How explicit is the Explicit Formula?

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March 21, 2013

(*Notes for our 20+20 minute talk at the AMS Special Session on Arithmetic Statistics in San Diego, January 2013*)

Preface

Any 'Explicit Formula' in analytic number theory deals with an *arithmetically interesting quantity*, often given as a partial sum $F(X) \cdot \sum_{p<X} G(p)$ of locally defined quantities $G(p)$ attached to prime numbers p, summed up to some cutoff value, $p < X$ and normalized, for convenience, by an elementary factor $F(X)$. There is a distinction to be made between such sums taken over *primes* and similar sums, but over *prime powers*. As is well known, in the classical study of the placement of prime numbers, a somewhat smoother 'sum of local data' is gotten if one passes from consideration of summation over the primes $\leq X$, such as in

$$\pi(X) := \sum_{p \leq X} 1$$

to summation over powers of primes $\leq X$, as in

$$\psi(X) := \sum_{n \leq X} \Lambda(n)$$

where $\Lambda(n)$ is the von Mangoldt function (defined to be $\log p$ if $n = p^e$ is a power of the prime number p with $e \geq 1$, and defined to vanish otherwise). We'll mention this again in the appendices, but in the main body of the text, we keep to the problems posed by partial sums $F(X) \cdot \sum_{p<X} G(p)$ of locally defined quantities $G(p)$ attached to prime numbers p.

The Explicit Formula expresses such function of X as a *dominant term*, plus an easily controlled error term, plus an interesting third term that might be called the *oscillatory term* that, in most cases, is only conjecturally controllable.

Usually the *dominant term* is computed by knowing the order of specific zeroes 'at central points' of relevant (global) L-functions, the 'easy error term' is related to the so-called *trivial zeroes* of those relevant L-functions, while the *oscillatory term* is a specific function of (X, and of) the infinitely many remaining nontrivial zeroes of those L-functions.

There are theoretical and computational challenges in working out the numerical contributions of these terms of the formula in concrete cases. We have no new results here, but our aim in this half hour plus ten minutes of discussion, is to offer numerical *visualizations* of the analytic formula in various interesting cases to advertise the need for some precise conjectures and computational projects regarding this problem and to recount some recent work. More for the future will be our plans for a web-accessible resource: a repository of some of the numerics for the cases related to elliptic curves that interest us.

Here we will focus on *issues of bias* following the classical 'Explicit Formula," and the work of: Sarnak, Granville, Rubenstein, Martin-Watkins, Fiorilli, Bober, Conrey-Snaith, and others. The example-problem we consider is related to the question—given an elliptic curve over the rational numbers and letting p range through prime numbers—of how often $p+1$ is an *over-count* or an *under-count* for the number of rational points on the curve modulo p? The rough answer is 50/50, but there can be a 'bias' similar to the classical Chebyshev bias [13]. For such finer statistics one resorts to 'Explicit Formulas.' Here, computation can even outstrip theory in that people have algorithms to make such computations whether or not the holomorphicity of the L-functions in question have been proved. Computations have depended on the work of many people, notably Mike Rubenstein's data base: `http://oto.math.uwaterloo.ca/~mrubinst/L_function_public/L.html`.

We have used these notes to provide material for *unusually short lectures*. Specifically, the notes were initially written for each of our 20 minute lectures at the San Diego Conference, as we have already mentioned, but also one of us replayed it all[1] in a 6 minute and 40 second lecture at Harvard[2]. We say this to make it clear that this article is meant to be somewhat light reading; it contains no new results, as we have mentioned, and no proofs of anything. But we feel that the issues here are a good source of student projects, and that our text might be a companion to any reader of the beautiful expository articles on classical Chebyshev bias, such as [13], or on the biases we discuss, as in [14].

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[1] Well, not quite all...

[2] This was part of the event sponsored by the complex dynamics seminar at Harvard University on 3/6/13: *PechaKucha: Mathematics, 20x20* Lectures by B. Gross, O. Knill, S. Koch, B. Mazur, C. McMullen, E. Riehl. For the meaning of PechaKucha, see http://www.math.harvard.edu/ ctm/sem/

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1 Brief Introduction

One of us (B.M.) having recently taught the classical *Explicit Formula* in a standard graduate course in analytic number theory, and having proved that eponymous formula, garnished—as it usually is— by a number of so-called "effective constants," c_1, c_2, etc., felt that, for some applications, this lettered effectiveness left us still too far from the statistical phenomena behind the formula. To get a closer bead on things, one would do well to work with the numbers behind these c_1's and c_2's, etc. It is natural then to see how 'actual data'—cutting off the terms of the formula at suitably large values of X given the range of currently feasible computations—compare with the expected results for arbitrarily large X.

Happily, the other of us (W.S.) has produced relevant computations that do exactly that (for applications of the Explicit Formula to certain problems of current interest to both of us). This, then, is a phenomenological talk, with (at least the beginning of) a corpus of graphs that offer some illustration of the effectiveness—or non-effectiveness—of the explicit formula in the specific instances of interest to us.

We offer no new theoretical results but use this occasion to mention some interesting recent work and conjectures (of other people) that might warrant more such computations and that raise a host of questions, both theoretical and computational. For example, to do some systematic numerical computations related to an elliptic curve E attached to a newform f_E (along the lines of what has already been done in this paper) it would be very useful to have a much larger data-set of the arithmetic function

$$n \mapsto r_E(n)$$

where $r_E(n)$ is the order of vanishing of the L-function of the automorphic forms $symm^n f_E$ for odd values of n. Regarding this arithmetic function, aside from having control of the parity of $r_E(n)$ (e.g., see [3])) hardly anything else is known. Nor do we (at least, the authors of this paper) yet have enough experience—when E has no complex multiplication—even to formulate a proper conjecture.

We might also mention that when making these numerical experiments one seems to be in a situation that is not entirely dissimilar from the type of slightly annoying mismatch between conjecture and data that one encounters in more traditional studies of Mordell-Weil statistics that was the subject of the survey article B-M-S-W. But this may be unavoidable, given that even the so-called 'easy error term' in the explicit formula will tend to converge only $O(1/\log X)$ fast.

We should say at the outset that for simplicity, and sometimes for necessity, we'll be assuming GRH throughout—without any further mention. In fact, at times we'll also be assuming (*with* explicit warning) some further conjectures.

2 A qualitative look at the Explicit Formula

As mentioned in the Preface, here is the shape of the explicit formula, given in even more qualitative vocabulary:

$$\textit{Sum of local data} \;=\; \textit{Global data} + \textit{Easy error term} + \textit{Oscillatory term}.$$

Before getting started, some general comments. We will be dealing with examples where each of these terms are given as functions of X, a cutoff value, where

- We want the term on the LHS, the "*Sum of local data* cut off at X" to be a finite sum of the form:

  $$\delta(X) \;=\; F(X) \cdot \sum_{p \le X} G(p)$$

  where the rules of the game (in this paper) are as follows:

  - We require the value $G(p)$ to be determined by only *local* considerations at the prime p. The simplest example of such a *Sum of local data* is given by taking $F(X)$ to be the constant 1, and $G(p)$ to be 1 for all p, giving us the classical $\pi(X) := \sum_{p \le X} 1$.
  - The normalizing factors $F(X)$ will be elementary smooth functions of the cutoff X. We sometimes choose this normalizing factor so that the resulting $\delta(X)$ has (conjecturally) finite mean.

  We will be concentrating on *sums of local data* attached to elliptic curves over $\mathbf{Q}$,

  $$\delta_E(X) := F(X) \sum_{p \le X} g_E(p)$$

  where the weighting function

  $$p \mapsto g_E(p)$$

  is a function of $a_E(p)$, the p-th Fourier coefficient of the eigenform of weight two parametrizing the elliptic curve.

  We will specifically be interested in issues of bias. This is what we mean: thanks to the recent resolution of the Sato-Tate Conjecture in this context, one knows that—roughly—half the Fourier coefficients $a_E(p)$ are positive and half negative. Indeed, the numbers of positive values and negative values look very close:

Curve	Rank	Negative $a_E(p)$ for $p < 10^9$	Positive $a_E(p)$ for $p < 10^9$	Difference
11a	0	25422268	25423101	-833
14a	0	25422229	25421074	1155
128b	0	25420641	25425608	-4967
816b	0	25424848	25421229	3619
2379b	0	25417900	25427007	-9107
5423a	0	25420479	25425242	-4763
29862s	0	25420525	25425197	-4672
37a	1	25423396	25422448	948
43a	1	25421536	25424196	-2660
160a	1	25424446	25421488	2958
192a	1	25418843	25426859	-8016
2340i	1	25425512	25419660	5852
10336d	1	25421245	25423628	-2383
389a	2	25427014	25418738	8276
433a	2	25425902	25419896	6006
2432d	2	25423818	25421900	1918
3776h	2	25422350	25422750	-400
5077a	3	25426985	25418831	8154
11197a	3	25429098	25416702	12396

To study, then, the weighted sums that directly reflect finer statistical issues related to this symmetric distribution, we will be concentrating on weighting functions $p \mapsto g_E(p)$ that have the property that

- for all primes p, $g_E(p)$ is an *odd* function of the value $a_E(p)$, and
- the *sum of local data*

$$\delta_E(X) := \sum_{p \le X} g_E(p)$$

  has—or can be convincingly conjectured to have—a finite mean[3] relative to multiplicative measure dX/X.

In such a context the mean of $\delta_E(X)$ can be interpreted as a *bias*!

For example, to consider the problem highlighted in our *Preface* (above) form the 'sum of local data'

$$\frac{\log X}{\sqrt{X}} \sum_{p \le X} \gamma_E(p)$$

where $\gamma_E(p) = 0$ if p is a bad or supersingular prime for E and is otherwise is $+1$ if E has less that $p+1$ rational points over $\mathbf{F}_p$; and $\gamma_E(p) = -1$ if more. Then this sum, which will be denoted $\Delta_E(X)$ below, measures exactly the difference between over-count and under-count, as formulated in the *Preface*.

