Hilbert's Hotel and other encounters with infinity

Barry Mazur

How do we measure infinity?

How do we compare infinite sets?

If someone tells you that

> *"it is equally likely for a prime number to have residue* $+1$ *as it is likely for it to have residue* -1 *modulo* 4*"*

In how many ways might one formulate (correct) rigorous mathematical statements that are in accord with that claim?

forgetting, for the moment, any proof of this claim . . .

Density Questions

$$\text{infinite set} \longrightarrow S \subset T \longleftarrow \text{infinite set}$$

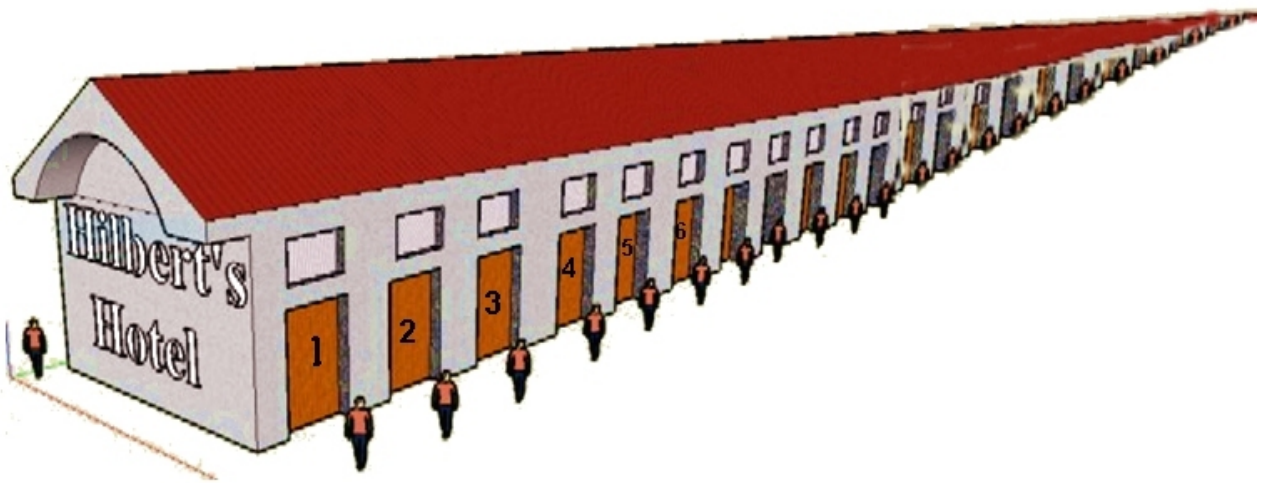
What might the density of a subset contained in an infinite set mean?

Of course a related, perhaps prior, question is:

What is an infinite set?

Infinite Real Estate

Oh the virtues of that unending corridor[1] of rooms in Hilbert's Hotel!



[1] https://thechristcollegemathblog.wordpress.com/2014/12/03/hilberts-hotel/

Hotel Hilbert

Even if every room is occupied and some new guest arrives looking for a room: no problem. The manager "just" asks all the guests to move to the next room in the corridor,

$$\boxed{1} \to \boxed{2} \to \boxed{3} \to \boxed{4} \to \boxed{5} \to \dots$$

leaving the first room free for the new guest.

Defining Infinity:

Here are four possible definitions of *infinite set*—they are all minor variants of one another. (The first is due to Dedekind and resonates with that curious corridor in *Hilbert's Hotel*.)

What do we think is the difference between the following four possible definitions of *infinite set*?

