A Foreword for: *Fearless Symmetry: Exposing the Hidden Patterns of Numbers* by Avner Ash & Robert Gross

(to be published by Princeton University Press)

At some point in his or her life every working mathematician has to explain to someone, usually a relative, that mathematics is hardly a finished project. The mathematicians know, of course, that it is far too early to put the glorious achievements of their trade into a big museum and become happy curators. Our subject has, in certain respects, hardly begun. But, at least in the past, this seems not to have been universally acknowledged.

Recent successes (most prominently: Fermat's Last Theorem) have advertised to a wide audience that math remains humanity's grand "work-in-progress" where mysteries abound and profound discoveries are yet to be made. Along with this has come a demand from a larger public for genuinely expository, but serious, accounts of current exciting themes in mathematics.

It is a hard balancing act: to explain important and beautiful mathematical ideas–to *truly* explain them–to people with a general cultural background but no technical training in math, and yet not to slip away from the full seriousness and ambitious goals of the subject that is being explained.

Avner Ash and Robert Gross do a wonderful job at this balancing act in *Fearless Symmetry.* On the one hand the substance of their book is honestly–fearlessly, even– faithful to the great underlying ideas of the mathematical story that they tell. On the other hand, the authors are keenly sensitive to the basic, almost pre-mathematical, issues that would occur to, that would beset, a newcomer to those ideas, and they treat these issues with a level of thoughtfulness that is–in my opinion– something of an exemplary model.

Our authors also bring out the *eternally unfinished* aspect of math, its open-ended quality. The resolution of any part of mathematics invariably modulates the subject into a different key, and makes a new and deeper set of questions vital. One theorem having been proved, more further-reaching problems come to prominence. Fermat's Last Theorem, posed over 350 years ago, has been proved; the curious *Problem of Catalan*, conceived over a century ago, to prove that 8 and 9 are the only two consecutive perfect powers ($8 = 2^3; \quad 9 = 3^2$) has recently been solved. But you need only glance at the last chapters of this book to see how, in the wake of the resolution of older problems, a new, and possibly richer, repertoire of interesting problems that would have astounded ancient Diophantus has come to occupy center stage.

And beyond that, waiting for future generations, are the sweeping expectations posed by celebrated problems like the ABC conjecture, and the Riemann Hypothesis.

Fearless Symmetry begins where few math books do, with an enlightening discussion of what it means for one "thing" to *represent* another "thing." This action –deeming A a "representation" of B–underlies much mathematics; for example, *counting,* as when we say that *these two mathematical units "represent" those two cows.* What an extraordinary concept *representation* is, and has always been, from early in its history! In Leibniz's essay *On the universal science: characteristic* where he sketched his scheme for a universal language that would reduce ideas "to a kind of alphabet of human thought" Leibniz claimed his characters (i.e., the ciphers in his universal language) to be manipulable *representations* of ideas.

> All that follows rationally from what is given could be found by a *kind of calculus,* just as arithmetical or geometrical problems are solved.

Nowadays, whole subjects of mathematics are seen as represented in other subjects, the "represented" subject thereby becoming a powerful tool for the study of the "representing" subject, and vice versa.

The mathematics of *symmetry* also has had an astounding history. It timidly makes appearance in Euclid's *Elements* under the guise, for example, of the notion of similarity (*to homoion*). In somewhat homey terms, the more modern attitude towards a symmetry is that it is a geometric transformation that you can perform on an object that makes the object ending up looking as if it were exactly the same, and in the same position, afterwards as before. For example, if we are working in Euclidean geometry, the symmetries of an equilateral triangle in the plane, consists in the three flips about the angle bisectors through each of its vertices, and also the three (yes there are three!) rotations that preserve the figure (rotate about the center by 120 degrees, 240 degrees, and 0 degrees). By the end of the nineteenth century, with the emergence of Klein's "Erlangen program," the general notion of symmetry has established itself as the very foundations of geometry; for all homogenous geometric structure had been come to be viewed as but a consequence of a study of the groups of their symmetries.

Groups of symmetries of a geometrical object possess an intrinsically algebraic structure, if we take the view that the *product* $S \cdot T$ of two symmetries S, T consists of the new symmetry that you get by first performing the symmetry T and and following that by performing the symmetry S. The surprise is that this kind of "multiplication structure" on the collection of symmetries of a geometry holds the key to a fuller understanding of that geometry. From an understanding, for example, of the continuous family of all *congruences,* of Euclidean geometry, together with knowledge of the corresponding multiplication as described above, we can (re-)construct all of Euclid's geometry, with its straight lines, its angles, its circles!

Once, however, that we have wrested these purely algebraic structures, *groups of symmetries,* from their geometric origins we are entitled to consider them entirely as creatures in algebra, where they are called, simply, *groups.* And, we can go the other way: to seek to *re-present* such an algebraic structure, a group, as a group of symmetries of some geometry; and, even more revealing perhaps, as a group of symmetries of a geometry *different* from the one from which it initially arose.

Viewed from this perspective, the bare algebraic notion of **group** establishes itself as an emissary, of a sort, between different geometries: the same group might account for the symmetries of two disparate geometries. Even more relevant to the substance of *Fearless Symmetry* is the great legacy of Evariste Galois in the nineteenth century, that these algebraic entities, groups, may bridge the even wider divide between the algebra of equations and geometry: *a group of symmetries of some system of algebraic equations might be represented as the group of symmetries of some geometry.* This development is the underpinning of much modern number theory.

The first two parts of this book are devoted to all these underlying algebraic ideas, including an introduction to the wonderful world of modular arithmetic opened up to us by the genius of Gauss, modular arithmetic being the beginnings of what we will call, below, "local number theory."

The third, and last, part of this book points the reader to the frontiers. *Reciprocity laws* play a big role in this story, for they form the backbone of what present-day number theorists call "global number theory." A *local problem* is one that concerns itself with issues regarding divisibility by a single prime number p, or by its powers. "Global problems," in contrast, constitute the basic hard questions we wish to answer about whole numbers. Reciprocity laws, when available, represent the extra glue, the further constraint, in a problem of global number theory that ties together all the corresponding problems in the various local number theories connected to each of the prime numbers $p = 2, 3, 5, 7, 11, \ldots$

Ash and Gross end their book with some comments on the Fermat's Last Theorem. The celebrated proof of this theorem depends upon the realization that a solution to the equation

$$X^p + Y^p = Z^p$$

with p an odd prime number and X, Y, Z nonzero integers leads us to be able to find a very distinctive finite group of symmetries of algebraic numbers, and to be able to represent this group *also* as a group of symmetries of a specific finite geometry, this representation having *peculiar* properties. The demonstration of the impossibility of a nonzero integer solution X, Y, Z follows from the proof that these peculiar representations cannot, in fact, exist.

Fearless Symmetry can be read, at one level, by a reader who may have no particular mathematical experience but is interested in the important concepts that frame the mathematical viewpoint (e.g., the concept of *representation* in Chapter 1, of *modular arithmetic* in Chapter 2). Readers with some background in basic mathematics who are happy to do a few calculations and to make a few numerical experiments, will also gain much as they accompany the authors further in their examination of some of the mathematical structures that play a role in this fine book.