigvee_{M' \subset B} S_{M'} \longrightarrow \partial X$ where the wedge is taken over a point in $e(B)$. This is a homotopy equivalence. Moreover, each $S_{M'}$ is homotopic to the geometric realization of the apartment associated to the torus M' in the Tits building;

By choosing an orientation of the "basic" apartment associated to $M \subseteq B$ we obtain a "consistent" orientation ~~of each of~~ of the geometric realizations of each of the apartments occurring in Σ. Thus we may view $H^{m-1}(|T|;A)$ as isomorphic to Functions(Σ, A)

sion of this isomorphism.

We shall show that $\overline{X}(u,M)$ is compact, isomorphic to the closed m-dimensional disc, and that its boundary maps (by a homotopy equivalence) to the subspace $S_{uMu^{-1}}$ in $\overline{X}$.

~~mmxxmmxkxmxxxbexsimplifiedxinxxariousxways~~

We need only consider the case u=1, for ~~mmxmmxid~~ the case of a general u then follows by comsidering the natural action of U(Q) on X and $\overline{X}$.

Let $\overline{X}_M \subset \overline{X}$ be the subspace $\overline{X}_M = \coprod_{P \supset M} e(P)$. Thus ~~$\overline{X}M$~~ $\overline{X}_M = \overline{X} \cup S_M$.

We must show that the closure of ~~x~~ $M(R)^+$ in X_M is compact, and that itsboundary is homotopically equivalent to S_M. For this, ~~mm mxmmimmxtmmx~~ set $A = M(R)^+$, and $A(P)$ = the closure of A in X(P). ~~mm~~ set $A = M(R)^+$ R_+^{*m}

we recall the definition of X(P), the corner associated to the parabolic subgroup P. ~~Kirstxxpxx~~ To

Note: This is now done in the Ash-Rudolph paper, so we needn't redo it.

~~Putting~~ Composing this isomorphism ~~together~~ with the isomorphism i of Lemma 1, we obtain

$$\psi : H^m(\bar{Y}, \partial\bar{Y}; A) \xrightarrow[\cong]{} \mathrm{Functs}(U(Q); A)^{\Gamma}$$

~~Proposition: If let denote the image of If $H^m(\bar{Y}, \bar{Y}; A)$, let denote the element in $\mathrm{Funct}(U(Q); A)$~~

The modular symbol Φ_B engenders another map from $H^m(\bar{Y}, \partial\bar{Y}; A)$ to $\mathrm{Funct}(U(Q); A)^{\Gamma}$; namely, if $\alpha \in H^m(\bar{Y}, \partial\bar{Y}; A)$, let $h_\alpha : H_m(\bar{Y}, \partial\bar{Y}; \mathbb{Z}) \longrightarrow A$ denote cap-product with α and set $\varphi(\alpha) = h_\alpha \cdot \Phi_B$.

Proposition: There is a choice of orientation of the basic apartment so that $\psi = \varphi$.

Proof: Since $\Gamma \cap M$ is trivial, the cycles $Y(u,M)$ lift to X and we must study their closure in the Borel-Serre bordification of X. Explicitly, there is a commutative diagram

$$\begin{tikzcd}
R_+^{*m} \cong M(R)^+ \arrow[r, hook] \arrow[d, "\cong"] & X \subseteq \bar{X} \arrow[d] \\
Y(u,M) \arrow[r] & Y \subseteq \bar{Y}
\end{tikzcd}$$

and let $\bar{X}(u,M)$ denote the closure of $M(R)^+$ in $\bar{X}$.

§4 Interpreting cohomology classes as functions on lattices

Keep to the notation of but make the following hypothesis:

$$\Gamma \cap B(Q) = B(Z).$$

To any cohomology class $\alpha \in H^m(\bar{Y}, \partial\bar{Y}; R)$ we obtained a function φ_α on $B(Q) \cap \Gamma \backslash B(Q)/M(Q)$ with values in R. After our hypothesis, the ~~latter~~ double quotient may be written $B(Z)\backslash B(Q)/M(Q)$. Let $\underline{L}$ denote the set of Z-lattices in the vector space Q^n $(n=m+1)$. There is a natural injection:

$$\mathrm{Funct}(B(Z)\backslash B(Q)/M(Q)\, ;\, R) \hookrightarrow \mathrm{Funct}(\underline{L}\, ;\, R)$$

enabling us to pass from φ_α to a function $f_\alpha : \underline{L} \longrightarrow R$.

