Bridges between Geometry and. . . Number Theory

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What is it that unifies Mathematics?

Sidenote: I keep thinking about different aspects of this question, and have given some lectures—all to general audiences—on this theme.

(For an account of my attempts at this, see my web-page).

I want to thank Olivia Caramello for inviting me to this conference, and suggesting that I reflect on the unifying elements in geometry.

But before one can even discuss the question:

What is it that unifies Mathematics? (or more specifically: Geometry?)

one has to deal with the question:

What *is* Mathematics? (or more specifically: Geometry?)

A friend of mine, a physicist, tells me that she believes that *anything* a physicist labels as *physics*. . . is **physics**.

For biologists, **Life** is the single word that points to the unification of the essence and substance of their subject.

—from proteins. . . to the behavior of elephants. . . to medical applications—

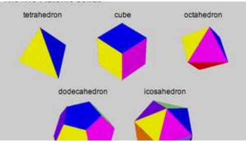
What is the unifying anchor for mathematics?. . . for geometry?

Is geometry unified? Does it matter? And are there useful ways to think about these questions?

Euclid's Elements. . . of Geometry

is a foundational unifying treatise for the subject geometry, if ever there was one,

and it ends neatly in the construction of the regular platonic solids.



But even in this foundation book of geometry, the tentacles of the subject geometry reach out to formulate foundations of other subjects. . .

E.g., Number Theory. . . as in Euclid's Book VII:

Definitions:

11. A prime number is that which is measured by an unit alone.

12. Numbers prime to one another are those which are measured by an unit alone as a common measure.

But. . . with some geometric language helping it along:

16. And, when two numbers having multiplied one another make some number, the number so produced is called plane, and its sides are the numbers which have multiplied one another.

17. And, when three numbers having multiplied one another make some number, the number so produced is solid, and its sides are the numbers which have multiplied one another.

Euclid. . . returns. . . to Number Theory in **Book IX:**

Proposition 20: "Prime numbers are more than any assigned multitude of prime numbers."

(*I really like Heath's English translation of Euclid in this vintage English*)

Geometry reaches out to other subjects either providing applications to them, or taking tools from them:

Two of countless examples—

- (1972) Charles Fefferman used the solution of the Kakeya Needle Problem to give an elegant characterization of the L^2-norm (focusing on it among the L^p-norms for general p).
- (2016) Maryna Viazovska solved the sphere packing problem in dimension 8 (also in dimension 24 with collaborators) by using a certain mock-modular form. Such objects are the invention of Ramanujan; they also play a role in the current study of black holes).

Topology and Physics...

- The vibrant current relationship between geometry and physics was given an important nudge, a quarter of a century ago when Michael Atiyah asked the seemingly harmless question:

"What is the physical interpretation of the Jones polynomial?"

This was answered a few years ago by Ed Witten in his fundamental paper:

"Quantum Field Theory and the Jones Polynomial."

Topology and Arithmetic

Nowadays one might go in yet a different direction and ask

What is the *Arithmetical-Algebraic-Geometric* interpretation of the Jones polynomial?

or of Chern-Simons' Theory?

or of TQFT?

Much current work addresses these bridges of fields:

1. Hee-Joong Chung, Dohyeong Kim, Minhyong Kim, George Pappas, Jeehoon Park & Hwajong Yoo; and of
2. Hikaru Hirano, Junhyeong Kim & Masanori Morishita.

They discuss questions of exactly that sort.

What ‘unifies’ Geometry’? What enables it to connect with other fields?

- Common language, common definition, and modes of expression, common foundations.
- parallel or surprisingly compatible structures,
- ‘moduli,’ or parametrized families of structures… of the same genre,
- the powerfulness of metaphors, analogies…

… man is an analogist and studies relations in all objects.

Emerson; *Nature*, Ch IV on ‘Language’

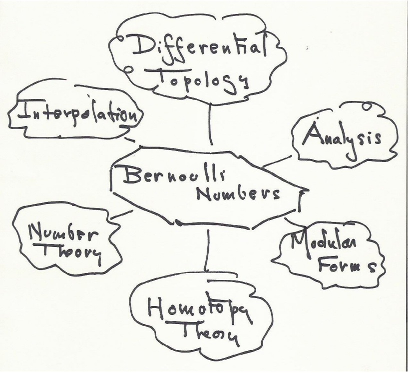
Ties, analogies, links, bridges

To talk about unifying themes, let's give this ad hoc vocabulary different (metaphorical) cohesive strengths.

