History of Mathematics as a tool

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As an introductory lecture (and, I hope, discussion session) for this History of Mathematics seminar[1] it seems appropriate to ask the question: *Why history?*

Well, the essence of *history* is *story*, and a good story is an end in itself.

But here-as a sample–are a few issues that may motivate us to make fundamental use of the 'historical tool.' A tool rather than an end.

- Dealing with *ideas in the form they were first discovered* often shines a light on the primal motivation for them, and shows them (sometimes, but not always) in their least technical dress. *Why did anyone dream up the notion of homotopy, and homotopy groups?*
- Studying the *evolution of concepts* is sometimes the best way of coming to an understanding of those concepts. The concept may be quite a moving target; learning how it came to be what it now is may be the most efficient route for understanding it, and especially for gauging how it may evolve in the future. *What was the route that the simple idea of 'dimension' took, to include—as it does today—concepts ranging from invariants of C^*-algebras to fractals? How did the notions of 'topology' and 'sheaf' get shaped? How did the notions of 'group' and 'group representation' evolve, or did they spring fully formed into being?*
- Becoming aware of how certain evident, natural—trivial *to us*—concepts of mathematics constituted great leaps of insight and intuition when they first emerged on the mathematical scene. (Along with its other virtues, this awareness certainly helps us teach better!) E.g., the notion of *linearity* is so intuitive to us. But if you could only whisper the one word *linear* to John Wallis while he was writing his *The Arithmetic of Infinitesimals* you could have saved him some 30-50 pages of text.
- Coming to grips with any large research agenda in the mathematical tradition; e.g., the *Erlangen program*, or *Bourbaki*, or the project of *Classification of Finite Simple Groups*[2].

[1] I thank Anthony Liu and Lucia Mocz for the invitation to give this lecture. If I ever publish this, it will be in memory of John Murdoch who taught the history of Greek (and medieval) Mathematics and would often choose the topics of his seminars only after asking people *What do you want to know?*

[2] For this, see on-going work of Alma Steingart, currently writing her PhD thesis at MIT.

- Achieving close familiarity with the intellectual development of a person[3] or with an intellectual climate—now gone—that had nurtured a viewpoint, that was important to our predecessors, and might well become again important, to us or to our successors.

- Recognizing that mathematics as an activity might be described as *one long conversation* stretching over millennia.

In my lecture I'll try to hint at the last item of this ad hoc taxonomy.

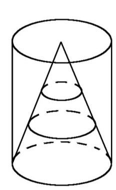
1 Classical Quadrature Problems

The term *quadrature* (of a figure) loosely refers to the problem of finding its area if it is a plane figure, or its volume if it is a solid. But that doesn't really describe the shape of the Classical Quadrature Problems that have appeared in the mathematical literature dating all the way back to the 5th century BC, and all the way forward to recent times.

The serious classical *quadrature problems* are phrased in a format that expresses profound *generic* area (or volume) relationships in geometry. This usually means expressing—as some simple numerical (i.e., *rational number*) ratio—the proportion between the area (or the volume) of an interesting species of object, a figure described only in generic terms, and the area (respectively, the volume) of another 'simpler' object dependent—by a clear construction—on the original figure[4].

There are quite a number of classical problems that fit this mold, e.g., that express the proportion of the areas of two planar configurations, or volumes of two solids (described in general terms) to a specific rational number. For example, Proposition 10 of Book XII of Euclid's *Elements* tells us that

the ratio of the volume of a cone to a cylinder that have the same base and the same height is $1/3$.



[3] *Euler* comes to mind as in Professor Dunham's class!

[4] A more modern pursuit, not entirely different in intent, is the study—in arithmetic algebraic geometry—of relations among periods of motives.