[3]See Section 6.1 below

3 The three terms on the RHS of these Explicit Formulas

- The first term on the RHS of our Explicit Formulas, i.e. the term we labeled *Global data*, is a constant independent of X determined by the certain 'central' (real) zeros, or the poles, of the relevant L-function or collection of L-functions. The collection of L-functions *relevant* to the problem highlighted in our *Preface* consists of the L functions attached to *all* the *odd* symmetric powers of the newform f_E, and the Global data is a function of the multiplicity of the zeroes of these L-functions at their central (real) points, where central refers to the functional equation that they enjoy.

  Often, and under GRH, these *real, central* zeroes of the relevant L-functions will have—conjecturally—a clean *global arithmetic interpretation* (e.g., via BSD and its variants) so that's why we call that term simply "*Global data*." In the problems we will be discussing this "global data" will be showing up as a certain *bias* in the arithmetic statistics of elliptic curves that hearkens back to the early work of Birch and Swinnerton-Dyer, but in the context of the vocabulary we will be using, was first written down by Peter Sarnak; this is in the spirit of the classical Chebyschev bias, and 'prime races;' an 'Explicit Formula' account of this classical theory can be found in [10].

- Often the 'easy error term' converges (like $O(1/logX)$) to a value (perhaps zero) as X tends to infinity.

- The *Oscillatory term* is determined by the (infinitely many) complex ("nontrivial") zeroes. It is usually an infinite sum, where (after appropriate conjectures) the summands are of the form $X^{i\gamma}/f(\gamma)$ where γ runs through the imaginary parts of the complex zeroes of the relevant L-functions, and $f(y)$ is some natural function. Numerically, this oscillatory term will indeed oscillate—as we shall amply see—but often one is tempted to, at the very least, conjecture some control over this wild card. In actual computations we are surprised by how small it is.

Part I: Setting up

4 Bias Questions

Let E be an elliptic curve over $\mathbf{Q}$ with no complex multiplication, associated to a newform whose p-th Fourier coefficient for p a prime is denoted, as usual, $a_E(p)$. Given the recent work on Sato-Tate, the probability distribution determined by the normalized values $\frac{a_E(p)}{2\sqrt{p}}$ is known to be symmetric about the origin for a large class of such elliptic curves. To repeat our starting question:

> Given an elliptic curve over the rational numbers, and letting p range through prime numbers, how often is $p+1$ an over-count or an under-count for the number of rational points on the curve modulo p?

As mentioned above, for a large class of elliptic curves, as a consequence of recent work on the Sato-Tate Conjecture, the answer is grossly *equally often* in the sense that, putting

$$N_E(p) = 1 + p - a_E(p) := \text{ the number of rational points on } E \text{ over } \mathbf{F}_p,$$

the ratio

$$\frac{\#\{p < X \mid N_E(p) < p+1\}}{\#\{p > X \mid N_E(p) > p+1\}} \;=\; \frac{\#\{p < X \mid a_E(p) > 0\}}{\#\{p > X \mid a_E(p) < 0\}}$$

tends to 1 as X goes to infinity, and we will be considering more delicate *bias questions* by examining a variety of "rough," and "smooth," ways of measuring the preponderance of positive—or of negative—$a_E(p)$'s. We wish to actually make such measurements, and take a look at their graphs.

This type of question, of course, bears on Birch's and Swinnerton-Dyer's initial "hunch" that the statistical preponderance of solutions modulo p of an elliptic curve is a predictor of whether or not the elliptic curve has infinitely many rational points.

5 The LHS of our Explicit Formulas

To give some ad hoc terms for variant partial sums of *Local Arithmetic Data* that measure such preponderances, let us refer to

- (the slightly doctored version of) the straight difference,

  $$\Delta_E(X) := \frac{\log\ X}{\sqrt{X}}\big(\#\{p < X \mid a_E(p) > 0\} \;-\; \#\{p < X \mid a_E(p) < 0\}\big),$$

  as the **raw data,**

- and to

  $$\mathcal{D}_E(X) := \frac{\log\ X}{\sqrt{X}} \sum_{p \le X} \frac{a_E(p)}{\sqrt{p}}$$

  as the **medium-rare data**, and

- $$D_E(X) := \frac{1}{\log\ X} \sum_{p \le X} \frac{a_E(p)\log p}{p}$$

  as the **well-done data**.

5.1 The statistical distinctions between the three formats

Not to build up too much suspense here, the reason for selecting these three formats for the "Local data" and for the specific normalizations chosen (i.e., the factor $\frac{\log X}{\sqrt{X}}$ occurring in the first two, and the factor $\frac{1}{\log X}$ in the third) is that they each are amenable to analysis via "an" *Explicit Formula*

$$(*) \quad \textit{Sum of local data} \;=\; \textit{Global data} \;+\; \textit{Easy error term} \;+\; \textit{Oscillatory term}$$

and such that if (GRH plus) certain interesting conjectures hold—then all three Sums of Local Data,

$$\Delta_E(X),\ \mathcal{D}_E(X),\ \text{and}\ D_E(X)$$

will have finite *means* (relative to the measure dX/X on $\mathbf{R}^+$), their 'means' being equal to the term *Global data* in their corresponding Explicit Formula; and furthermore, what distinguishes these three formats is that conjecturally[4]—

- the raw data will have *infinite* variance,
- the medium-rare data will have *finite variance*, and
- the well-done data will actually achieve its mean as a limiting value.

For a picture gallery of graphs of these Sums of Local Data, see Part III below. For a more extensive data base of such pictures, see ****

6 The RHS of our Explicit Formulas

Here are some brief comments on each of the 'terms' on the RHS of the Explicit Formula for our three variants, where we write that RHS of–for example–the well-done variant above for an elliptic curve E as given below:

$$D_E(X) := \frac{1}{\log X}\sum_{p\le X}\frac{a_E(p)\log p}{p} \qquad = \qquad \rho_E \quad + \quad \epsilon_E(X) \quad + \quad \frac{1}{\log X}S_E(X).$$

For the definition of $S_E(X)$, see section 7 below.

6.1 The '*Global Data*' or—conjecturally– the *'Mean'*

Recall that if $X \mapsto \delta(X)$ is a (continuous) function of a real variable, to say that $\delta(X)$ **possesses a limiting distribution** μ_δ **with respect to the multiplicative measure** dx/x means that

[4]as described in a letter of Sarnak; see subsection?? below.

for continuous bounded functions f on $\mathbf{R}$ we have:

$$\lim_{X\to\infty} \frac{1}{\log X}\int_0^X f(\delta(x))dx/x \;=\; \int_{\mathbf{R}} f(x)d\mu_\delta(x).$$

Recall that the **mean** of the function $\delta(X)$ (relative to dX/X) is defined by the limit

$$\mathcal{E}(\delta) := \lim_{X\to\infty} \frac{1}{\log X}\int_0^X \delta(x)dx/x \;=\; \int_{\mathbf{R}} d\mu_\delta(x).$$

The depressing thing here is that if you take a function $\delta(X)$ that is anything you want up to $X = 4,000,000$ and equal to 5 for $X > 4,000,000$ then the mean of δ is equal to 5, so what in the world can it mean[5] to compute data up to $4,000,000$? But we press on.

The standard conjectures for the terms in our three formats above tell us that—in all three of our examples—the values of *means* are given by the 'global data.' In particular, for the well-done variant, the mean is conjectured to be ρ_E. More specifically:

- **The well-done data:** the mean is (conjecturally) $\rho_E = -r_E$ where r_E is the *analytic rank* of E.
- **The medium-rare data:** the mean is (conjecturally) $1 - 2r_E$ and
- **The raw data:** the mean is (conjecturally)

  $$\frac{2}{\pi} - \frac{16}{3\pi}r_E \quad + \quad \frac{4}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\Big[\frac{1}{2k+1} + \frac{1}{2k+3}\Big]r_E(2k+1).$$

  where

  $$r_E(n) := \; r_{f_E}(n) \;=\; \text{the order of vanishing of } L(symm^n f_E, s) \text{ at } s = 1/2,$$

  with f_E := the newform of weight two corresponding to the elliptic curve E; and where we have normalized things as the analysts love to do, so that $s = 1/2$ is the central point. **NOTE:** For a discussion of the numerics of the values $r_E(2k+1)$, see Section **??** below.

6.2 The '*Easy error term*'

Let us leave any analysis of this term, $\epsilon_E(X)$, as an interesting student-project.

Project 6.1. Work out theoretically in general, and computationally for a few specific elliptic curves, the nature of the easy error term $\epsilon_E(X) = O(1/\log X)$ and estimate the explicit constants.

