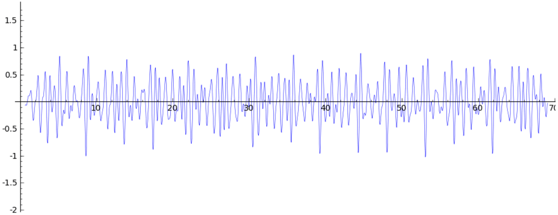
How Explicit is the Explicit Formula?

my co-author William Stein and I asked

To play with this, it occurred to us to experiment with the **Oscillatory term** in the *Explicit Formula* of analytic number theory:



The Explicit Formula:

$$\text{LHS} = \text{RHS}$$

LHS

LHS is a function of a variable X that is a sum of local (interesting arithmetic) data at each prime p, this being summed for all $p < X$.

For example, consider the question: **how often does the equation**

$$E \quad y^2 + y = x^3 - x$$

have *more than* p **solutions mod** p**, and how often** *less***?**

Build a 'sum of local data' that reflects this question:

Let $N_E(p)$ be the number of solutions mod p, and you could fashion a *raw measure*:

$$\Delta_E(X) :=$$

$$= \frac{\log X}{\sqrt{X}} \big(\#\{p < X \mid N_E(p) > p\} \; - \; \#\{p < X \mid N_E(p) < p\} \big),$$

Or, you could try to get to the same question via a smoother "sum of local data":

$$D_E(X) :=$$

$$= \frac{1}{\log\ X} \sum_{p \leq X} \frac{(N_E(p) - p) \log p}{p}.$$

(Conjecturally: the mean of $D_E(X)$ is the Mordell-Weil rank of E.)

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What are the RHSs of the Explicit Formulas?

Nothing more than a fancy Fourier analysis of these sums of local data, as functions of X.

The RHS of the Explicit Formula Is a fancy *Fourier analysis* of $D_E(X)$.

It is an infinite sum of functions

$$\sum_{\sigma+i\gamma} f_{\sigma+i\gamma}(X),$$

each summand, $f_{\sigma+i\gamma}(X)$ being attached to a point $\sigma + i\gamma \in \mathbf{C}$ in the "fancy Fourier spectrum" of $D_E(X)$.

Riemann et al: This *fancy Fourier spectrum* is, miraculously discrete (!) and can be expressed as the zeroes of an appropriate L-function.

These “zeroes” come in three distinct packages:

(I) A possible “central zero” at the complex number

$$\sigma + i\gamma \; = \; 1 + i \cdot 0 \; = \; 1.$$

The associated summand attached to this point of the spectrum,

$$f_{\sigma+i\gamma}(X) = f_1(X)$$

is a constant (as “function” of X) and conjecturally equal to:

$$r_E := \textit{the Mordell-Weil rank of } E$$

.

Call r_E the **global signal.**

(II) The "trivial zeros" (a discrete orderly set of real zeroes $\sigma_\nu + i \cdot 0$ for $\nu = 1, 2, 3, \dots$).

They contribute to what we'll call the **easy error signal**

$$\epsilon_E(X) = O(1/\log X).$$

Its *mean* is 0.

(III) The "nontrivial (complex) zeros" (a far less orderly set of zeroes— *conjecturally!* lying on a vertical line $1 + i \cdot \gamma_\nu$ for $\nu = 1, 2, 3, \dots$).

Call their contribution the **oscillatory signal.**

$$S(X) = \frac{1}{\log X} \sum_\nu \frac{X^{i \cdot \gamma_\nu}}{i \cdot \gamma_\nu}.$$

Its convergence is slightly problematic—and somewhat tricky to graph numerically. Conjecturally, its *mean* is 0.

An inexplicit view of the Explicit Formula

(as a sum of three different kinds of 'signals')

$$D_E(X) \; = \; r_E \; + \epsilon_E(X) \; + \text{oscillatory signal}$$

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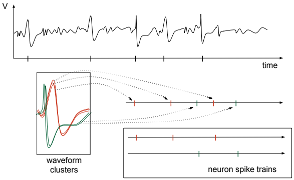
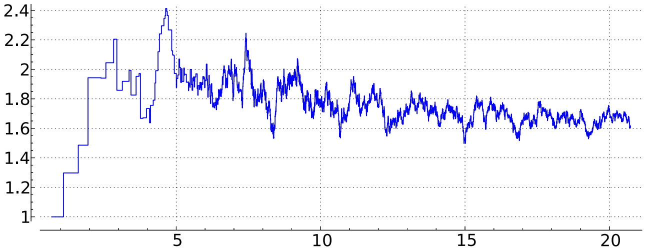
The spike-sorting problem in neuroscience

- Electrodes in the brain record the combined signal of multiple neurons
- Different neurons have different characteristic waveforms
- Spike-sorting consists of clustering the recorded spikes to retrieve individual neuron signals



(Neuro slide thanks to Sonia Todorova)

Here's a graph of $D_E(X)$ for $E = 389a$ whose $r_E = 2$.