This "1/3" reoccurs as the ratio of the volume of a conical solid built on *any* base to the cylindrical solid built on the same base, and of the same height. The earliest text I know that "explains" the "1/3" in this more general context is *Arithmetica Infinitorum* by John Wallis who did his work before the full-fledged invention of Calculus. To us, of course, this is just an immediate observation, the $\frac{1}{3}$ coming directly from the formula

$$\int_0^x t^2 dt = \frac{1}{3}x^3.$$

2 Archimedes and Quadrature

Archimedes' work on quadrature is an extraordinary story in itself, ranging from

- his treatise *On the Sphere and Cylinder I* where he showed that

  > *The ratio of the volume of a sphere to that of the cylinder that circumscribes it is* 2/3,

  to

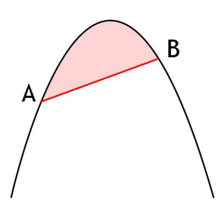
- *the problem of quadrature of the parabola* this being dealt with in (not one, but) two of his treatises in quite different ways:

  - Propositions 14-16 in *The Quadrature of the Parabola*

    and

  - Proposition 1 in *The Method*.

The goal of 'quadrature of the parabola' is to "find" the area of a segment of a parabola bounded by a chord.

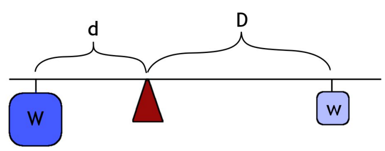


This means—following the format of a classical quadrature problem—giving explicitly the proportion of (the area of) the above kind of figure, $\mathcal{P}$, to that of some other, simpler, figure (in this case: a triangle) constructed in terms of $\mathcal{P}$.

This problem is especially illuminating in that Archimedes offers two approaches to it. The method in his treatise *Quadrature of the Parabola* is via exhaustion, i.e. approximation by polygons. This does prove what he wants. But the more curious method is the one that he himself refers to as a *mechanical method*—a mode of reasoning to which he *does not* give the full authority of proof: it's an example of a *heuristic*[5]—perhaps, chronologically, the first such example *explicitly* labeled as not-quite-a-proof that we have any record of.

A major tool Archimedes will use is his famous "law of the lever"[6] which proclaims that if weights W and w are placed on the plank that is the lever, at opposite sides of the fulcrum but at distances D and d from the fulcrum, respectively, then the lever will balance *if and only if*

$$D \cdot w \quad = \quad d \cdot W.$$



Now what in the world does this have to do with *area*? you might ask. The answer is that Archimedes is engaged, here, in an ingenious thought-experiment, where the rules of the game are dictated by some basic physical truths, and the link to *area* (he will also treat *volume* problems this way as well) is by a profound analogy: in the figure below, imagine the point K as the fulcrum of a lever.

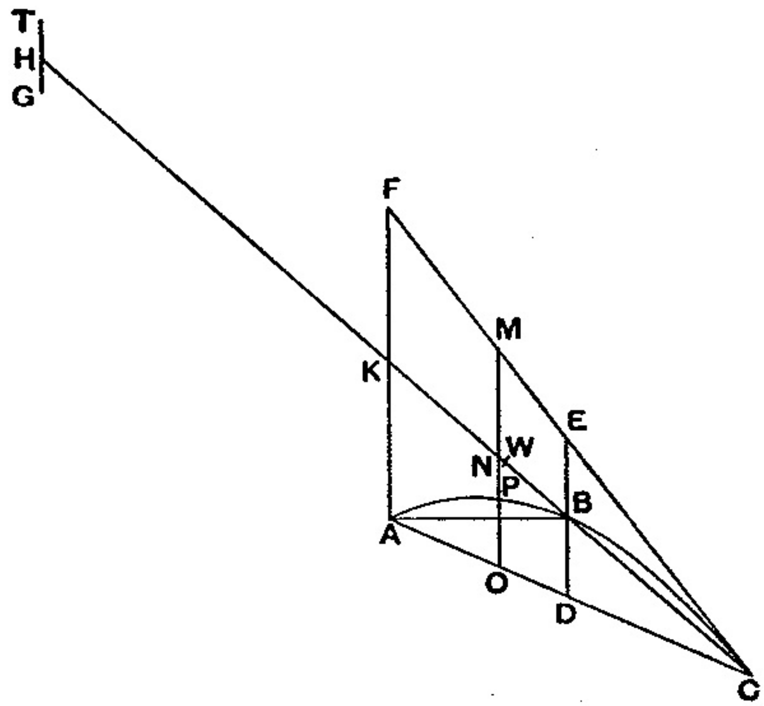
[5] very fitting, coming from the pen of the celebrated shouter of "Eureka."