[5] poor pun intended

7 The '*Oscillatory term*'

Although this oscillatory sum is similar for all three formats, here let us concentrate on this term as it appears in the Explicit Formula for the 'well-done data,' $D_E(X)$. We write it as $\frac{1}{\log X} S(X)$ where $S(X) = S_E(X)$ is the limit, as T tends to infinity, of the trigonometric series:

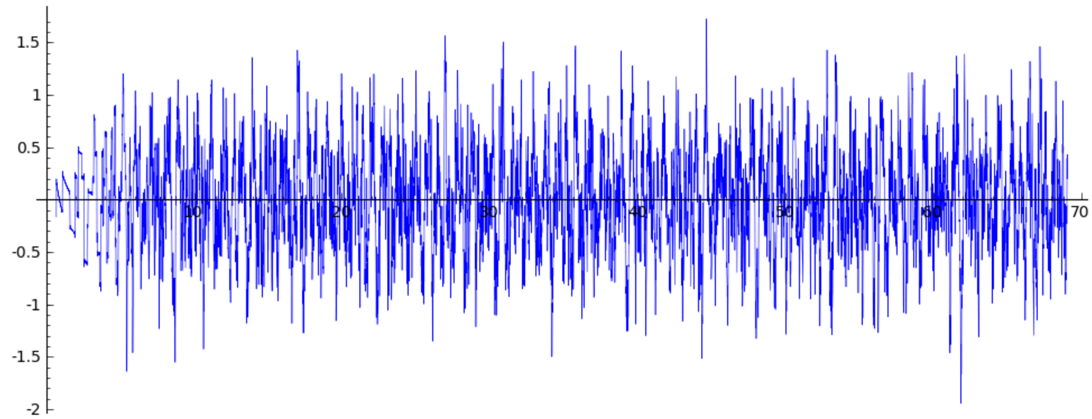
$$S_E(X,T) = \sum_{0<|\gamma|\leq T} \frac{X^{i\gamma}}{i\gamma},$$

where the sum is over the imaginary parts of the complex zeroes of $L(f_E, s)$ at $s = 1/2$. It is a consequence of the explicit formula that $S_E(X)$ does (conditionally) converge and it has been tentatively suggested (e.g., see [[14]]) that

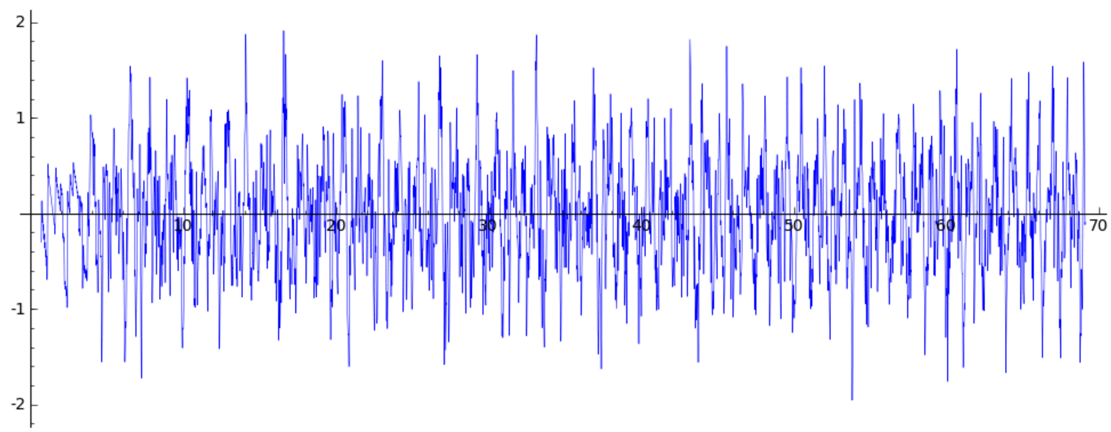
Conjecture 7.1. $S_E(X) \stackrel{?}{=} o(\log X)$.

The analogous oscillatory term for the classical Riemann zeta function has an extensive literature. See, for example [9], [**?**] and the bibliography there. Here are some pictures to convey a sense of how our $S_E(X,T)$ behaves, at least in the currently computable range which (roughly) allows T to be only as high as 10^4.

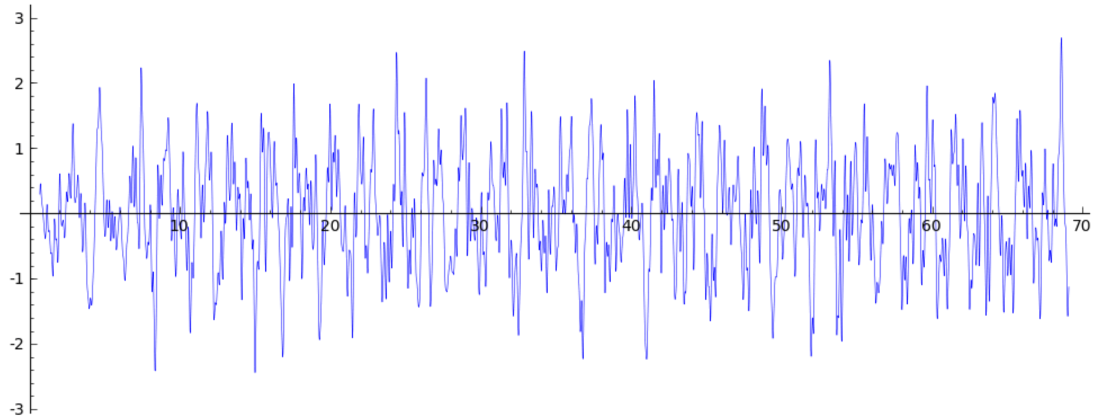
$E = 11a$



$E = 37a$



$E = 389a$



From these examples, one might imagine that for 'most arguments X' the range of actually achieved values of $S_E(X)$ may be even more restricted than Sarnak's suggestion, i.e., $o(\log X)$ as quoted above. That is, even if the function $X \mapsto S_E(X)$ is in fact unbounded, it might be the case that it spends most of its time having a very restricted upper bound for its values. To study this, let us consider the distribution of values of $S_E(X,T)$ for any fixed (X,T) with T large.

7.1 Distributions of values

Let $\mathbf{R}_{>0}$ be the multiplicative group of positive real numbers, and $\mathbf{R}$ the additive group of reals. For $I \subset \mathbf{R}_{>0}$ a Haar measurable set, let $|I|$ denote a Haar measure. Let $\mathcal{S} : \mathbf{R}_{>0} \to \mathbf{R}$ be a real-valued Lebesgue-integrable function. Fixing $I \subset \mathbf{R}_{>0}$ a subset of finite measure, for every measurable subset $J \subset \mathbf{R}$, form the probability measure on $\mathbf{R}$

$$J \mapsto \mu_{\mathcal{S},I}(J) := \frac{|I \cap \mathcal{S}^{-1}(J)|}{|I|}.$$

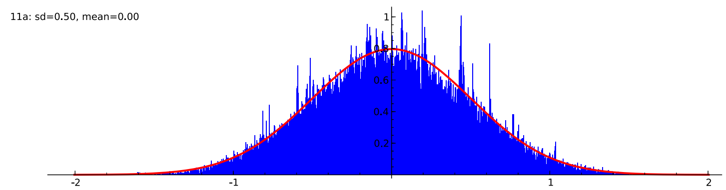
So, $\mu_{\mathcal{S},I}(J)$ is the *probability that the function $\mathcal{S}$ achieves a value in the range J over the gamut of arguments in I.* Say that $\mathcal{S}$ has a **normal distribution of values** if, for $X > 0$ setting $I_X = (0, X]$, the limit

$$\mu_{\mathcal{S}} := \lim_{X \to \infty} \mu_{\mathcal{S},I_X}$$

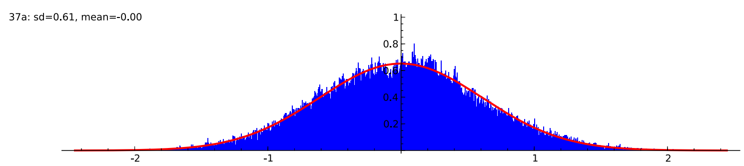
exists. These definitions are particularly relevant to the oscillatory terms $\mathcal{S}(X) := S_E(X)$ that we are currently studying. The data seems to indicate convergence to a limiting distribution (the mean value being 0) with a strikingly small (variance, or equivalently: strikingly small) standard deviation of values.

Here, then, are some pictures of what seems to be data 'converging' to a limiting distribution μ_E of the values of the oscillatory terms $S_E(X)$ for a few elliptic curves E:

$$E = 11a$$



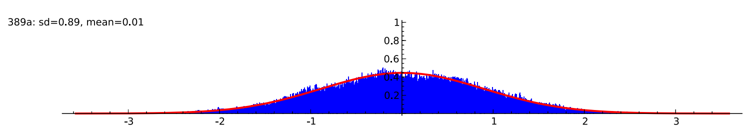
$$E = 37a$$



The red curve is the normal distribution with mean 0 and standard deviation given by that of the data.

Note: Conditional on the conjecture $LI(E)$ (see section 12 below) μ_E exists (see 14).

It is interesting to compare μ_E to the limiting distributions connected to the bias of nonresidues to residues mod q, as in [13]. There one has the added feature that these limiting distributions themselves tend to the normal distribution as the modulus q tends to infinity.



Definition: The **bite**, β_E, of the oscillatory term $S_E(X)$ is the standard deviation of the distribution μ_E of values of $S_E(X)$.

Note: For discussion of variance of distributions related to the medium-rare data, see [14]. In view of that discussion and parallel comments in [13], it is tempting to think of rescaling our measures $\mu_E(y)$ substituting $y \cdot \log \mathrm{cond}(E)$ for y in hopes of getting a convergent 'rescaled bite' as $\mathrm{cond}(E)$ tends to infinity, and asking whether (after such a rescaling) these distributions converge to the normal distribution.

Here are a few examples comparing the bite to the conductor. We also compare this data this to the quotient

$$\lambda_E := \frac{\log \text{cond}(E)}{\beta_E},$$

and to the Mordell-Weil rank r_E:

E	11a	37a	389a	431b1	443c1	5002c1	5021a1	5077a
$\beta_E \approx$	0.5	0.61	0.89	1.38	1.40	1.57	1.94	1.19
$\lambda_E \approx$	4.8	5.9	6.7	4.3	4.4	5.4	4.4	7.1
$r_E =$	0	1	2	0	0	0	0	3

Project 7.2. Continue the computations above to be able to get good approximations to the absolute constant c.

But there is a finer structure to the behavior of the oscillatory term. For that, one must zoom in and focus attention to the values of X that are close to powers of prime numbers. We will now do that.