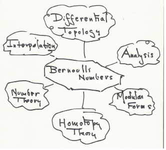
- By a tie let's mean a concept that simply brings together disparate fields, or points of view, or concepts.
- By an analogy let's mean something more: that it connects two disparate concepts by some similarity in their structure.
- By a link we meant there might be some genuine joining, possibly by having the different concepts be part of a single larger framework.
- A bridge will include ideas about how to actually prove this joining and produce the encompassing larger framework.

E.g., Bernoulli Numbers as a "tie"

I once gave a lecture on how such a clean simple notion as *Bernoulli numbers* (strangely) ties together a constellation of different mathematical subjects—and does it in a way that one actually can experience the profound kinship of these subjects— 'Bernoulli numbers' being the keystone:

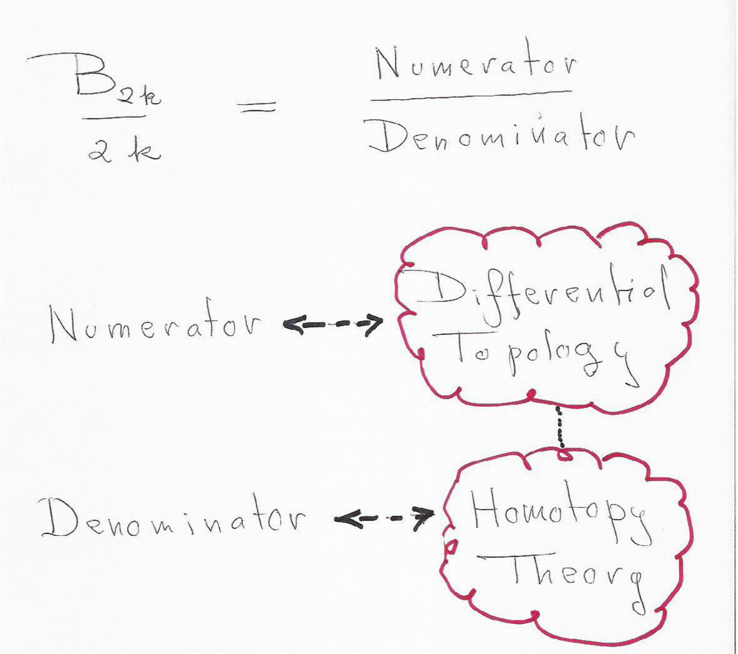


Differential Topology $\overset{\textit{Bernoulli}}{\longleftrightarrow}$ Homotopy Theory



can be illustrated by the Kervaire-Milnor formula:

Numerator and Denominator of Bernoulli Numbers



Here's that unsettling comment of André Weil:

> *Nothing is more fruitful—all mathematicians know it—than those obscure analogies, those disturbing reflections of one theory in another; those furtive caresses, those inexplicable discords; nothing also gives more pleasure to the researcher. The day comes when the illusion dissolves; the yoked theories reveal their common source before disappearing. As the Gita teaches, one achieves knowledge and indifference at the same time.*

Indifference? There are metaphorical bridges, begun in ancient mathematics, that connect subjects and viewpoints cajoling us to view one field from the perspective of another—for example: *geometry as algebra*.

The ‘bridge’ between Geometry and Algebra

Geometry as Algebra and *Algebra as Geometry*—these metaphors have been with us since ancient times and and the sheer wonder has never faded.

There were. . . naysayers:

> *It is not possible to arrive at a demonstration by using for one’s proof a different genus from that of the subject in question; e.g.one cannot demonstrate a geometrical problem by means of arithmetic.*

Aristotle: Posterior Analytics I.16

René Descartes,

commenting about his merger of algebra and Euclidean geometry, said:

> *I would borrow the best of geometry and of algebra and correct all the faults of the one by the other.*

René Descartes, of course, offered a vital step to this metaphor, turning it into a true synthesis.

And now there is Algebraic Geometry

with its remarkable evolution in the past century starting, perhaps with the Italian school—led by Francesco Severi.

The temper of that school was *non-rigor*. They were very focused on the 'geometry' —of 'algebraic geometry'—as their primary source of intuition.

Among the freedoms they took for themselves was to often assume that the objects they were dealing with could be put "in general position"—and they would give no formal justification for this.