[6] There is an extensive earlier tradition of discussion about equilibrium and disequilibrium on a balance; and on the action of levers of all sorts. For example,

> A small change occurring at the centre makes great and numerous changes at the circumference, just as by shifting the rudder a hair's breadth you get a wide deviation at the prow.

This is from Part 7 of Aristotle's *On the motion of animals*; I want to thank Jean de Groot for conversations about this; I look forward to her forthcoming commentary on Aristotle's *Mechanics*.

There is also, to be sure, an extensive later tradition of discussion; notably Ernst Mach's very strange critique of the "law" itself, in the Introduction and first few chapters of his wonderful *The Science of Mechanics*.



The plank of the lever is the line segment HKC. But to understand the above figure, we should proceed in the order of its construction.

So, begin with the *figure of interest* here; namely, the parabolic segment, $\mathcal{P}$, which is the region bounded by the parabolic arc labeled ABC and the chord AC. The point D is the bisector of the chord AC.

The 'vertical' lines AF and DE are lines parallel to the *diameter* of the original parabola that forms our parabolic segment $\mathcal{P}$. So is OM, meant to signify a general such line segment (parallel to the *diameter* of the original parabola).

The line FC is tangent to the parabola at the point C (and extended to meet the line AF. This constructs a triangle FAC which will play the role of the *simpler figure* in our quadrature problem.

The point B is the intersection of the 'vertical' line DE and the parabola. We also then have another triangle in our picture; namely: $\Delta := ABC$. The triangle Δ might equally well play the role of *simpler figure* in our quadrature problem, for one can show:

$$FAC \;=\; 4 \cdot \Delta.$$

Also–although this won't enter in the logical argument—it is immediate that

$$\Delta \subset \mathcal{P} \subset FAC.$$

Now extend a straight line through BC to a point H as indicated in the drawing. Note that we haven't yet said how far we should extend the line BC; i.e., except for the fact that H is required to lie on that line, we haven't yet specified H. The point K is the intersection of this straight line that extends BC and AF. Our **lever** or **plank** is the straight line HKC and the **fulcrum** is K.

We will be "weighing" (yes: weighing) the parabolic segment $\mathcal{P}$ by placing $\mathcal{P}$ in the precise position on the lever (which will turn out to be the point H) so that it balances perfectly with the triangle FAC deployed along the plank as it is shown in the diagram. We do this is a very curious way: we laminate both FAC and $\mathcal{P}$ by viewing each as made up of a continuum of line segments parallel to the diameter of the parabola; call them *slivers*. A typical sliver of FAC would be the line segment OM and the corresponding sliver of $\mathcal{P}$ would be OP.

We now imagine that each sliver has "weight" proportional to its length. And we proceed to move (sliver-by-sliver) the slivers of $\mathcal{P}$ on the plank to the left of the fulcrum K so that they balance—thanks to the 'law of the lever'—the corresponding unmoved sliver of FAC which remains wherever it happened to be on the plank (on the right side of K). A fact that Archimedes proves is that this process will put *all* the slivers of $\mathcal{P}$ on exactly the same point of the lever, left of K; and this point we call H. Now replace the triangle FAC, deployed as it is in the spread-out manner given in the picture, by a weight $\mathcal{W}$ at the point on the plank corresponding to the position of the center of gravity of FAC. Call that point W. It remains to work out the distances HK and KW which Archimedes does, to find:

$$KW \; = \; \frac{1}{3} \cdot HK.$$

Archimedes thereby concludes his heuristic argument to claim the quadrature:

$$\mathcal{P} \; = \; \frac{1}{3} \cdot FAC \; = \; \frac{4}{3} \cdot ABC.$$

3 Squaring the Circle

The *most* notorious of the classical quadrature problems, *squaring the circle* is, perhaps, also the most natural 'first problem' you might dream up if you want to relate the area cut out by some *curvilinear* planar figure to the area of a related *rectilinear* figure. As we shall see, the era of Euclid's *Elements of Geometry* is quite late in the history of this problem, although there are only glimmers in Euclid's text of an interest in this *circle-squaring* problem, as in Proposition 2 of Book XII:

> Circles are to one another as the squares on their diameters.