7.2 The Gibbs Phenomenon in the oscillatory term

The Explicit Formula for $D_E(X)$ tells us that we might well expect discontinuities of the function $S_E(X)$ for prime number values of X. The analogous question has been examined in the case of the classical Riemann zeta-function. Here is a brief resumé of information one finds about this in the literature. Let

$$S_{\text{Riemann}}(X) := \sum_{|\gamma|<X} X^{i\gamma}/i\gamma,$$

where γ ranges through the nontrivial zeroes of $\zeta(s)$, the Riemann zeta-function. This oscillatory term has been embedded in what one might call a 'Lerch spectral zeta function,' defined by the Dirichlet series:

$$Z_{\text{Riemann}}(X, s) := \sum_{|\gamma|<X} X^{i\gamma}/i\gamma^s,$$

where again γ ranges through the nontrivial zeroes of the Riemann zeta function. For fixed $X \geq 0$ the function $Z_{\text{Riemann}}(X, s)$ extends to a meromorphic function of s on the complex plane, and for $X > 0$ it is entire[6]. The special case of $Z_{\text{Riemann}}(1, s)$ fits into the immense literature regarding 'spectral zeta-functions,' that extends to asymptotic distributions of eigenvalues for oscillating membranes, and to Zeta-functions of Laplacians [7]. As for the Gibbs phenomenon, Theorem 3 of [7] offers the following jump-discontinuity analysis (in the variable X) for the analytic continuation of the Dirichlet series $Z_{\text{Riemann}}(X, s)$ at real points $0 < s = \sigma < 1$.

[6] See [7] for the latter statement, and [5], [6] for its proof.

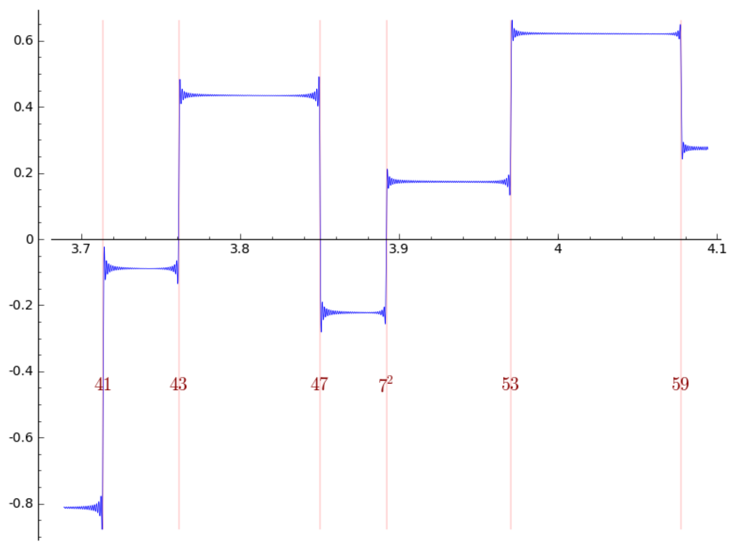
[7] For this, see [17] and its bibliography, which *remains* a useful, and delightful, thing to read!

$$\lim_{X \to p^k \pm 0} Z_{\text{Riemann}}(X,\sigma)|\log X - \log(p^k)|^{1-\sigma} \;=\; \mp \frac{\log p}{2\pi p^{k/2}} \int_0^\infty \frac{\sin}{t} t^\sigma dt.$$

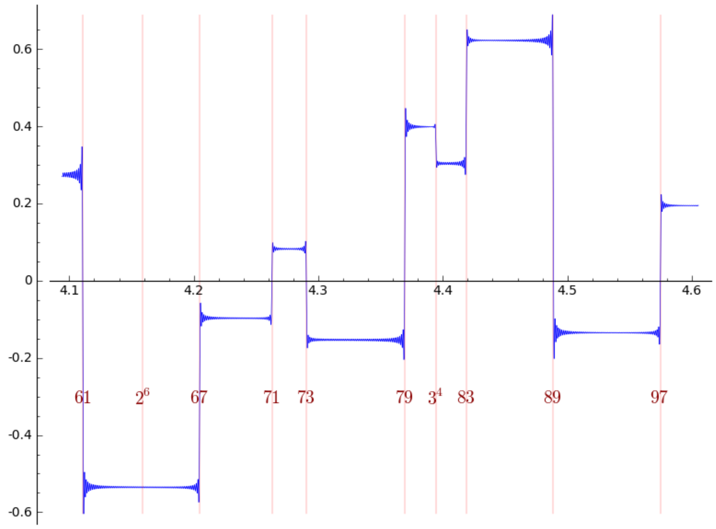
Project 7.3. Rework this theory to cover the case of $S_E(X)$.

Here is a small picture exhibition of the Gibbs phenomenon for our oscillatory terms $S_E(X)$ when roughly 10^4 terms are used. It is striking how roughly linear these oscillatory term appear, around—of course—the discrete jumps at powers of prime numbers.