. . . and then the move towards focusing on the ‘algebra’ —of ’algebraic geometry’

—while, concomitantly, coming up with a rigorous approach to the subject was successfully done by Oscar Zariski (and others)

who yoked the powerful commutative algebra of Wolfgang Krull and Emmy Noether (and van der Waerden and others) to the intuition of the Italians.

And this initiated, or came in parallel with, broad foundations, such as . . .

foundations focusing on *fields of rational functions*. E.g.,

- André Weil's *Foundations of Algebraic Geometry* (1946)

  and at approximately the same time

- *Ultra-algebraic* approaches to aspects of—at least—the algebraic geometry of curves, such as Claude Chevalley's *Introduction to the Theory of Algebraic Functions of One Variable*.

  without a picture in the book, or even pictorial language—it is all fields and extensions of fields— it's a curious tour de force with no hint of geometric intuition.

But. . . in contrast:

there was the *Séminaire Chevalley* (1957/58) where Chevalley developed his novel ideas about the foundations of algebraic geometry: a view of the subject that incorporated commutative rings directly into the structure, and was a precursor to:

Grothendieck's *Langage des Schémas*

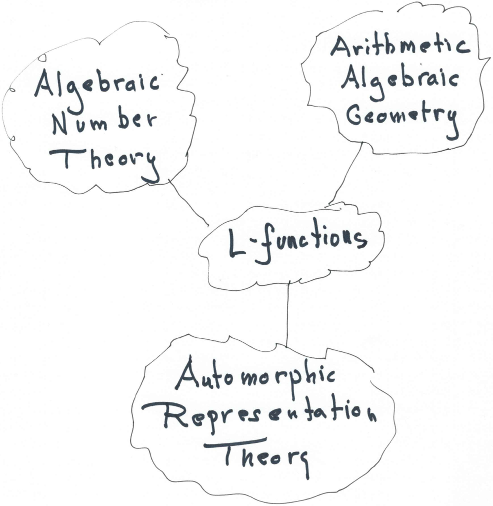
which provided a unification even more striking, allowing for the development of:

Arithmetic Algebraic Geometry

and even better, today's:

Arithmetic and Automorphic Algebraic Geometry

with L-functions tying this all together:



Two interlacing grand programs—

The Langlands program:

Automorphic Rep'ns $\overset{L-\text{functions}}{\leftrightarrow}$ Galois Rep'ns

Main Conjectures:

Galois Rep'ns $\overset{\text{Euler Systems}}{\leftrightarrow}$ Arithm.(Cohom.) Invariants

An Euler System forms a bridge

between a Galois Representation and fundamental Arithmetic Invariants of the algebraic geometry behind that Galois representation.

The constellation of algebraic cycles of an Euler System goes beyond merely reinforcing the analogical connection between the two sides of a "Main Conjecture" that links the two sides but:

also allows us to *actually prove* the conjectured connection.

Knots analogous to Primes

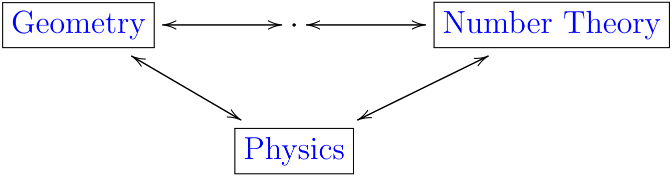
For example (what I will focus on for the rest of this hour):

Knots in 3-manifolds $\leftrightarrow$ Primes of number fields

an analogy that unites

topological structures with *arithmetic structures,*

and more broadly:



See the video lecture on the analogies tying this triumvirate together, starting with the analogy between knots and primes:

Masanori Morishita (Kyushu Uni.)/ Knots and Primes - Hyperbolic Geometry and Galois Deformation
`https://www.youtube.com/watch?v=tnLstfIeQm4`

Why do we think of $\mathcal{S} := \mathrm{Spec}(\mathbb{Z}) \sqcup \{\infty\}$ as the arithmetic "three-dimensional sphere."

First, any connected finite extension of the ring of integers $\mathbb{Z}$ is ramified—so:

$\mathcal{S}$ is simply connected.

As for the cohomology of $\mathcal{S} := \mathrm{Spec}(\mathbb{Z}) \sqcup \{\infty\}$ one needs some class field theory, but reformulated in the vocabulary of étale (and some other Grothendieckian) cohomology theories.