To describe the natural injection, let G^+ denote the subgroup of $GL_n(Q)$ consisting in matrices with positive determinant. Let B^+ denote the subgroup of upper triangular matrices in G^+; M^+ the diagonal matrices in G^+, and $G^+(Z) = G(Z) \cap G^+$. There is the following chain of mappings, all induced by inclusion or quotient homomorphisms:

$$B(Z)\backslash B(Q)/M(Q) \xrightarrow{\cong} B(Z)\backslash B^+/M^+ \twoheadleftarrow B(Z)\backslash B^+ \xrightarrow{\cong} G^+(Z)\backslash G^+$$

where the latter bijection can be deduced from . Since G^+ acts transitively on $\underline{L}$ by $g\Lambda = ({}^t g)^{-1}\cdot\Lambda$, and $G^+(Z)$ is the isotropy subgroup of the standard lattice $Z^n \subset Q^n$, we have a natural identification

$$G^+(Z)\backslash G^+ \cong \underline{L}.$$

§5 Hecke operators on lattices.

If p is a prime number, r an integer $0 \leq r \leq n$ and $\Lambda \subseteq Q^n$ a lattice, by a p^r-modification of Λ is meant a lattice Λ' such that $\Lambda \subseteq \Lambda' \subseteq p^{-r}\Lambda$

and Λ'/Λ is of order p^r.

Define the Hecke operator $T_{p,r}$ on $Z[\underline{L}]$ to be the endomorphism given by $\Lambda \longmapsto \sum \Lambda'$ where the Λ' range through all p^r-modifications of Λ.

Proposition (compatibility) If p is a prime number not dividing the level of Γ, then the square

$$\begin{array}{ccc}
H^m(\bar{Y}, \partial\bar{Y}; R) & \longrightarrow & \text{Functs}(\underline{L}; R) \\
{\scriptstyle T_p(r)}\downarrow & & \downarrow{\scriptstyle T_{p,r}} \\
H^m(\bar{Y}, \partial\bar{Y}; R) & \longrightarrow & \text{Functs}(\underline{L}; R)
\end{array}$$

is commutative.

Proof:

§6 The basic p-adic manifold.

If E is any set, and $\Pi : E \to E$ a map, define the Π-adic completion of E, E_{Π}, to be the projective limit of the system

$$E_{k+1} \xrightarrow{\pi} E_k \xrightarrow{\pi} E_3 \xrightarrow{\pi} E_2 \xrightarrow{\pi} E_1 = E$$

where each of the E_i's is E and the mappings, as indicated, are all π. We view E_{π} as mapping to E by the natural projection to E_1.

Now let Π denote the automorphism ~~of~~ $U(Q)$ obtained by conjugation with the matrix

$$\begin{pmatrix} p^n & & & & \\ & p^{n-1} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & p \end{pmatrix}$$

$$(a_{ij}) \longmapsto (p^{i-j} a_{ij}) \quad .$$

Since Π brings $U(Z)$ into itself, it induces ~~an~~ a mapping on the quotient space $U(Z) \backslash U(Q)$ which we denote by π. Clearly, π is surjective, and has finite pre-images. ~~If one decomposes the finite~~ ~~ring of~~ ~~adeles, and finite integral adeles into~~ ~~xxprodxxtxxfxxcompxnentxxpx~~

If we write $\mathbb{A}_f = \mathbb{A}_f' \times \mathbb{Q}_p$

$\hat{\mathbb{Z}} = \hat{\mathbb{Z}}' \times \mathbb{Z}_p$

then $U(Z)\backslash U(Q) \cong U(\hat{Z}')\backslash U(A_f') \times U(\mathbb{Z}_p)\backslash U(Q_p)$.

~~and~~ The π-adic completion of $U(\mathbb{Z})\backslash U(Q)$ is then immediately seen to be

$$\{U(\hat{Z}')\backslash U(A_f')\}_{\pi} \times \{U(\mathbb{Z}_p)\backslash U(Q_p)\}_{\pi} = (U(Z')\backslash U(A_f')) \times U(Q_p).$$

~~Denote this topologi~~CA

~~Denote this to~~B

The first factor in the product decomposition above is a discrete space, while the second is taken with its usual p-adic topology.

Denote this topological ~~xxx~~ space $\underline{M}$. It is a p-adic manifold of dimension ~~xxdimxU(Q_p)x~~ equal to the dimension of $U(Q_p)$ (i.e. $n(n-1)/2$).