First, recall the concepts: **injective** and **surjective** map:

Injective

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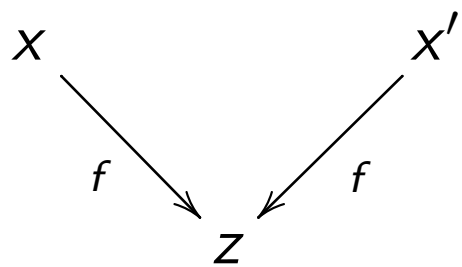
I.e., if for $x, x' \in X$ we have

$$\{x \neq x'\} \implies \{f(x) \neq f(x')\}.$$

I.e., no collapsing:



An example:

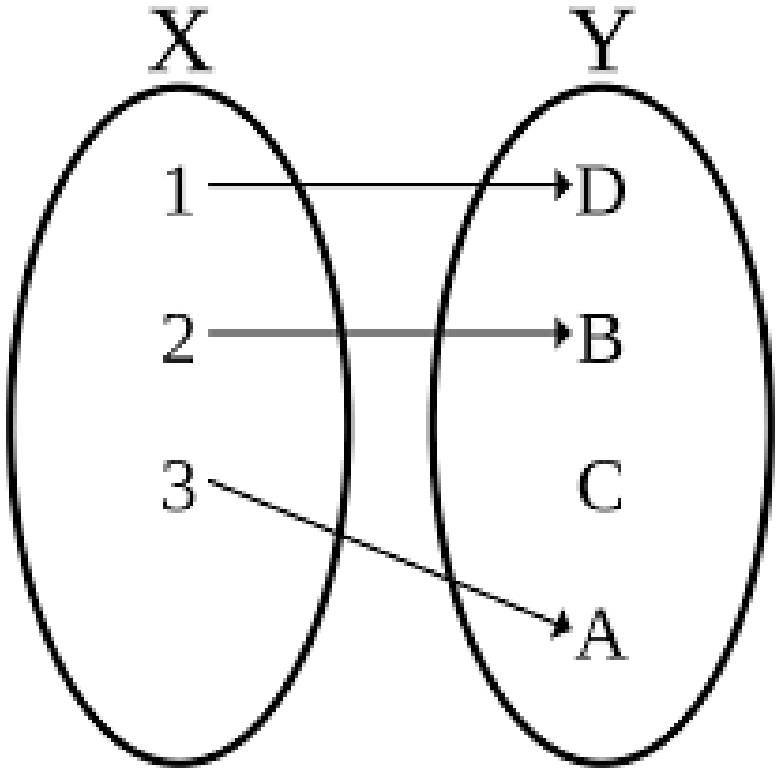


Figure: injective

Surjective

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I.e.,

$$\forall y \in Y \ \exists x \in X \text{ such that } f(x) = y.$$

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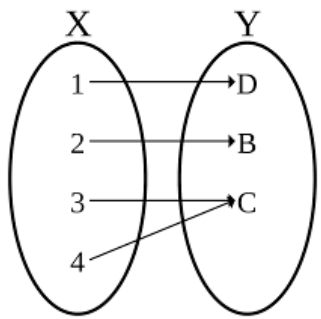

Figure: surjective

Now for the four different candidate-definitions of this concept "infinite set":

Definition ∞_1 :

A set S is **infinite** if there exists an *injective* mapping $f : S \to S$ (i.e., from the set S to *itself*) that is not surjective (equivalently: *injective* but not a one:one correspondence between the set S and itself).

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This definition is given in Richard Dedekind's (1888) essay:

"Was sind und was sollen die Zahlen?"

Definition ∞_2 :

A set S is infinite if there exists a *surjective* mapping $f : S \to S$ (i.e., from the set S to *itself*) that is not injective (equivalently: *surjective* but not a one:one correspondence between the set S and itself).

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(The set of natural numbers is what you think it is:

$$\mathsf{N} := \{1, 2, 3, 4, \dots\},$$

even though the ancients were dubious about the number 1 as being in the same category as the other whole numbers.

To actually *define* this set N without making use of the *dot-dot-dots* requires some apparatus—e.g., mathematical induction.)

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E.g., it is a set S such that any finite subset of S has a non-empty complement in S.

So, what is a finite set?

Definition or Characterization?