One of the later great contributors to the lore of circle-squaring is John Wallis (1656) whose *Arithmetic Infinitorum*[7] is an extensive account—I think of it as almost a diary—of his attempt to, in effect, square the circle. Circle-squaring, of course, he did not do (or at least not in the manner in which the problem was usually framed) but his consolation prize was pretty marvelous. His Proposition 190 (in modern notation[8]) is the claim:

$$4/\pi \;=\; \frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{7}{8}\cdot\frac{9}{8}\cdot\frac{9}{10}\cdot\frac{11}{10}\cdot\frac{11}{12}\cdots$$

Wallis writes, immediately before Proposition 190:

> For, although here no small hope seemed to shine, nevertheless this slippery Proteus *(Wallis is talking about π)* whom we have in hand ... frequently escapes and disappoints hope.

Wallis knows well that he hasn't 'squared the circle' via the expected route. He writes:

> And indeed I'm inclined to believe (what from the beginning I suspected) that this ratio we seek *(Wallis is still talking about π)* is such that it cannot be forced out in numbers according to any method of notation so far accepted, not even by surds (of the kind implied by Van Schooten in connection with roots of certain cubic equations ... or in the thinking of Viète, Descartes and others) so that it seems necessary to introduce another method of explaining a ratio of this kind, than by true numbers or even by the accepted means of surds.

Along the way in this treatise, Wallis is molding a sort of ad hoc proto-Calculus designed to establish quadrature of more and more geometric figures, boasting that he has beaten the ancients at this game, and taking issue (in pretty brutal language) with the somewhat amateurish objections that Thomas Hobbes has about these proceedings.

Pressing the point made at the beginning of this session, that *mathematics is a long conversation*, let's pass to yet another vastly interesting would-be circle-squarer working almost *two millennia* before Wallis; namely:

4 Hippocrates of Chios

Hippocrates of Chios is believed to be the earliest Greek mathematician of whom we possess some actual pieces of text—although these pieces come to us, at least, fourth-hand; the relay chain here is

[7] Jaqueline Stedall's English translation, *The Arithmetic of Infinitesimal* was published by Springer (2004)

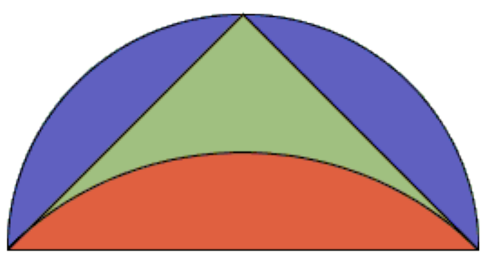
[8] Wallis's symbol for $4/\pi$ is $\square$ so—at the very least—he has brilliantly squared the circle in his notation.

Hippocrates → Eudemus → Simplicius → the near-contemporary historian Becker → us (via Netz).

What is actually available to us as text is even more difficult to describe; as with many ancient texts it seems to be something of a confection; see Reviel Netz's book *Eudemus of Rhodes, Hippocrates of Chios and the Earliest form of a Greek Mathematical Text*[9]. Nevertheless, what we have, is enough to paint—or at least suggest—a vivid picture of an early mathematician at work. (For people who want to take a look at the Greek text, I'll put it on my web page.)

Hippocrates effects a quadrature—in the strict sense that we described it—of certain lunes. Ancient commentators seem to construe Hippocrates work as a type of squaring the circle. Take a look at the Reviel Netz's translation of the text: pp-248-252 in loc.cit.

5 The starting lune



From the available text one is led to bet that the very starting point of Hippocrates investigations has to do with the observation that if you circumscribe a semicircle about an isosceles right-angle triangle—about a semi-square, so to speak—so that the hypotenuse of the triangle is the diameter of that semi-circle, an amusing give-and-take occurs.