Cohomology with coefficients in the multiplicative group $\mathbb{G}_m$:

$$i = 0: \qquad H^0(\mathcal{S}, \mathbb{G}_m) = \mathbb{Z}^* = \{\pm 1\}$$

$$i = 1, 2: \qquad H^i(\mathcal{S}, \mathbb{G}_m) = 0$$

$$i = 3: \qquad H^3(\mathcal{S}, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$$

$$i > 3: \qquad H^i(\mathcal{S}, \mathbb{G}_m) = 0$$

“Poincaré Duality”

If F is a finite flat group scheme over $\mathcal{S}$ and $F^* :=$ $\underline{Hom}(F, \mathbb{G}_m)$ its (Cartier) dual finite flat group scheme, then cup-product:

$$H^i(\mathcal{S}, F) \otimes H^{3-i}(\mathcal{S}, F^*) \longrightarrow H^3(\mathcal{S}, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}.$$

induces a perfect pairing of cohomology groups (for the flat—fppf—topology over $\mathcal{S}$).

and a perfect pairing

$$H^i(\mathcal{S}, F) \otimes Ext_{\mathcal{S}}^{3-i}(F, \mu_n) \longrightarrow H^3(\mathcal{S}, \mu_n) = \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

for any finite flat group scheme F of exponent n.

So: $\mathcal{S}$ is morally 2-connected and enjoys a 3-dimensional Poincaré duality "oriented" by the coefficient sheaf $\mathbb{G}_m$.

and contains a 'Cebotarev" family of disjoint "knots;" i.e., $2, 3, .5, 7, 11, 13, \ldots$

The "Knot" p

Let p be a prime; consider reduction mod p:

$$\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$$

and put

$$\mathcal{K}_p := \mathrm{Spec}(\mathbb{F}_p) \quad \hookrightarrow \quad \mathcal{S} = \mathrm{Spec}(\mathbb{Z}) \sqcup \{\infty\}$$

and $\mathcal{K}_p$ is what we want to call

The "Knot" attached to a prime p

$$\begin{array}{ccc} \bar{\mathbb{F}}_p & & \mathrm{Spec}(\bar{\mathbb{F}}_p) \\ \uparrow & & \downarrow \\ \mathbb{F}_{p^\nu} & & \mathrm{Spec}(\mathbb{F}_{p^\nu}) \\ \uparrow & & \downarrow \\ \mathbb{F}_p & & \mathrm{Spec}(\mathbb{F}_p) \end{array}$$

The fundamental group of $\mathcal{K}_p$— i.e. $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$—is (canonically) isomorphic to $\hat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$. From the étale homotopy perspective, $\mathrm{Spec}(\bar{\mathbb{F}}_p)$ is contractible, and therefore:

$\mathcal{K}_p = \mathrm{Spec}(\mathbb{F}_p)$ is homotopically a $K(\hat{\mathbb{Z}}, 1)$-space.

Before we get into specifics, these are correspondent objects:

The analogy (very local)

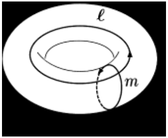
$$
\begin{array}{ccc}
K \simeq S^1 & \xleftrightarrow{\text{like}} & \mathcal{K}_p \\
\updownarrow \pi_1 & & \updownarrow \pi_1 \\
\mathbb{Z} & \xleftrightarrow{\text{like}} & \hat{\mathbb{Z}}
\end{array}
$$

The topologist's Tubular Neighborhood

For $K \subset S^3$ a knot, we decompose the three sphere as follows:

$$S^3 = X_K \cup N_K$$

where N_K is 'the' tubular neighborhood of the knot $K \subset N_K$



and

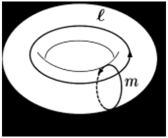
$$\partial X_K = X_K \cap N_K = \partial N_K := T_K.$$

Up to isotopy, the knot complement may be viewed as compact manifold with torus boundary, $T_K = \partial X_K$,

$T_K :=$ the torus around the knot

and on that torus, given that K is assumed to be an *oriented* knot—there's a normal ('meridianal') loop m defined up to homotopy,

and ℓ a 'radial loop,'—i.e., a *shadow* of the knot—only defined (in homotopy) up to the ambiguity of adding a multiple of m.