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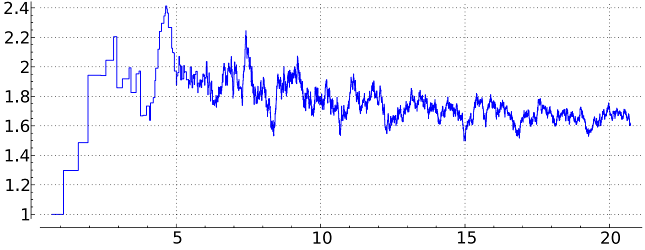
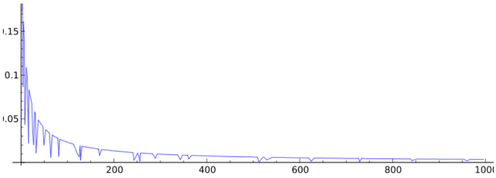


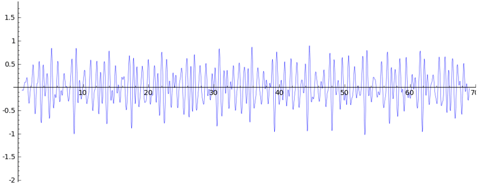
and here's a typical $\epsilon_E(X)$:



And here again is an example of the oscillatory term

$$S(X)/\log X = \frac{1}{\log X} \lim_{T\to\infty} \sum_{|\gamma \leq T|} \frac{X^{i\cdot\gamma}}{i\cdot\gamma} = \frac{1}{\log X} \sum_{\nu}^{\infty} \frac{X^{i\cdot\gamma_\nu}}{i\cdot\gamma_\nu}.$$

$S(X)$:



Suggested conjecture— in a letter by Sarnak

$$\lim_{X\to\infty} S(X)/\log X \stackrel{??}{=} 0$$

This is worth exploring!

- for the pure joy of numerical exploration (since it is tricky to compute),
- to search for a convincing guess of “the” quantitative rate of convergence to 0 (since this would also offer (conjectural) upper bounds for length-of-computation of analytic rank),
- to estimate the size of the peaks of $S(X)$ (since they are surprisingly small: < 3.0 as far as our computations go).

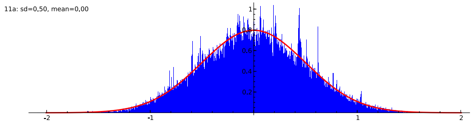
A probabilistic—somewhat Bayesian—analysis of error terms:

The oscillatory function $S(X)$ is (provisionally) conjectured to be $o(\log X)$

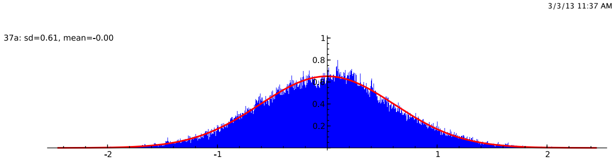
...but—most of the time as a measured by multiplicative measure $dX/\log X$—it seems to have quite small values.

So, form the distribution $\mu = \mu_E$ whose integral over any interval I gives the probability over all positive arguments X that $S(X)$ achieves a value in I.

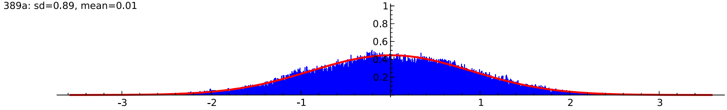
$E = 11a$:


11a: sd=0.50, mean=0.00
1
0.6
0.4
0.2
-2
-1
1
2

$E = 37a$:


3/3/13 11:37 AM
37a: sd=0.61, mean=-0.00
1
0.8
0.4
0.2
-2
-1
1
2

Question: Is μ_E a *normal* distribution (with mean 0)?



Definition: The **bite**, β_E, of the oscillatory term $S_E(X)$ is the standard deviation of the (conjecturally normal) distribution μ_E of values of $S_E(X)$.

Open-ended Problem: How is the *bite function*

$$E \to \beta_E$$

related to any of the other (more standard) invariants of E such as its conductor, analytic rank, etc.?

And as you can see, for my co-author and me,

this is *work in progress!*