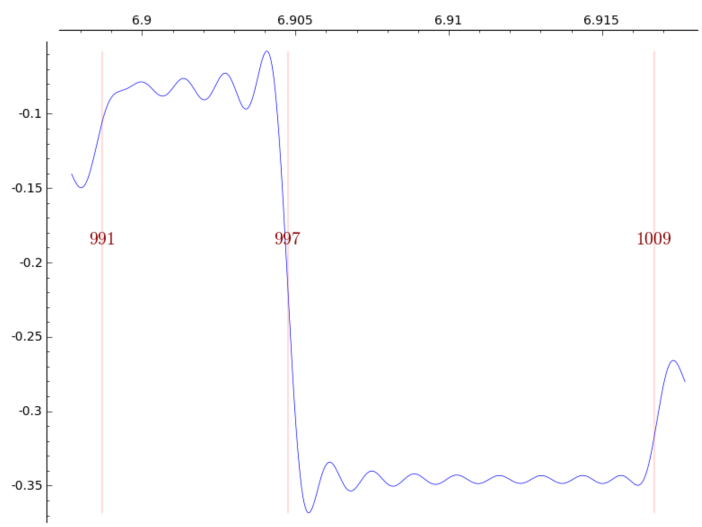
$E = 11a$ between 40 and 60



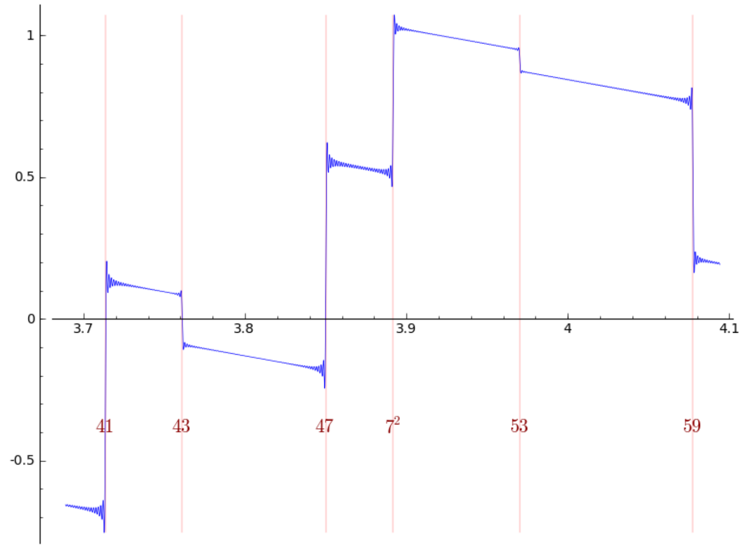
$E = 11a$ between 60 and 100



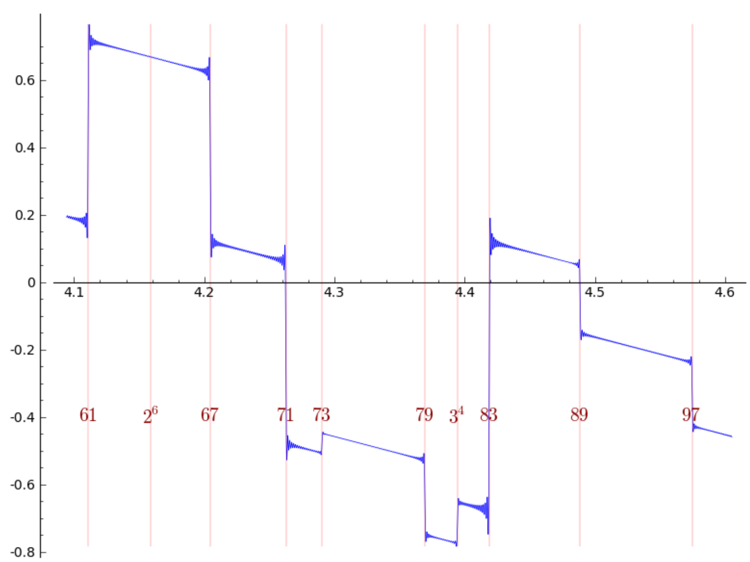
$E = 11a$ between 990 and 1010



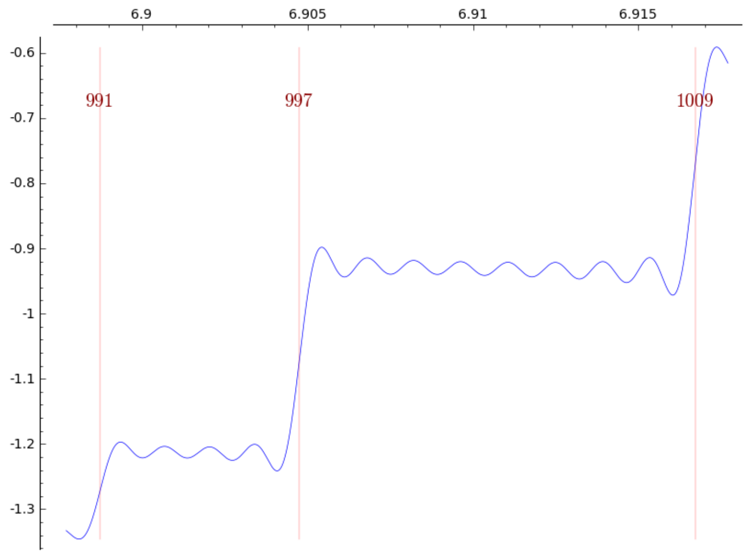
$E = 37a$ between 40 and 60



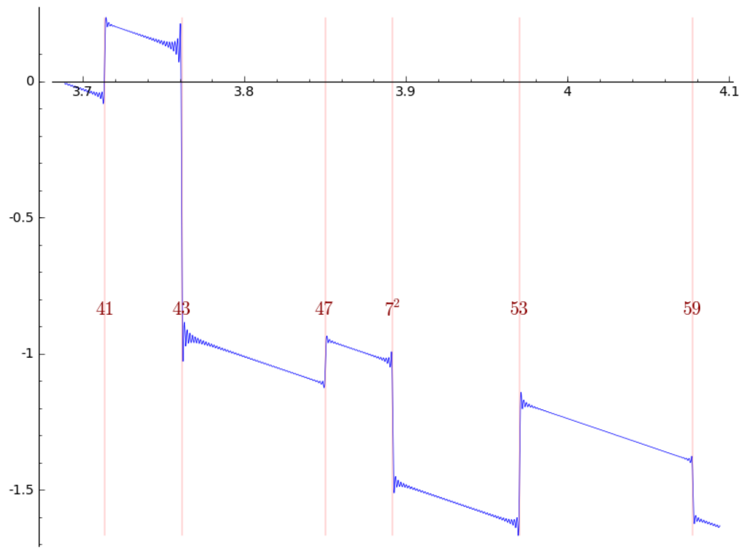
$E = 37a$ between 60 and 100



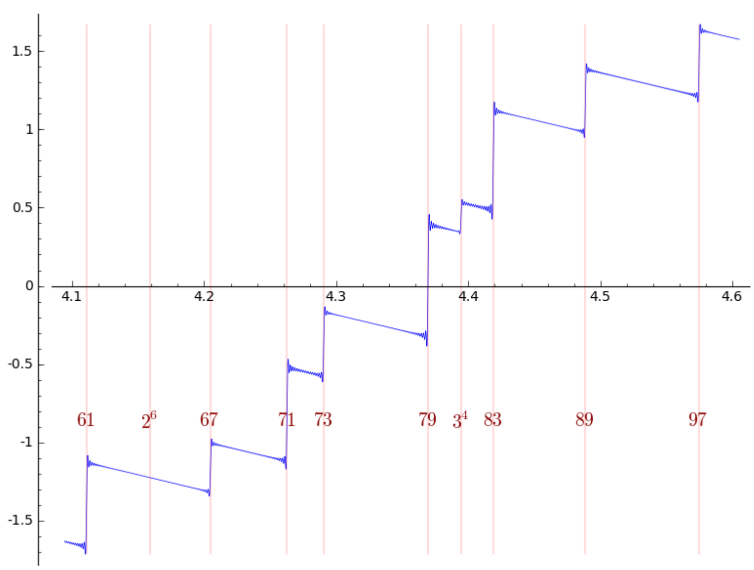
$E = 37a$ between 990 and 1010



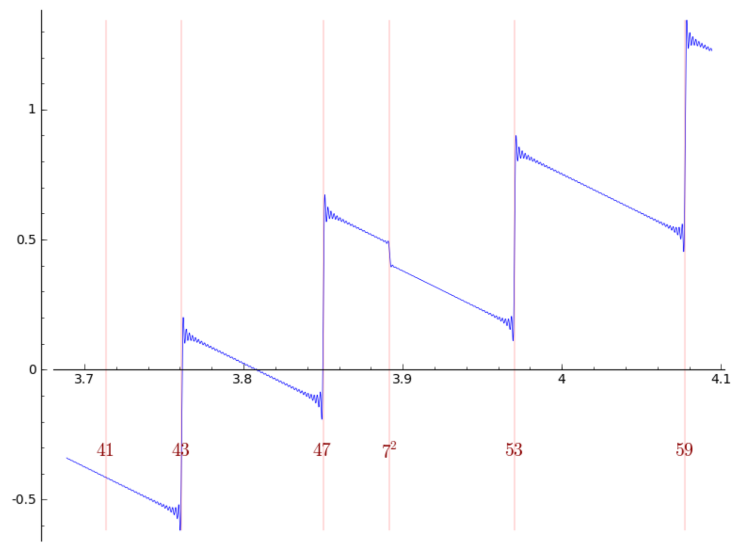
$E = 389a$ between 40 and 60



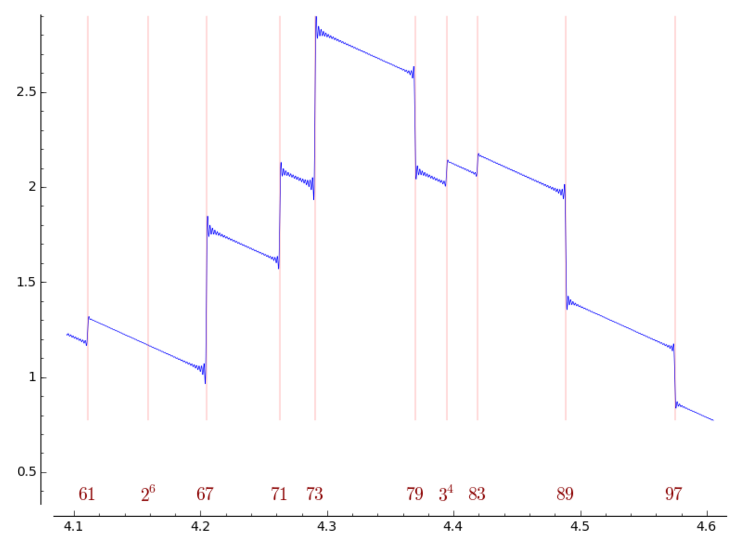
$E = 389a$ between 60 and 100



$E = 5077a$ between 40 and 60



$E = 5077a$ between 60 and 100



8 To summarize:

The Explicit Formula for the 'well-done data,' i.e., $D_E(X)$, is (conjecturally)

$$D_E(X) \; = \; -r + E \; + \; O(\frac{1}{\log X}) \; + \; \frac{1}{\log X} S_E(X),$$

where the rapidity of the conjectured convergence of $D_E(X)$ to $-r$ depends on a concrete understanding of the $O(\frac{1}{\log X})$ term, plus whether, and how rapidly, we expect $S_E(X)$ to decrease. Putting it somewhat archly, one measure of the ease of application of the Explicit Formula, or its 'explicitness,' is how large a value of X do you need for the following to be a true equation:

$$r_E \; = \; \text{the closest integer to } \; -D_E(X)?$$

Part II: Some theory

9 The letter of Peter Sarnak

In a letter [14] to one of us (to B.M.) Peter Sarnak sketched reasons for the statements made about the three formats for sums of local data that we introduced above. As we understand it, the computations in that letter was, at least in part, the fruit of conversations with Andrew Granville and also an outgrowth of [13]. We are grateful for that letter, and for illuminating discussions with Granville, Rubinstein, and Sarnak. Assuming a list of standard conjectures about the behavior of L-functions, together with some very plausible but less standard conjectures, Sarnak begins by showing—as we mentioned above—that (conditional on standard conjectures) the medium-rare local data, $\mathcal{D}_E(X)$, has a limiting distribution with *mean* equal to $1 - 2r_E$.

The *variance* of this limiting distribution is the sum of the squares of the reciprocals of the absolute values of the non-real zeroes of the L-function of E. The argument for these (and related) facts follows Mike Rubenstein's and Peter Sarnak's line of reasoning in the article *Chebyshev's Bias* [[13]]. For another expository account of number theoretic issues related to biases, see [10]. Similar reasoning works for other formats, including the *raw* sum of local data as will be depicted in our graphs below; i.e.,

$$\Delta_E(X) := \frac{\log\ X}{\sqrt{X}} \big(\#\{p \leq X;\ a_E(p) > 0\} \ - \ \#\{p \leq X;\ a_E(p) < 0\}\big),$$

which (given reasonable conjectures, and guesses) one discovers to have infinite *variance* so whatever bias we will be seeing in our finite stretch of data will eventually wash out[8].

[8]All this is specific to elliptic curves E with no complex multiplication, as our examples below all are. The non-finiteness of the variance is related to the fact that the (expected) number of zeroes—in intervals $(1/2, i/2 + iT)$ $(T > 0)$—of the L function of the n-th symmetric power of the newform f_E attached to E grows at least linearly with n.

10 'Explicit Formula' statistics

Let E be an elliptic curve over $\mathbf{Q}$ without complex multiplication associated to a newform f with Fourier expansion:

$$f(q) = q + \sum_{n \geq 2} a_E(n) q^n.$$

For p a prime, write

$$\frac{a_E(p)}{\sqrt{p}} := \alpha_p + \beta_p, \tag{10.1}$$

with $\alpha_p = e^{i\theta_p}$ and $\beta_p = e^{-i\theta_p}$ and

$$\theta_p \in [0, \pi]). \tag{10.2}$$

Our basic data consists of the function

$$p \mapsto \theta_p \tag{10.3}$$

To have some vocabulary to deal with its statistics, consider

$$U_n(\theta) := \frac{\sin(n+1)\theta}{\sin\theta}$$

and note that the set $\{U_n\}$ for $n = 0, 1, 2, \ldots$ forms an orthonormal basis of the Hilbert space $L^2[0, \phi]$.