~~It is on $\underline{M}$ that our p-adic measures will be constructed, a description of which is given below:~~

~~Theorem: Let be a function on $B(Z)\,B(Q)/M(Q)$ with values in R. Suppose that the associated function f on $\underline{L}$ is an eigenfuction for the Hecke operators $T_{p,r}$ (for a fixed p, and 0 r n) with eigenvalues $t_{p,r}$. Suppose further that the eigenvalues $t_{p,r}$ are p-adic units (0 r n). Let $_0, \dots, _{n-1}$ be the n roots of the polynomial $P(X) = \sum_r (-1)^r p^{p(r-1)/2} t_{p,r} X^{n-r}$ indexed ~~xxxxxxx~~ so that $\mathrm{ord}_p\ _i = i$.~~

~~We prepare to state our main theorem.~~

It is on $\underline{M}$ that our p-adic measures and distributions, will be constructed. Indeed, if φ is a function on $U(Z)\backslash U(Q) \cong B(Z)\backslash B(Q)/M(Q)$ with values in a p-adically complete ring R, let f denote the associated function on $\underline{L}$, the space of lattices in Q^n. Suppose that f is an eigenfunction for the $T_{p,r}$ $(0 \leqslant r \leqslant n)$ with eigenvalues $t_{p,r}$ which are p-adic units (f is an "ordinary" eigenfunction). ~~Let $_0 \dots _{n-1}$~~

~~Let~~

let $\alpha_0, \dots, \alpha_{n-1}$ be the n roots of the polynomial

$$P(X) = \sum_{r=0}^{n} (-1)^r . p^{r(r-1)/2} . t_{p,r} X^{n-r}$$

organized in such a way that $\mathrm{ord}_p \, \alpha_i = i$. This is possible since the $t_{p,r}$ are p-adic units.

To define a ~~measure~~ distribution μ on $\underline{M} = U(Z)\backslash U(Q)$ it suffices to define μ on the basic open sets $O(x,k) = pr_k^{-1}(x)$ where $k \geq 0$ is an integer, $x \in U(Z)\backslash U(Q)$, and $pr_k : \underline{M} \longrightarrow U(Z)\backslash U(Q)$ is projection to the k-th copy of $U(Z)\backslash U(Q)$ in the projective system defining $\underline{M}$. For $m \geq k$ ~~Define~~ $\mu_m(O(x,k))$ by the formula

$$(*) \qquad \mu_m(O(x,k)) = \left(\frac{\prod_{i=0}^{n-1} \alpha_i^{n-i}}{p^{w(n)}} \right)^{-m} \sum_{\substack{\pi^{m-k}.x' = x \\ x' \in U(Z)\backslash U(Q)}} \varphi(x')$$

where $w(n)$ is such that the parenthetical expression (in which it appears) is a p-adic unit. Explicitly, $w(n) = \sum_{i=0}^{n-1} i \cdot (n-i)$.

If φ is an R-valued function on ~~$B(Z)\backslash B(Q)/M(Q)$~~ $U(Z)\backslash U(Q)$ which is an ordinary eigenfunction for the $T_{p,r}$, let μ_m be defined by * above.

Set $\mu(O(x,k)) = \lim_{m \to \infty} \mu_m(O(x,k))$.

<u>Theorem:</u> This limit exists defining a distribution μ on $\underline{M}$. If R is a p-adic field, and φ is a bounded funtion, then μ is a measure.

§7. Construction of p-adic measures

Let K be a nonarchimedean local field, with ring of integers $\mathcal{O}$, uniformizer π, residue field $k = \mathcal{O}/\pi\cdot\mathcal{O}$ of order $q = p^d$, and whose normalized valuation is denoted $|\ |$ $\quad (|\pi| = q^{-1})$.

Let $n \geq 2$ be an integer, and V an n-dimensional vector space over K. A lattice in V has the usual meaning: it is a free $\mathcal{O}$-module Λ contained in V such that $\Lambda \otimes_{\mathcal{O}} K \cong V$.

Let M denote the K-vector space of (formal) finite K-linear combinations of lattices in V.

If Λ is a lattice, Λ_k will denote $\pi^{-1}\cdot\Lambda/\Lambda$ viewed as n-dimensional vector space over k. If r is an integer $(0 \leq r \leq n)$ by a q^r-modification of Λ we shall mean a lattice Λ' such that

$$\Lambda \subset \Lambda' \subset \pi^{-1}\cdot\Lambda$$

and Λ'/Λ is a k-vector subspace of Λ_k of dimension r.