The homotopy type of the knot complement:

$$X = X_K := \; S^3 - N_K \; \stackrel{\sim}{\hookrightarrow} S^3 - K \hookrightarrow \; S^3,$$

Mention: The Gordon-Luecke Theorem.

The cohomology of the knot complement:

Alexander duality establishes a $\mathbb{Z}$-duality between $H^1(X_K;\mathbb{Z})$ and

$$\partial : H_2(S^3, K;\mathbb{Z}) \xrightarrow{\simeq} H_1(K;\mathbb{Z}) = \mathbb{Z},$$

giving us a canonical isomorphism:

$$H^1(X_K;\mathbb{Z}) = \mathbb{Z}$$

which tells us that

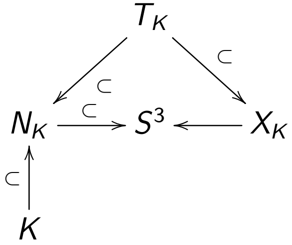
- all finite abelian covering spaces of S^3 branched at the knot, but unramified outside it, have *cyclic* groups of deck transformations,
- that these cyclic groups have canonical compatible generators,
- and that $X_K^{\mathrm{ab}} \to X_K$, the maximal abelian covering space of X_K, has group of deck transformations Γ canonically isomorphic to $\mathbb{Z}$.

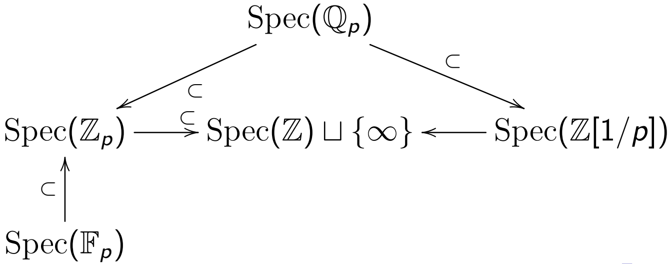
The arithmetician's "complement of the Knot" attached to a prime p

This comes from adjoining the inverse of p: $\mathbb{Z} \subset \mathbb{Z}[1/p]$, giving:

$$\boxed{\mathcal{X}_p := \mathcal{S} \setminus \mathcal{K}_p = \mathrm{Spec}(\mathbb{Z}[1/p]) \subset \mathrm{Spec}(\mathbb{Z}) \sqcup \{\infty\} =: \mathcal{S}}$$

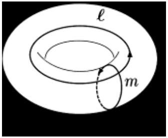
The topologist's picture and the arithmetician's picture





The analogy (local)

$$\begin{array}{ccccc} K \subset \{\textit{Tubular nbd of } K \textit{ in } S^3\} & \longleftrightarrow & \mathrm{Spec}(\mathbb{Z}_p) & \hookleftarrow & \mathrm{Spec}(\mathbb{F}_p) \\ \uparrow & & \uparrow & & \\ \partial\{\textit{Tubular nbd of } K\} & \longleftrightarrow & \mathrm{Spec}(\mathbb{Q}_p) & & \end{array}$$



The analogy (global)

$$
\begin{array}{ccc}
\text{a } \textit{Knot } K & \xleftrightarrow{\text{analogous to}} & \text{“}\textit{a Prime p}\text{”} = \mathrm{Spec}(\mathbb{F}_p) \\
\downarrow & & \downarrow \\
\{K \subset S^3\} & \longleftrightarrow & \{\mathrm{Spec}(\mathbb{F}_p) \hookrightarrow \mathrm{Spec}(\mathbb{Z}) \sqcup \{\infty\}\} \\
\uparrow & & \uparrow \\
S^3 \setminus K & \longleftrightarrow & \mathrm{Spec}(\mathbb{Z}) \sqcup \{\infty\} \setminus \{p\} = \mathrm{Spec}(\mathbb{Z}[1/p]) \sqcup \{\infty\}
\end{array}
$$

allowing us to frame possible statistical analogies. . .