For $V(\theta)$ a smooth function on $[0, \pi]$, write $V = \sum_{n=0}^{\infty} c_n U_n$ with $c_n := \langle V, U_n \rangle$.

Just to cut down to the essence as rapidly as possible, and just for this lecture:

Definition 10.1. Say that our data (10.3) has **'Explicit Formula' statistics** if there is a sequence of non-negative integers $\{r_n\}_n$ for $n = 1, 2, 3, \ldots$ such that for all smooth functions $V(\theta)$ as above with $c_0 = 0$, the "V-weighted average of the data"

$$S_V(X) := \frac{\log X}{\sqrt{X}} \sum_{p \leq X} V(\theta_p) \tag{10.4}$$

- possesses a limiting distribution[9] μ_V with respect to the multiplicative measure dX/X,

[9] Recall that, as in subsection 5.1 above, $S_V(x)$ **possesses a limiting distribution** μ_V **with respect to the multiplicative measure** dx/x if for continuous bounded functions f on $\mathbf{R}$ we have:

$$\lim_{X \to \infty} \frac{1}{\log X} \int_0^X f(S_V(x)) dx/x = \int_{\mathbf{R}} f(x) d\mu_V(x). \tag{10.5}$$

- μ_V has support on all of $\mathbf{R}$ is continuous and symmetric about its mean, $\mathcal{E}(S_V)$, and

$$\mathcal{E}(S_V) \ = \ -\sum_{n=1}^{\infty} c_n\big(2r_n + (-1)^n\big). \tag{10.6}$$

One can also compute—given some plausible conjectures—the behavior of the **variance** (i.e., the measure of fluctuation of the values of $S_V(X)$ about the mean) as well; the variance is defined by the formula

$$\mathcal{V}(S_V) := \mathcal{E}\big([S_V - \mathcal{E}(S_V)]^2\big).$$

Remark 10.2. If some standard conjectures[10] and some non-standard conjectures[11] hold, then our data (10.3) would indeed have *'Explicit Formula' statistics*; for details, see [14]. The integers r_n, which by the previous footnote are (conjecturally) the orders of vanishing of specific L-functions at their central points, are expected to have the large preponderance of their values equal to 0 or 1, depending on the sign of the functional equation satisfied by the L-function to which they are associated, so the *mean* for a given V as computed by equation (10.6) stands a good chance of being finite.

11 The bias between under-counts and over-counts

We will assume that our data has 'Explicit Formula' statistics, and—copying Sarnak ([14])— apply this to the question we began with, i.e., what is the "bias" in the race between under-counts and over-counts?

$$\Delta_E(X) := \frac{\log X}{\sqrt{X}}\big(\#\{p < X \mid N_E(p) < p+1\} \ - \ \#\{p < X \mid N_E(p) > p+1\}\big).$$

Let $H(\theta)$ be the Heaviside function, i.e., the function with value

$$H(\theta) \ = \ +1 \tag{11.1}$$

for $\theta \in [0, \pi/2)$ and -1 for $\theta \in [\pi/2, \pi)$. So

$$\Delta_E(X) = \frac{\log X}{\sqrt{X}} \sum_{p \le X} H(\theta_p) \tag{11.2}$$

[10] that (for $n = 1, 2, \dots$) the L-functions of the symmetric n-th powers of the elliptic curve,

$$L(s, E, \mathrm{sym}^n) := \prod_p \prod_{j=0}^{n} (1 - \alpha_p^{n-j} \beta_p,^j \, p^{-s})^{-1}, \tag{10.7}$$

have analytic continuation to the entire complex plane satisfying a standard function equation (and one can relax analyticity and require merely an appropriate meromorphicity hypothesis) and that they be holomorphic and non-vanishing up to $Re(s) = 1/2$ (i.e., GRH). The integer r_n (for $n = 1, 2, \dots$) is then the multiplicity of the zero of $L(s, E, \mathrm{sym}^n)$ as $s = 1/2$.

[11] LI(E); see 14, 4

For $n \geq 0$, set

$$c_n(H) \;=\; \langle H, U_n \rangle \;=\; \frac{2}{\pi}\Big[\int_0^{\pi/2} U_n \sin^2\theta d\theta - \int_{\pi/2}^{\pi} U_n \sin^2\theta d\theta\Big] \tag{11.3}$$

which is 0 if n is even and

$$(-1)^{(n-1)/2}\frac{2}{\pi}\Big[\frac{1}{n} + \frac{1}{n+2}\Big]$$

if n is odd.

For $N \geq 1$ let

$$H_N(X) := \sum_{n=1}^{N} c_n(H) U_n(\theta) \tag{11.4}$$

So H_N is a smoothed out version of $H(\theta)$ and $H_N(\theta) \to H(\theta)$ as N tends to infinity. Thus

$$S_N(X) := S_{H_N}(X) = \frac{\log X}{\sqrt{X}} \sum_{p \leq X} H_N(\theta_p) \tag{11.5}$$

is a smoothed out version of

$$S(X) := S_H(X) = \frac{\log X}{\sqrt{X}} \sum_{p \leq X} H(\theta_p) \tag{11.6}$$

Therefore, by formula (10.6), we would have:

$$\mathcal{E}(S_N) \;=\; \frac{8}{3\pi}(1-2r) + \frac{2}{\pi}\sum_{k=1}^{N}(-1)^{k+1}\Big[\frac{1}{2k+1} + \frac{1}{2k+3}\Big]\big(2r_E(2k+1) - 1\big). \tag{11.7}$$

Now one does have parity information concerning the arithmetic function $n \mapsto r_E(n)$. For a detailed study of the root numbers of l-functions of symmetric powers of an elliptic curve, consult [3]. For $n \geq 1$ let $\nu_E(n) \in \{0, 1\}$ be (zero or one) such that $\nu_E(n) \equiv r_E(n)$ modulo 2. Let $s_E(n)$ be the non-negative integer such that:

$$r_E(n) = \nu_E(n) + 2s_E(n)$$

(for $n \geq 3$, odd). Thus if the multiplicities of order of vanishing at the central point $s = 1/2$ of the odd symmetric n-th power L-functions attached to E (for $n \geq 3$) was never greater than 1, and hence entirely dictated by parity, then the conjectured mean, $\mathcal{E}(S_N)$, would be equal to

$$\mathcal{T}_E^{\{N\}}) \;:=\; \frac{8}{3\pi}(1-2r_E) + \frac{2}{\pi}\sum_{k=1}^{N}(-1)^{k+1}\Big[\frac{1}{2k+1} + \frac{1}{2k+3}\Big]\big(2\nu_E(2k+1)-1\big). \tag{11.8}$$

Now consider the limit:

$$\mathcal{T}_E := \lim_{N\to\infty} \mathcal{T}_E^{\{N\}}.$$

Project 11.1. Check if all the possibilities for parity as given in [3] leads, in fact, to convergent values of $\mathcal{T}_E$. Work out those values. E.g., In [3] one reads that for n odd and E semistable, the parities of $symm^n E$ are all the same; i.e., independent of (odd) n. So in the semistable case,

$$\mathcal{T}_E = \frac{8 \pm 2}{3\pi} - \frac{16}{3\pi} r_E,$$

where the sign depends on whether $\nu_E(2k+1)$ is 1 or 0.

Put

$$\mathcal{Z}_E^{\{N\}} := \frac{2}{\pi}\sum_{k=1}^{N}(-1)^{k+1}\Big[\frac{1}{2k+1} + \frac{1}{2k+3}\Big]\big(4s_E(2k+1)\big).$$

Questions: Does the limit,

$$\mathcal{Z}_E := \lim_{N\to\infty} \mathcal{Z}_E^{\{N\}}$$

exist? Does it converge to a finite value? If so, then the conjectured mean would be:

$$\mathcal{E}_E = \mathcal{T}_E \; + \; \mathcal{Z}_E.$$

Is s_{2k+1} bounded? Is the set of positive integers k such that $s_{2k+1} \neq 0$ of *density zero* set of positive integers k? Is that set finite?

Some data for higher order of vanishing for symmetric powers is given in the article of Martin and Watkins [16]. The following table is taken from their article:

E	k	s_{2k+1}
$2379b$	1	2
$5423a$	1	2
$10336d$	1	2
$29862s$	1	2
$816b$	2	1
$2340i$	2	1
$2432d$	2	1
$3776h$	2	1
$128b$	3	1
$160a$	3	1
$192a$	3	1

12 The relationship between bias and unbounded rank: the work of Fiorilli

Recall from Section 6.1 above that the **mean** of $\delta(X)$ is by definition:

$$\mathcal{E} := \lim_{X\to\infty} \frac{1}{\log X}\int_0^X \delta(x)dx/x \;=\; \int_{\mathbf{R}} d\mu_\delta(x).$$

In the work of Sarnak and Fiorilli, another measure for understanding 'bias behavior' is given by what one might call **the percentage of positive support** (relative to the multiplicative measure dX/X). Namely:

$$\mathcal{P} = \mathcal{P}_E := \liminf{}_{X\to\infty}\frac{1}{\log X}\int_{2\le x\le X;\delta(x)\le 0} dx/x$$

$$= \limsup{}_{X\to\infty}\frac{1}{\log X}\int_{2\le x\le X;\delta(x)\le 0} dx/x$$

It is indeed a conjecture, in specific instances interesting to us, that these limits $\mathcal{E}$ and $\mathcal{P}$ exist.