The standard Hecke operator $T_r : M \longrightarrow M$ $\quad (r = 0,\dots,n)$ is defined on lattices by the formula

$$T_r : \Lambda \longmapsto \sum_{\Lambda' :\, q^r\text{-modification of } \Lambda} \Lambda'$$

Thus, T_0 is the identity, and T_n sends Λ to $\pi^{-1}\cdot\Lambda$.

The natural operation of the operators T_r on the space of linear functionals on M is a "right action" denoted by $f \longmapsto f\,|\,T_r$.

The subalgebra of $\mathrm{End}(M)$ generated over $\mathbb{Z}$ by the standard Hecke operators T_r $\quad (r=0,\dots,n)$ will be denoted H, the Hecke algebra.

Let $\mathfrak{B}$ denote a choice of a <u>complete flag</u> of subspaces of V. That is,

$$\mathfrak{B}: \quad 0 = V_0 \subsetneqq V_1 \subsetneqq V_2 \subsetneqq \cdots \subsetneqq V_n = V.$$

We see that $\dim_K V_i = i$. For each lattice Λ, $\mathfrak{B}$ determines a complete flag on the k-vector space Λ_k.

Let $I \subset Z^n$ denote the set of n-tuples whose entries are either 0 or 1. If $a \in I$, $a = (a_1, a_2, \ldots, a_n)$ we shall define the <u>jump indices</u> of a to be the integers j such that $a_j = 1$. By the <u>length</u> of a we shall mean the number of its jump indices, or equivalently, $\sum_j a_j$.

If Λ' is a q^r-modification of Λ say that j is a <u>jump index</u> for Λ' if

$$\Lambda' \cap V_{j-1} + \Lambda \neq \Lambda' \cap V_j + \Lambda .$$

If Λ' is a q^r-modification of Λ and $a \in I$ we say that Λ' is of <u>type</u> a if Λ' and a have the same set of jump indices. If Λ' is of type a, one has that the length of a is r.

For $a \in I$ define the <u>partial Hecke operator of type</u> a, $T(a): M \longrightarrow M$ by the rule

$$T(a): \Lambda \longmapsto \sum_{\substack{\Lambda' \text{ modification of } \Lambda \\ \text{of type } a}} \Lambda'$$

Clearly,

$$(1) \qquad T_r = \sum_{\text{length } a = r} T(a) \qquad r = 0, \ldots, n$$

Set

$$e_j = (0,0,\ldots,\underset{\substack{\uparrow\\ j}}{1},\ldots,0) \in I \qquad (j=1,\ldots,n)$$

and $$A_j = (0,0,\ldots,\underset{\substack{\uparrow\\ j}}{1},1,\ldots,1) \in I \qquad (j=1,\ldots,n).$$

Thus $A_j = \sum_{i=j}^{i=n} e_i$.

Denote $T(e_j)$ by $E(j)$; we shall refer to it as the j-th <u>elementary partial Hecke operator</u>.

(2) <u>Formulas</u>

a) If $a \in I$ has jump indices $j_1 > j_2 > \cdots > j_r$, then

$$E(j_1)\cdot E(j_2)\cdot \ \ldots\ \cdot E(j_r) = q^{r(r-1)/2}\cdot T(a)$$

<u>Note</u>: The partial Hecke operators do not necessarily commute with one another, and so the order in the above formula is important.

b) $T(A_i)\cdot T(A_j) = T(A_j)\cdot T(A_i)$ for any $i,j \leq n$.

If $b = (b_1,b_2,\ldots,b_n) \in Z^n$ with $0 \leq b_1 \leq b_2 \leq \cdots \leq b_n$ we may write

$$b = m_1\cdot A_1 + m_2\cdot A_2 + \ldots\ldots + m_n\cdot A_n$$

with $m_i \geq 0$. For such a b, set

$$S^b = T(A_1)^{m_1} \cdot T(A_2)^{m_2} \cdot \ldots\ldots \cdot T(A_n)^{m_n}$$

where, in the above formula, the order of the factors does not matter, by [2.b].

We refer to the S^b as averaging operators.