In the colored figure above—that gives the circumscribed semicircle about an isosceles right-angle

[9] http://onlinelibrary.wiley.com/doi/10.1111/j 1600-0498.2004.00012.x/abstract.

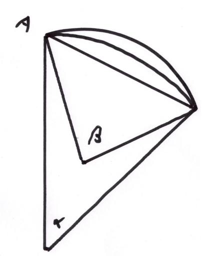
triangle—the only ‘colored region’ that remains to need explicit definition is the red region. This ‘red region’ is defined to have, as base line, the hypotenuse of the right-angle triangle, and to be similar to either of the two blue regions. This implies that the red area is simply *equal* to the sum of the two blue areas, thanks to our understanding the relationship between the diagonal of a square and its sides.

This implies that the *lune* which is the region enclosed by the two circular arcs has area *equal to* that of the green triangle. A perfect example of a ‘classical quadrature!’

But this is only Hippocrates’ first quadrature, about which he could have been exuberantly proud! Needless to say, true mathematician that he was, Hippocrates would have been keenly motivated to generalize this construction to encompass other lunes. This he did in at least two other ways[10]. Before describing his work any further, let us back up and ask:

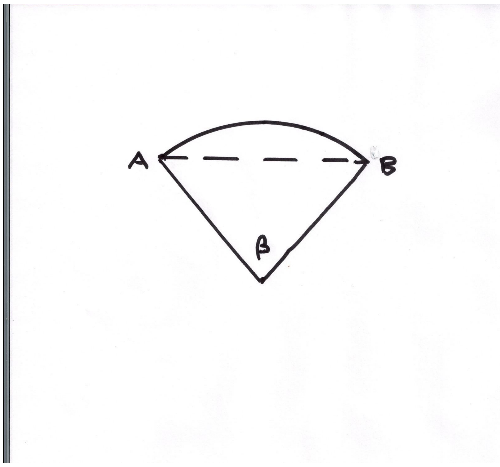
6 What is a lune?

A **lune** is given by two angles $0 < \alpha < \beta < \pi$, in circles of radius $r > s$ respectively, cutting out the same chord AB. We can denote a lune then by the data $\{\alpha, r; \beta, s\}$.



Now if we have an arc of a circle of radius s, cutting out an angle $0 < \beta < \pi$ from its center, subsuming a chord AB as in this figure below, the length of a chord AB is $2s\sin(\beta/2)$.

[10]although his overly enthusiastic later commentators—Simplicius might have been the origin of this misconception—claimed: “In this way Hippocrates squared *all* lunules, inasmuch as he squared the lunules having as their outer circumference: the semicircle, and the arc greater than a semicircle, and the arc smaller than a semicircle.” The *all* is my italics.



This puts the following relation on the data:

$$(\mathbf{1}) \qquad r\sin(\alpha/2) = s\sin(\beta/2).$$

To clear those denominators, we can write $\alpha = 2x, \beta = 2y$, so our relation is:

$$(\mathbf{1'}) \qquad r\sin(x) = s\sin(y).$$

Also we could, if we wish, normalize by scaling (r, s) by multiplying by the same positive number. Effectively, the lunes sit in a two-parameter moduli space given by the data

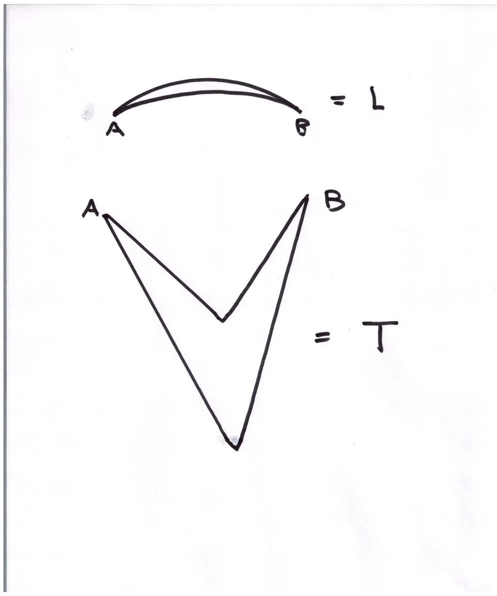
$$\{x, 1; y, \frac{\sin(x)}{\sin(y)}\}.$$