Within the appropriate axiomatic set-theoretic context, the four definitions of "infinite set" are equivalent, so we have a choice:

- We can choose one of them as our *primary definition*, and the other three can be thought of as *'characterizations'* of the then-defined concept—infinite set.
- We can simply say: these are all equivalent and any one can serve as "the" definition.

The relationship between these choices

depend on the ambient axiomatic context in which are working. For example, if you accept the 'Axiom of Choice' then if a set is infinite following Definition ∞_2 it is also infinite following Definition ∞_1.[2]

[2]For an excellent account of the issues that Dedekind confronted in his essay see *Notes on Richard Dedekind's "Was sind und was sollen die Zahlen?" by David Joyce,*
https://mathcs.clarku.edu/~djoyce/numbers/dedekind.pdf

Definition versus Characterization?

The question, then, (*What is an infinite set?*) depends on the choice: definition versus characterization. The same holds for the question:

What *are*[3] Prime Numbers?

[3] 'and what should be'—following the tone of Dedekind's *"Was sind und was sollen die Zahlen?"*

Two possible definitions

A prime number p is a (whole) number greater than one

1. that is not expressible as the product of two smaller numbers. (Unfactorable.)

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A prime number p is a (whole) number greater than one

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or

2. that has the property that if it divides a product of two numbers, it divides one of them.

Your Choice!

If you choose **(2)** (*p is prime if 'whenever' it divides a product it divides one of the factors*) as the fundamental definition...

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you are actually placing the notion of prime number in the broader context of 'prime'-ness as it applies to number systems more general than the ring of ordinary numbers—

and more specifically framing it in the context of *prime ideals* of a general ring.

Moreover, choosing **(2)** as *your* definition

casts **(1)** (*that primes are numbers that are 'unfactorable'*) as a specific feature that characterizes prime numbers, thanks to the theorem that guarantees the equivalence of these two formulations.

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i.e., focusing on **(1)**, the unfactorable quality of prime number, would then cast **(2)** as simply a more general feature also *characterizing* prime-ness (within the larger context of commutative rings with identity element).

What prime numbers **are there**?

New primes are discovered every few days... *Two weeks ago today:* A record prime of 50001 digits was computed:

$$10^{50000} + 65859$$

is the smallest prime with 50001 digits.[4]

[4] `https://www.multiprecision.org/downloads/ecpp/cert-50000.bz2` in PARI/GP format and `https://www.multiprecision.org/downloads/ecpp/cert-50000.primo.bz2` in Primo format as converted by PARI/GP code written by J. Asuncion, who is also the author of the fastECPP implementation in PARI/GP.

Proofs that there are infinitely many primes

The classical proof: ... is the one in Euclid's *Elements*[5].

It proves that the set of prime numbers is "non-finite." E.g., if you already know that $2, 3, 5, 7$, and 11 are primes, for example, then Euclid's proof notes that there exists yet another prime

$$\leq 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \ + \ 1 \ = 2311.$$

Well, 2311 is, in fact, a prime not on your list, but... so is 13.

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[5] "Elements" —'$\Sigma\tau o\iota\chi\epsilon\hat{\iota}\alpha$'—is quite a stark title. "Elements" of what?

Far out proofs:

Following Euler, the Riemann zeta function $\zeta(s)$ which has its additive and multiplicative formats—for $Re(s) > 1$:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

unique factorization theorem

The values of $\zeta(s)$ at positive even integers

For $s = 2n$ (with n any positive integer) Euler proved that

$$\zeta(2n) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^{2n}}\right)^{-1} = r_n \cdot \pi^{2n} \qquad (0.1)$$

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$$\boxed{r_n := (-1)^{n+1} \frac{2^{2n-1}}{(2n)!} \cdot B_{2n}}$$

where B_{2n} = the $2n$th Bernoulli number.

What if there were only finitely many prime numbers?

Now,

- *If there were only finitely many prime numbers*, then

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^{2n}}\right)^{-1} \tag{0.2}$$

would be a *rational* number.