Let $\tilde{H}^B$ denote the H-algebra generated by the "abstract elements" $T(a)$ for $a \in I$, subject to the relations (1) and (2.a, b) ($T(e_j) = E(j)$). Then $\tilde{H}^B$ operates on M (~~probably faithfully, but we have not bothered to check this~~). Let $\Sigma \subset \tilde{H}^B$ denote the subalgebra generated by the averaging operators S^b. We call the subalgebra Σ the algebra of averaging operators.

Our main construction will be to start with a linear functional f on M, which is an eigenfunction for H (~~and~~ whose eigenvalues satisfy a ~~mild~~ condition ~~property~~) and to produce a linear functional F which is an eigenfunction for Σ (indeed: which is a fixed vector for all the averaging operators). Such a linear functional F may be interpreted as a p-adic measure.

The following formula is immediate from (2.a):

$$(3.a) \qquad T(e_j)\cdot S^{A_{j+1}} = q^{(n-j)}\cdot S^{A_j} = q^{(n-j)}\cdot S^{A_{j+1}+e_j}$$

(as follows from the conventions we have set up e.g. $S^{A_j} = T(A_j)$ and (2.a)). Note: this works for $j=n$, if we make the convention that $A_{n+1} = 0$.

If $a \in I$ has the jump indices $j_1, j_2, \ldots, j_r$, let ~~$\delta(a) = rn - r(r-1)/2$~~

$$w(a) = rn - r(r-1)/2 - \sum_{i=1}^{r} j_i .$$

Let $a = (a_1, \ldots, a_n) \in I$

Lemma: Let $b = m_1 \cdot A_1 + m_2 \cdot A_2 + \cdots + m_n \cdot A_n$ with $m_i \geq a_i$

Then

$$(3.b) \qquad T(a) \cdot S^b = q^{w(a)} \cdot S^{a+b}$$

~~(2.a) commutation~~

Proof: We use commutativity of the $T(A_i)$ (2.b) to rearrange the factorization (*) of S^b in such a way so as to exploit (3.a) proceeding by induction on the number of jumps.

define $\|f\| = \max\limits_{\text{lattices}} |f(\Lambda)|$.

Eigenfunctions for H:

Let f be a K-valued linear functional on M which is an eigenfunction for the Hecke algebra H. Thus

bounded $\Leftrightarrow$ $\|f\| < +\infty$

$$(4) \qquad f \,|\, T_r = t_r \cdot f \qquad \text{for } r = 0, 1, \ldots, n$$

where $t_r \in K$ ($t_0 = 1$). We make the hypothesis that t_r is a unit in $\mathcal{O}$ for ~~for~~ r ~~≠~~ n, ~~and that t_n is a unit in $\mathcal{O}$.~~ (i.e.: f is an "ordinary" eigenfunction)

We obtain a homomorphism $H \xrightarrow{\varphi} K$ ($T_r \longmapsto t_r$, and in general an operator is sent to its eigenvalue). Moreover, it is evident from relations (1) and (2a) that φ extends ~~kimx~~ to homomorphisms

$$H^B \xrightarrow{\varphi} K$$

by arbitrarily sending the elementary partial Hecke operators $T(e_j)$ ($j = 1, \ldots, n$) to the n roots of the polynomial

$$P(X) = \sum (-1)^{r(r-1)/2} t_r \cdot X^{n-r}$$

(which, by our hypothesis, may be seen to lie in K and in [illegible] $i = 0, 1, 2, \ldots, n$)

It is convenient, if an operator is denoted by a capital letter, to let the corresponding small letter refer to its image under ϕ.

Thus

$$T(e_j) \longmapsto t(e_j) \quad \text{(the roots of (**))},$$ and we stipulate ϕ by requiring (over)

If $a \in I$, then

$$T(a) \longmapsto q^{-r(r-1)/2}\, t(e_{j_1})\cdots t(e_{j_r}) = t(a).$$

Moreover, we let the image of S^b be denoted s^b. which ~~may~~ can be seen to be a unit in $\mathcal{O}$ We obtain from (3.b):

$$(5) \qquad t(a)\cdot s^b = q^{w(a)}\cdot s^{a+b} \quad .$$

Remark: The $t(e_j)$ are none other than the "eigenvalues of Frobenius" of the local unramified n-dimensional Galois representation associated to the representation ψ of the Hecke algebra in the "standard" dictionary. ~~I am a bit mystified by their occurrence.~~

suitably normalized

~~Since t_n has been assumed to be a unit, it is nonzero, and consequently all the $t(a)$ are nonzero.~~

~~3. Say that the eigenfunction f is ordinary if the roots $t(e_j)$ have the distinct absolute values. This implies, by elementary arguments, that these roots lie in K. Moreover, we may determine the extension ϕ of ψ uniquely by requiring $t(e_1)\ t(e_2)\ \ldots\ldots\ t(e_n)$.~~ $|t(e_j)| = q^{-j}$

~~Call this choice of the canonical extension of~~ . Note the important fact that for this choice s^b is a unit in $\mathcal{O}$.