Fundamental group(s) $\rightarrow$ Homology group(s)

$$
\begin{array}{ccc}
\pi_1(\text{complement of } K \subset S^3) & \qquad & \pi_1(\text{compl. of } p \text{ in } \mathrm{Spec}(\mathbb{Z})) \\
\big\updownarrow {\scriptstyle =} & & \big\updownarrow {\scriptstyle =} \\
\pi_1(S^3 - K) & & \mathrm{Gal}(\mathbb{Q}^{\mathrm{unram}\setminus p}/\mathbb{Q}) \\
\big\downarrow {\scriptstyle \textit{abelianization}} & & \big\downarrow {\scriptstyle \textit{abelianization}} \\
H_1(S^3 - K) \simeq \mathbb{Z} & & H_1(\mathrm{Spec}(\mathbb{Z}) - \{p\}) \simeq (\mathbb{Z}_p)^*
\end{array}
$$

Maximal abelian ‘unramified’ extensions

$$(S^3 \setminus K)_\infty \xrightarrow{\Gamma := \mathbb{Z}} S^3 \setminus K$$

$$\mathrm{Spec}(\mathbb{Z}[\zeta_p^\infty]) \xrightarrow{\Gamma := \mathbb{Z}_p^*} \mathrm{Spec}(\mathbb{Z})$$

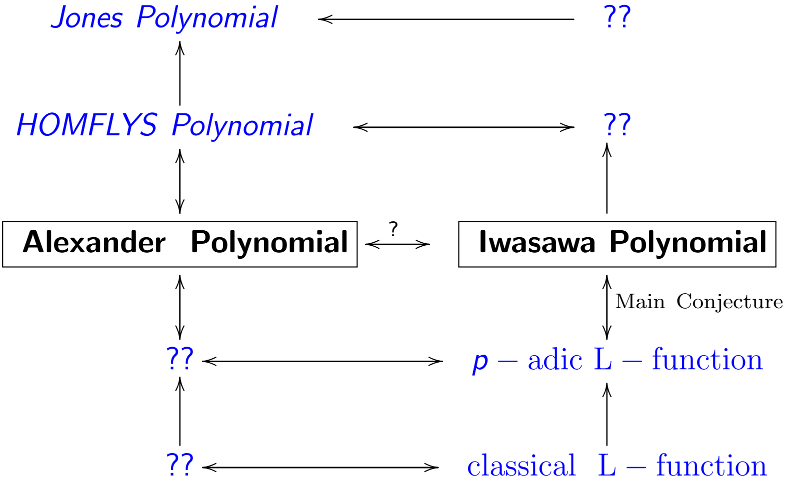
This is an invitation to compare:

- The Alexander Polynomial:=the characteristic polynomial of the action of Γ on the relevant cohomology of $(S^3 \setminus K)_\infty$ (the maximal abelian covering of S^3 ramified only along the knot K),

with

- The Iwasawa Polynomial:=the characteristic polynomial of the action of Γ on the relevant cohomology of $\mathrm{Spec}(\mathbb{Z}[\zeta_p^\infty])$.

These polynomials are linked to...



E.g.: "What is the *arithmetical algebraic* interpretation of the Jones polynomial?"

Brief comments on comparison and differences

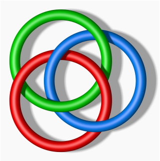
- If by **unknotted** one means that the fundamental group of the knot is abelian, every prime is **'knotted.'**
- A serious distinction between knots and primes has to do with what is called *wild inertia* a phenomenon that exists, and is of crucial importance in number theory, but there's no corresponding complexity in our analogous situation in knot theory.
- There is a duality in the structure of the Alexander polynomial (it is invariant under inversion $t \mapsto t^{-1}$; hence if θ is a root, so is θ^{-1}). But there is nothing like that for Iwasawa polynomials.
- *Discuss* links $\{p, q\} \sim \{K, L\}$ and . . . .

Reciprocity

$$QR \longleftrightarrow Skew - Symmetry$$

$$Triples\ of\ primes \longleftrightarrow Borromean\ Rings$$

$$Redei\ invariants \longleftrightarrow Massey\ Triple\ Prods. \sim Milnor\ invariants$$



More specificity to the analogy:

Prime Numbers p $\overset{\text{The analogy}}{\leftrightarrow}$ **Hyperbolic Knots** K

Definition: A hyperbolic knot or link is a knot or link whose complement is a hyperbolic manifold, complete and of finite volume.

Going Further?

Questions about

- The analogue of Chebotarev's Theorem . . . (see the note: *Chebotarev Questions*)
- Generalized Iwasawa Theory for knot groups. . .
- Eigenvarieties: deformations of representations of knot groups. . .

Thanks for offering me the opportunity to raise these questions!