7 A curious mode of quadrature

The area of the entire pie-shaped wedge of the first figure is equal to αr^2, and the area of the second pie-shaped wedge subsuming angle α is equal to βs^2. The difference between these two areas, i.e.,

$$\alpha r^2 - \beta s^2$$

is simply $T - L$, where L is the area of the lune, and T the area of the polygonal region depicted in the first figure above and the one below:



SO, if (... if ...) you want to perform the quadrature gotten by simply by arranging L *to be equal to* T, you would look for pairs α, β such that

$$(\mathbf{2}) \quad \beta s^2 = \alpha r^2,$$

or

$$(\mathbf{2'}) \quad y s^2 = x r^2,$$

Putting ($\mathbf{1}$) and ($\mathbf{2}$) together you see that you are looking for pairs $\alpha = 2x, \beta = 2y$ that satisfy the single equation:

$$(\mathbf{3}) \quad sin^2(x)/x = sin^2(y)/y,$$

In the range we are working, the function $\sin^2(x)/x$ has a single maximum (at the value $x = \gamma$ where $\gamma \tan(\gamma) = 2$; this γ is near $\pi/2$) and $\sin^2(x)/x$ vanishes at 0 and π. So for *every* x, there is a unique y in the appropriate range such that equation **(3)** has a solution.

One might say, then, that *the moduli subspace of lunes that allow for this type of quadrature is one-dimensional; i.e., a curve in the full two-dimensional moduli space of lunes.*

For *every positive real number* n there is a *unique* pair $0 < x \leq y < \pi/2$ such that $n := \beta/\alpha = y/x$ and the above equation holds. We get a curious function:

$$N(x) := y/x$$

where y is the unique solution of our equation **(3)** in the appropriate range.

Setting $y = nx$, equation **(3)** is equivalent to:

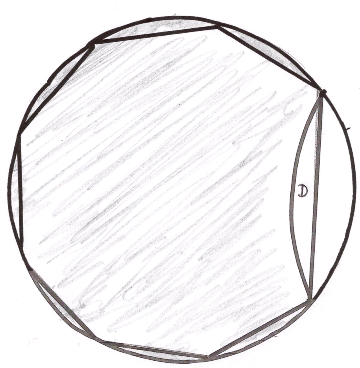
$$(\mathbf{3'}) \quad sin(nx) = \sqrt{n} \cdot sin(x).$$

But if we want to keep to the format of Hippocrates' quadratures, something deeper has to be posited; namely, that the angles α and β have to be rationally related.

It may be surprising that this *most simple curious quadrature* meshes with some of the *different* geometric quadratures related to polygons considered by Hippocrates himself, when x and y are solutions of**(3)** that have whole number (or near-whole-number) ratios (see Section 9 below).

8 Quadratures (vaguely) in the style of Hippocrates

In the special cases where, in fact, $N(x) = n \geq 2$ *is* an integer, we may construct a diagram generalizing some of the actual constructions of Hippocrates. Consider:



a polygon P of n *plus one* sides that has two properties:

1. it is inscribed in a circle,
2. all but one side (which I will call the "long side") have the same length.

For continuity reasons for each integer $n \geq 2$, there exists a continuum of such polygons (up to similarity equivalence). Let's denote these by P.

We now lay a circular segment D on the "long side" that is similar to the circular segments on any of the chords given by the short side of the polygon. This produces a lune. In the diagram above, $n = 7$ and the lune is the grayed area. When $n = 2$ and the 'long side' is a diameter, we get Hippocrates' "starting lune."

Question: For which of these lunes can we imagine a quadrature directly following the idea of Hippocrates' quadrature of his "Starting Lune?"

An answer: If the length of the "long side" is $\sqrt{n}$ times the length of any of the short sides. Call such polygons **polygons of Hippocrates**.