Contradiction!

Comparing

$$\zeta(2n) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^{2n}}\right)^{-1} = r_n \cdot \pi^{2n} \qquad (0.3)$$

with:

Theorem: (Lindemann-Weierstrass) π is transcendental,
and

choosing n to be some positive number you like:
$n = 1$ or $n = 2$, or ... $n =$ a million, we get a contradiction, since the central term in this equation (0.3) would be rational... but $r_n \cdot \pi^{2n}$ is irrational.

For any choice of n we get:

Corollary ($Proof_{\{n\}}$)

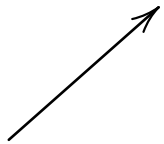
There are infinitely many prime numbers.

Query

1. *Assuming that these proofs aren't circular (and I'm guessing that they aren't) would you consider them to actually be different proofs for different choices of positive integer n?*

2. *Which, if any, of the definitions we listed above of the concept "infinite" do these proofs rely on?*

Prime Number Races



That is the title of a wonderful article[6] by Andrew Granville and Greg Martin that explains in detail the race between, for example, the team of prime numbers congruent to 1 mod 4 (color them blue) and the team congruent to -1 mod 4 (color them red).

[6] https://arxiv.org/pdf/math/0408319.pdf

Red versus Blue

$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41,$

$43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$

Primes congruent to 1 mod 4 (blue)

—versus—

Primes congruent to -1 mod 4 (red)

Score:

$$\{\text{ Score for primes } < 100\}: \qquad 11 \quad \text{to} \quad 13$$

The race is very close, but the team of Primes congruent to $-1 \bmod 4$ is ahead. This is often the case for larger packets of prime numbers:

$$\{\text{ Score for primes } < 100,000\}: \qquad 4783 \quad \text{to} \quad 4808$$

Flipping a coin

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That is, the ratio

$$|\{\text{red primes} < X\}|$$

to

$$|\{\text{blue primes} < X\}|$$

tends to 1 as X tends to ∞.

So, the race is tight

It is even tighter given:

1. a conjecture[7] that for 100% of the positive integers X the number of red primes less than X exceeds the number of blue primes less than X,

and

2. a theorem of Littlewood that says that there will be cutoffs X arbitrarily large for which the number of blue primes less than X exceeds the number of red primes less than X.

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[7] beginning with some comments of Tschebyscheff in a letter to the mathematician Fuss. This was sharpened and made explicit as a conjecture in 1962 by Knapowski and Turán. See the discussion in the article *Prime Number Races* of Granville and Martin cited above.

This discussion might lead us to the general question:

What ways are there to measure the 'density' of a subset S of the set $\mathcal{P}$ of prime numbers?

Such a notion, **density** should be defined on lots of (but not necessarily all) subsets S of $\mathcal{P}$,

$$S \mapsto \delta(S) \quad \in \quad [0,1] \subset \mathbb{R},$$

and should at least have these basic properties:

Requirements for a 'density function'

1. The density of the full set of primes $\mathcal{P}$ is 1 and
2. the density of any finite set of primes is 0.

3. *Density is finitely additive:* Let $S_1, S_2 \subset \mathcal{P}$ be disjoint subsets of primes and $S_3 := S_1 \cup S_2$ their union. If two out of the three subsets S_1, S_2, S_3 have defined densities then the third one has a defined density too: and

$$\delta(S_1) + \delta(S_2) = \delta(S_3).$$

Natural Density:

The most natural definition one can think of that might be called the density of a subset S of primes is:

$$\delta_{\text{natural}}(S) := \lim_{X \to \infty} \frac{|\{p \in S; p \leq X\}|}{|\{p \in \mathcal{P}; p \leq X\}|}, \tag{0.5}$$

if that limit exists.

And δ_{natural} satisfies all the requirements in the axioms above.

Natural Density depends on very little:

Note that "natural density" depends *only* on the structure of $\mathcal{P}$ viewed as an ('abstract') ordered set, and not at all on the placement of the prime numbers within the set of real numbers.