If $b = (b_1,\ldots,b_n) \in Z^n$, let $\|b\|$ denote $\min_{j=1,\ldots,n}(b_j - b_{j-1})$.
where b_0 is taken to be bigger

Proposition:

Let f be an ordinary bounded, eigenfunction of H, and let ϕ be the canonical extension of ψ.

$$|t(e_j)| = |q^{n-j}|.$$

Under our hypothesis the roots of $P(X)$ have absolute value

If $b = (b_1, \ldots, b_n)$ with $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$ set

$$F_b = s^{-b}\cdot f \,\Big|\, S^b \; .$$

Then $\lim\limits_{\|b\| \to \infty} F_b$ converges to yield a linear functional $\tilde{F}$

such that $\tilde{F}|S^\beta = s^\beta \tilde{F}$ ~~, where the limit is taken over those b such that $b_1=0$, and~~ for all $\beta \in \mathbb{Z}^n$ with $\|\beta\| > c$

Proof: Combining (1) and (4), we have (for $r=1,2,\ldots,m$) and b as in the proposition)

$$\sum_{\text{length } a = r} f \,\Big|\, T(a)\cdot S^b = \sum_{\text{length } a = r} t(a)\cdot f \,|\, S^b \; .$$

Using (3.b) and (5) we get:

$$(6) \qquad \sum_{\text{length } a = r} t(a)\cdot(F_{a+b} - F_b) = 0 \; .$$

For $r=n$, (6) yields:

$$(7) \qquad F_{A_1+b} = F_b$$

(recall: $A_1 = (1,1,\ldots,1)$) .

~~Now set $c = \max\limits_{i=1,\ldots,n-1} \left(t(e_{i+1}) / t(e_i) \right)$ 1.~~

<u>Lemma</u>: If $\|b\| \geqslant nk$, and $a \in I$, then

$$(8) \qquad \left| F_{a+b}(\Lambda) - F_b(\Lambda) \right| \leqslant q^{-k}. f$$

for any (lattice) Λ in M.

<u>Proof</u>: Fix Λ and write $G(b) = F_b(\Lambda)$. We proceed by induction on k, the case k=0 holding because s^b is a unit. Suppose the lemma true for k ($\geqslant 0$); we prove it for k+1.

Note that the assumption of ordinariness gives us:

$$(9) \qquad t(a) \leqslant q^{-1}. t(A_r)$$

for any $a \neq A_r$ in I, of length r.

Combining (6), (9) and (8) [for the case k] we see that

$$(10) \qquad |G(A_r+b) - G(b)| \leqslant q^{-(k+1)}. f$$

for any b such that $\|b\| \geqslant nk$ and $r=1,2,\dots,n$.

Since

$$G(e_r+b)-G(b) = G(e_r+b)-G(e_r+b+A_{r+1})+ G(b+A_r)-G(b)$$

and $\|e_r+b\| \geqslant \|b\| -1$, we apply (10) to get:

$$(11) \qquad |G(e_r+b) - G(b)| \leqslant q^{-(k+1)}. f$$

for any b such that $\|b\| \geqslant nk+1$.

Now suppose that $\|b\| \geqslant nk+n$, and $a = e_{j_1}+e_{j_2}+\dots+e_{j_r} \in I$ is of length r (and by (7) we may suppose that $r < n$). For $i= 0,1,\dots,r$, write

$$b^{(i)} = e_{j_1}+e_{j_2}+\dots+e_{j_i} + b \ .$$

Then $\| b^{(i)} \| \geqslant nk+n-i \geqslant nk+1$; consequently (11) applies yielding

$$(12) \quad | G(b^{(i+1)}) - G(b^{(i)}) | \leqslant q^{-(k+1)} \cdot f$$

and therefore:

$$(13) \quad | G(a+b) - G(b) | \leqslant q^{-(k+1)} \cdot f$$

for all $a \in I$, and b such that $\|b\| \geqslant k \cdot (n+1)$.