The standard conjecture (that we have been making all along) is GRH. But here, one includes the further conjecture (given in Sarnak's letter, and the article of Fiorilli) that the the set of nontrivial complex zeroes of the relevant L-function $L(E,s)$ with positive imaginary part is a set of complex numbers that are *linearly independent* over $\mathbf{Q}$. Such a conjecture Rubenstein and Sarnak refer to in [13] as the it Grand Simplicity Hypothesis (GSH). Fiorilli calls his version of it **Hypothesis LI(E)**. For recent, somewhat related, work on such linear independence questions, see citeM-N. Fiorilli, following the work of Sarnak, proves:

Theorem 12.1. *Assume GRH and LI(E). Then the following two statements are equivalent:*

1. *The set of (analytic) ranks $\{r_E\}_E$ ranging over all elliptic curves over $\mathbf{Q}$ is it unbounded.*
2. *The l.u.b of the set of percentages of positive support $\{\mathcal{P}_E\}_E$ is equal to* 1.

13 The relationship between bias and bounding the rank: the work of Bober

In [1], Jonathan Bober establishes a conditional upper bound on the ranks of various known elliptic curves of (relatively) high Mordell-Weil rank, notably Noam Elkies' elliptic curve E_{28} for which 28 linearly independent rational points have been found; Bober shows, conditional on the Birch-Swinnerton-Dyer conjecture and GRH, that the Mordell-Weil rank of E_{28} is either 28 or 30. He does this by a nice 'bias' computation using the Explicit Formula.

14 Further finer questions: conditional biases

In summary, given the conjectures discussed, the *theory of the means* of the general weighted sums of local data we have been examining related to an elliptic curve E is determined by the orders of vanishing at the central point of the L-functions of the symmetric powers of the modular eigenform attached to E: and conversely: knowledge of the means of all such weighted sums determines all those orders of vanishing.

$$\{\text{Weighted biases}\} \quad \leftrightarrow \quad \{\text{Central zeroes}\}$$

This leads to various issues needing conjectures, and computations. What might we reasonably conjecture about:

1. *the arithmetic function $k \mapsto r_E(2k+1)$?*
   - Is it unbounded?
   - Is $r_E(2k+1) \geq 2$ for only a set of values of k of density 0?
   - Is $r_E(2k+1) \geq 2$ for all but finitely many k's?
2. *the collection of weighted biases that have finite mean?* I.e., for which weighted biases does Equation 10.6 have a convergent RHS?

3. *the detailed statistical behavior of the function $S_E(X,Y)$?*

4. *an effective version of LI(E)?* I.e., can we put our fingers on an explicit positive function $F(H,T)$ such that for every linear combination of the form

$$\sum_{j=1}^{\nu} \lambda_j \gamma_j$$

with the λ_j 's rational numbers of height $< H$ and the γ_j 's positive imaginary parts $< T$ of the complex zeroes of the L function $L(E,s)$, we have an inequality of the form

$$|\sum_{j=1}^{\nu} \lambda_j \gamma_j| > F(H,T)?$$

5. *conditional biases?* For example, given two elliptic curves E_1, E_2 over $\mathbf{Q}$ (that are not isogenous), say that a prime p is of type $(+,+)$ if both $a_{E_1}(p)$ and $a_{E_2}(p)$ are positive, of type $(+,-)$ if $a_{E_1}(p)$ is positive and $a_{E_2}(p)$ negative, etc.

   Now *race* the four types of primes against each other! What is the ensuing statistics, and how much of the analytic number theory regarding zeroes of L functions attached to

$$symm^m(f_{E_1}) \otimes symm^n(f_{E_2})$$

   do we need to compute biases, if such biases exist?

15 Appendix A: an example of a very classical 'explicit formula' ($\psi(X)$ versus $\pi(X)$)

Let $\Lambda(x)$ be the *Von Mangoldt Lambda-function.* That is, $\Lambda(x)$ is zero unless $x = p^k$ is a power of a prime—$(k \geq 1)$—in which case $\Lambda(p^k) := \log p$. Consider

$$\psi_0(X) := \frac{1}{2}\Lambda(X) + \sum_{n<X} \Lambda(n).$$

Although one might argue whether or not $\psi_0(X)$ fits into the mold of what we have been calling a 'sum of local data,' it is certainly *not* one of our bias sums of local data, which has been our principal concern. Nevertheless, it will serve, and will sit, appropriately normalized, on the LHS of Theorem 15.1, our example of an 'explicit formula.' Let $\rho = \frac{1}{2} + i\gamma$ run through the zeroes in the line $Re(s) = \frac{1}{2}$ of the Riemann zeta function. (For expedience, we assume RH here).

Theorem 15.1. (Explicit Formula)

$$\frac{1}{X} \cdot \psi_0(X) \;=\; 1 \;-\; \sum_{|\gamma| \leq T} \frac{X^{i\gamma}}{(\frac{1}{2} + i\gamma)\sqrt{X}} \;+\; C(X,T)$$

where–following the format of explicit formulas discussed above—we view

- *the term on the LHS of the above equation as our 'sum of local data';*
- *the first term on the* RHS,*—i.e.,* 1*—as the 'global term' corresponding to the pole of* $\zeta(s)$ *at* $s = 1$ *with residue* 1*; it is the mean of the* LHS*, our sum of local data;*
- *the second term on the* RHS, $\sum_{|\gamma|\le T} \frac{X^{i\gamma}}{(\frac{1}{2}+i\gamma)\sqrt{X}}$ *as a cutoff at* T *of the 'oscillatory term' while*
- *the third term,* $C(X,T)$ *is a cutoff at* T *and at* X *of the 'easy error term.' It converges to zero if the limits are taken in the order*

  $$\lim_{X\to\infty}\lim_{T\to\infty} C(X,T).$$

  This $C(X,T)$ *has the following shape:*

  $$C(X,T) := \frac{-\log(2\pi) - \log(1-1/X^2)/2}{X} + \epsilon(X,T),$$

  where:

  $$\epsilon(X,T) \;<<\; \frac{\log X}{X}\cdot\min\big(1, \frac{X}{T\langle X\rangle}\big) \;+\; \frac{\log^2(XT)}{T}.$$

Here, $\langle X\rangle$ is the distance between X and the nearest prime power, and with all this, the $<<$ would still need explicitation—even if that word is non-standard. This result and its proof is given, for example, as Theorem 12.5 in [12].

We should note in passing that there is, of course, a massive literature on this type of formula relating the zeroes of the Riemann zeta function and $\pi(X)$, and relating $\pi(X)$ to the zeroes[12], in all its variants.

16 Appendix B: $\psi(X)$ versus $\pi(X)$ in the context of newforms

As mentioned in the Preface and as noted in the previous Appendix, to achieve relatively smooth data (and for other reasons) one often considers sums of local arithmetic data over prime powers, with a cut-off at some large real number X rather than just sums over primes. The "other reasons" are, for example, that one naturally takes as the starting point of Explicit Formulae the application of Perron's formula to the logarithmic derivative of an L-function, and this naturally entails sums of local arithmetic data over prime powers. For example, consider modular newforms (of even weight $k = 2w \ge 2$). Let

$$\omega = q + a_2q^2 + a_3q^3 + \cdots + a_nq^n + \ldots$$

be the Fourier expansion of a cuspidal newform of weight k on $\Gamma_0(N)$ for some N. Put

$$D^\flat_\omega(X) = \frac{1}{X^w}\sum_{n\le X} a_n\Lambda(n)$$

[12]For that, see, for example, [9].

(comparing this with section 5.5 [11]).

In ***** Simon Spicer studies the Explicit Formula in this context. See the forthcoming work of Simon Spicer regarding Explicit Formulas for $D^{\flat}_{\omega}(X)$ (and its associated picture gallery) where ω is the newform attached to an elliptic curve E over $\mathbf{Q}$.

Part III: Some pictures of the LHS

17 The well-done data: $D_E(X)$

(Graphs of $X \mapsto D_E(X) = \frac{1}{\log\ X} \sum_{p \le X} \frac{a_E(p)\log p}{p}$**)**

Rank $r = 0$**:** $\mathcal{E}$ =**11A.**

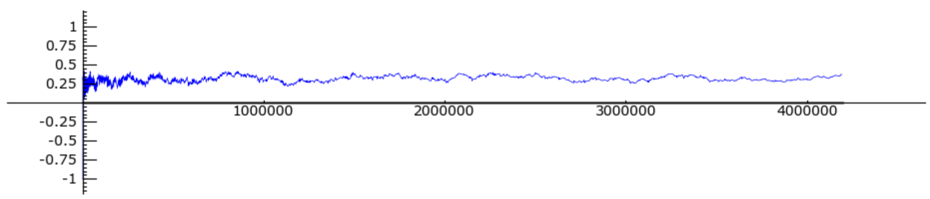


Figure 17.1:

Rank $r = 1$: $\mathcal{E}$ =37A.

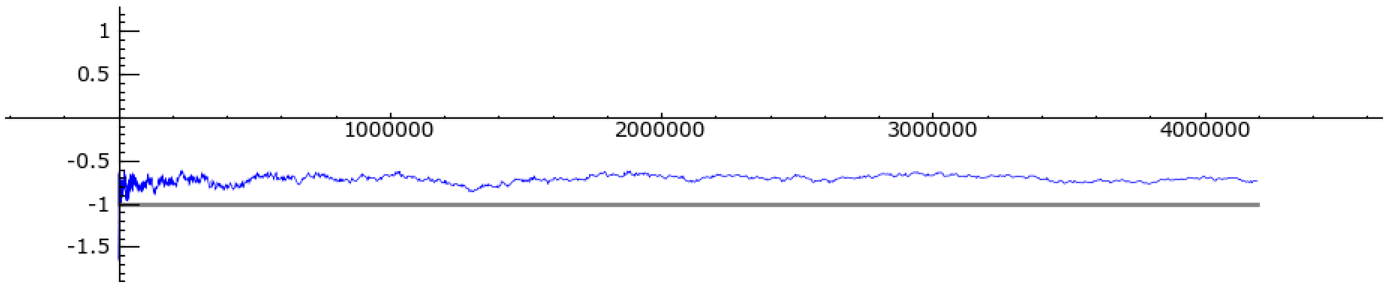


Figure 17.2:

Rank $r = 2$: $\mathcal{E}$ =389A.