Also if you are willing to replace "lim" by "liminf" you have a definition of 'proportion' $\underline{\delta}(T)$ of any subset $T \subset \mathrm{N}$ of natural numbers N:

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$$\underline{\delta}(T) := \liminf_{X \to \infty} \frac{|\{n \in T;\ n \leq X\}|}{X}, \tag{0.6}$$

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For each real number $x > 1$ and any set of primes S, let's define the "x-**Density**" of S to be

$$\delta_x(S) := \frac{\sum_{p \in S} p^{-x}}{\sum_{p \in \mathcal{P}} p^{-x}} \tag{0.7}$$

What's neat about "x-**Density**":

$$\delta_x(S) := \frac{\sum_{p \in S} p^{-x}}{\sum_{p \in \mathcal{P}} p^{-x}} \tag{0.8}$$

is that (for any $x > 1$) $\delta_x(S)$ is defined for every subset $S \subset \mathcal{P}$, and is finitely additive.

It has the defect, though, of assigning a *non zero density* to absolutely every nonempty subset—including finite subsets.

Dirichlet Density

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Define the **Dirichlet density**, $\delta_{\text{Dirichlet}}(S)$, of a subset $S \subset \mathcal{P}$ to be the limit of its x-Density (as x tends to 1)—*if that limit exists*.

$$\delta_{\text{Dirichlet}}(S) := \lim_{x \to 1} \delta_x(S) \qquad (0.9)$$

Note that if S is any finite subset of $\mathcal{P}$ the numerator,

$$\sum_{p \in S} p^{-x},$$

of the equation

$$\frac{\sum_{p \in S} p^{-x}}{\sum_{p \in \mathcal{P}} p^{-x}}$$

has a finite limit as x tends to 1, while:

Lemma: The denominator,

$$\sum_{p \in \mathcal{P}} p^{-x}$$

tends to infinity as x tends to 1

So,

the Dirichlet density of any finite set of primes is 0.

Dirichlet density then satisfies all three requirements desired for a 'density' listed above.

The relationship between Natural and Dirichlet Density

Theorem

If a subset $S \subset \mathcal{P}$ has a Natural density, then S also has a Dirichlet density, and the two densities are the same.

The non-correlation of prime numbers and congruence

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The same sort of thing is true (proved again by Dirichlet) more generally for any other reasonable congruence condition. I.e., distinguishing primes by how they "behave' modulo m for a fixed integer m.

Take m any positive number.

Let $x \mapsto \Phi(x)$ be Euler's Phi-function. There are $\Phi(m)$ congruence classes mod m containing elements relatively prime to m.

Given such a congruence class, Dirichlet showed that the probability that a 'random prime' falls into that congruence class is

$$\frac{1}{\Phi(m)},$$

i.e., the primes are uniformly distributed over those congruence classes mod m.

A very new result about proportions of numbers having interesting properties

Question

What is the proportion of integers that are the sum of two rational cubes?

This happens frequently; e.g. . . .

$$1, 2, 6, 7, 8, 9, 12, 13, 15, 16, 17, 19, 20, 22, 26, 27, 28, 30, 31, 33, 34, 35, \ldots$$

For example:

$$6 = (17/21)^3 + (37/21)^3$$

Is it as 'likely' that an integer is the sum of two rational cubes as not?

A very recent result of Levent Alpöge, Manjul Bhargava and Ari Shnidman—with full proofs forthcoming is an important advance in that direction:

Theorem

(Alpöge, Bhargava, Shnidman) , A positive proportion of integers are not[8] *the sum of two rational cubes, and a positive proportion of integers are*[9]*.*

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[8] See —

[9] proof forthcoming

Welcome to Arithmetic Statistics!

These kinds of questions are both utterly traditional... but also are in a currently burgeoning, rapid-moving, part of number theory where the aim is to get the (statistical) 'lay of the land' regarding basic issues in number theory.