For then a directly analogous give-and-take, as we did with Hippocrates' starting lune, would express our lune L (that grayed-out area) as having the same area as the inscribed polygon P.

Of course you might ask: which of these construction can be actually constructed with ruler and compass? (*You* might ask that, and *Euclid* could ask that, but it isn't quite clear to me that Hippocrates actually did ask it, even though his construction might hint that way.)

But let's set up some vocabulary to talk about this question.

9 The Galois theory of Hippocrates

The natural modern way of describing an *n-plus one-gon of Hippocrates* is to imagine the polygon to be inscribed in the unit circle in the complex plane with the real axis as its axis of symmetry. For specificity (but for no other reason) restrict to the case where $n \geq 3$ is an *odd* integer. So then the vertices of the *n-gon of Hippocrates* will be given by complex numbers of the form $e^{\pm i(2a+1)\theta}$ for some unique $0 < \theta < \frac{1}{2}$ and where a ranges through integers $1 \leq a \leq k$, with $n = 2k+1$. Setting $\eta = e^{i\theta}$ we have that

- The n-gon of Hippocrates can be constructed entirely once you know η, and
- The algebraic number η is a solution to the following (more or less polynomial) relation:

  $$(\mathbf{4}) \quad X^n - X^{-n} = \sqrt{n} \cdot (X - X^{-1}).$$

  Avoiding the trivial solutions $X = \pm 1$ we have the equation:

  $$(\mathbf{4'}) \quad X^{n-1} + X^{n-3} + \cdots + X^{3-n} + X^{1-n} \;=\; \sqrt{n},$$

  so (if $n \geq 2$ is a rational integer) the roots are algebraic integers of degree $\leq 4n-4$.

As hinted above, except for notation, equation **(4)** is nothing more than equation **(3')** again; I'm not sure what to make of this.

Noting that this relationship **(4)** is invariant under the action of the (Klein) group of automorphisms

$$X \mapsto \pm X^{\pm 1}$$

and that $X = \pm 1$ are the only solutions of **(4)** that are fixed by nontrivial elements of that Klein group, we can clean up this problem by concentrating on the $4k$ solutions other than $\{\pm 1\}$ and viewing them as the union of k distinct orbits $\{\pm \eta_j^{\pm 1}\}$ for $j = 1, 2, \ldots, k$ where the η_j all lie in the "first quadrant" and where—say—$\eta_1 = \eta$ distinguishes itself among all the other η_j's by also lying on the unit circle (η is a kind of "*relative Salem number*"). It is this η that gives us the data that allows us to construct the n-gon of Hippocrates.

If you set $\nu_j := \eta_j^2 + \eta_j^{-2}$ you get that the collection $\{\nu_1, \nu_2, \ldots, \nu_k\}$ are the roots of a polynomial of degree k with coefficients in the field $F := \mathbf{Q}[\sqrt{n}]$. It might be fun to look at the Galois theory of this field, but even without doing this, set $F_1 = \mathbf{Q}[\sqrt{n}, \nu_1]$ and $L := \mathbf{Q}[\sqrt{n}, \eta = \eta_1]$. we see that F_1/F is of degree $\leq k$ and L/F_1 is of degree dividing 4.

Well, if $k = 1$ or 2 it follows that $L/\mathbf{Q}$ is a field extension obtainable by successive extraction of square roots, and therefore the 3-gon and the 5-gon of Hippocrates are each constructible (in the manner of Euclid) via compass and straight edge. One could then go on to "square" the lunes (of Hippocrates) that are built from the 3-gon and the 5-gon of Hippocrates as we've described above.

Question: What happens, for example if $n = 7$ or higher? Are there more examples of *constructible* quadratures coming from polygons of Hippocrates? What are the Galois groups (over $\mathbf{Q}$) of the polynomials given by (**4**)?

Note: there is some literature that relates to this—all in slightly less modern terms—but none of the results I've seen are at all definitive!

These questions might repay study; if we do this, we—as modern mathematicians—would then be in something of a 'conversation' with the work of a mathematician who lived two millennia earlier than us, regarding issues of significant, and certainly interesting, substance.