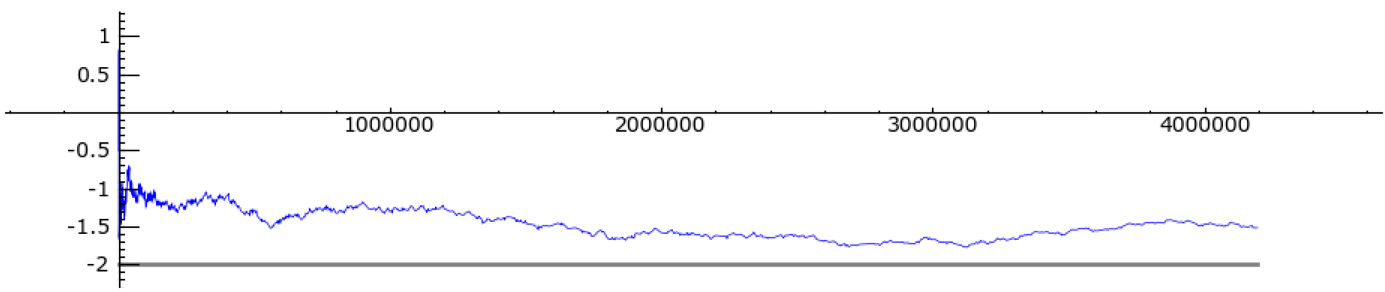


Figure 17.3:

Rank $r = 3$: $\mathcal{E}$ =5077A.

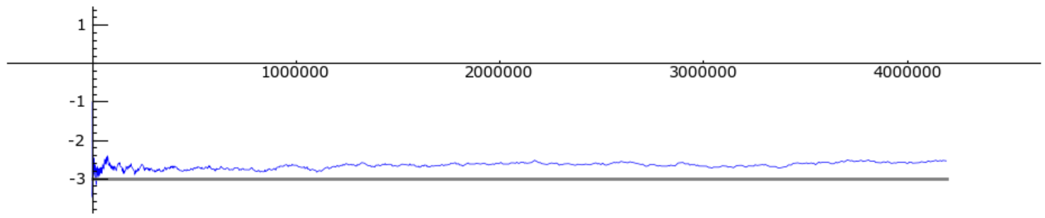


Figure 17.4:

Rank $r = 4$.

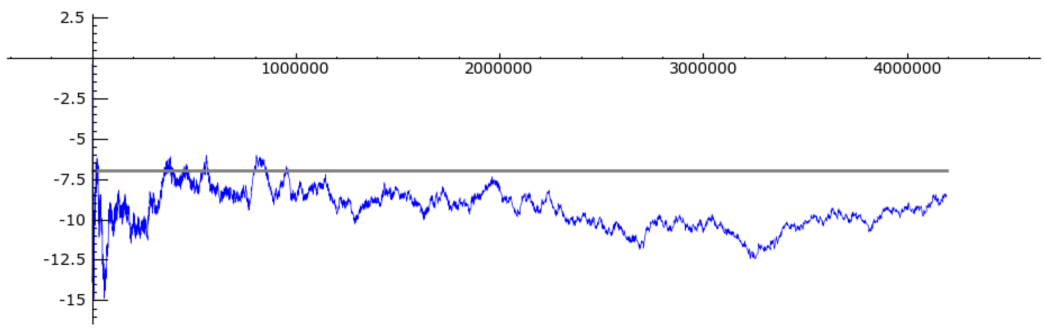


Figure 17.5:

Rank $r = 6$.

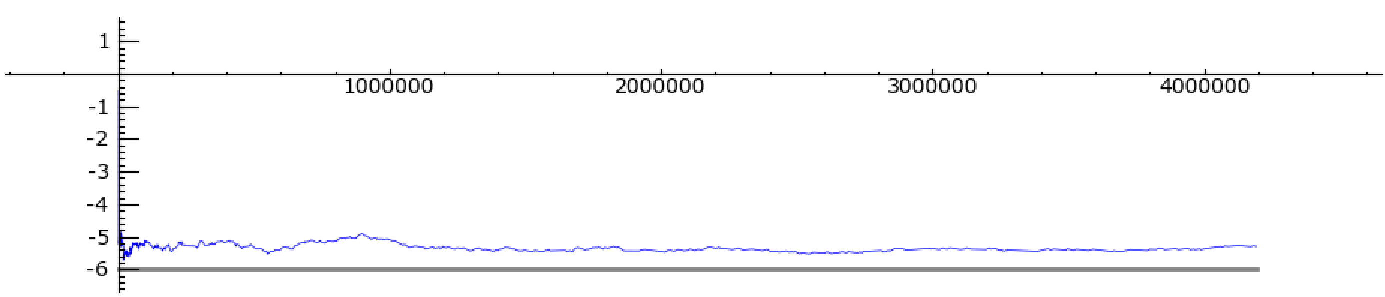


Figure 17.6:

18 The medium-rare data: $\mathcal{D}_E(X)$

(Graphs of $X \mapsto \mathcal{D}_E(X) = \frac{\log X}{\sqrt{X}} \sum_{p \leq X} \frac{a_{\mathcal{E}}(p)}{\sqrt{p}}$**)**

Rank $r = 0$**:** $\mathcal{E} =$**11A.**

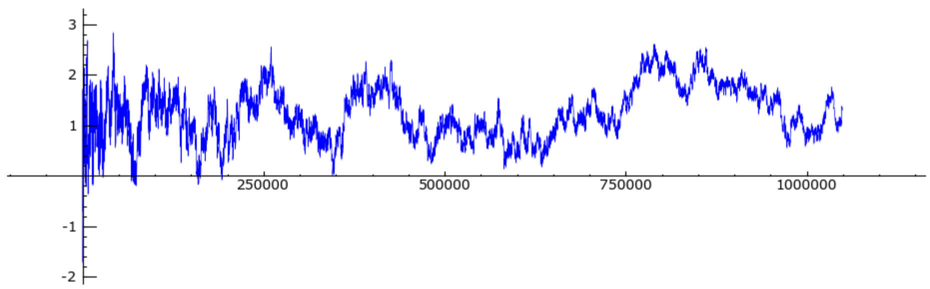


Figure 18.1:

Rank $r = 1$**:** $\mathcal{E} =$**37A.**

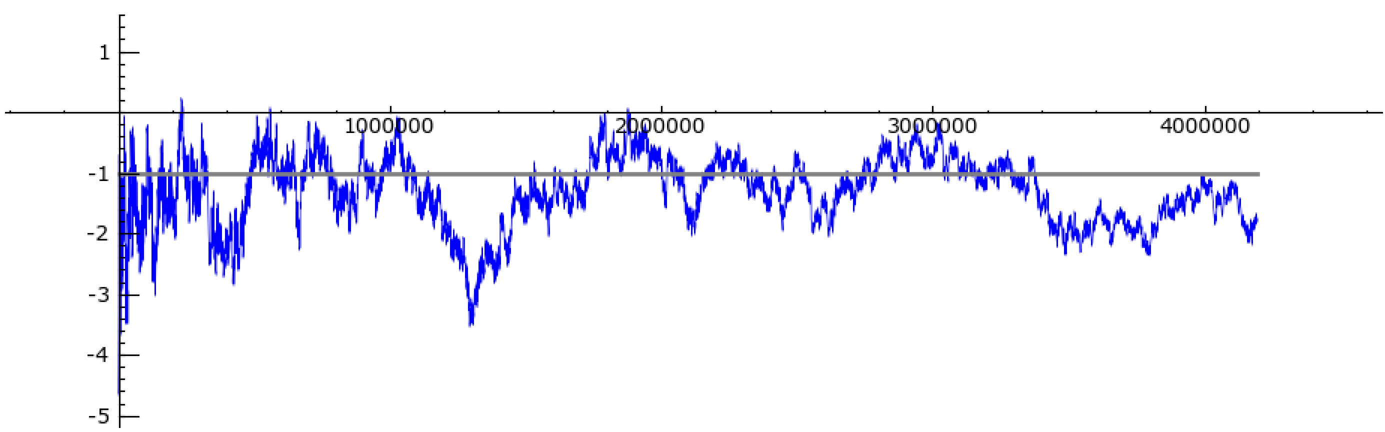


Figure 18.2:

Rank $r = 2$: $\mathcal{E}$ =389A.

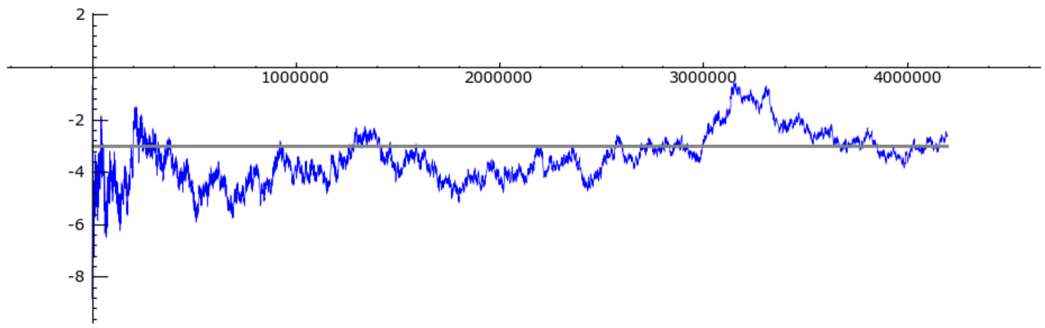


Figure 18.3:

Rank $r = 3$: $\mathcal{E}$ =5077A.

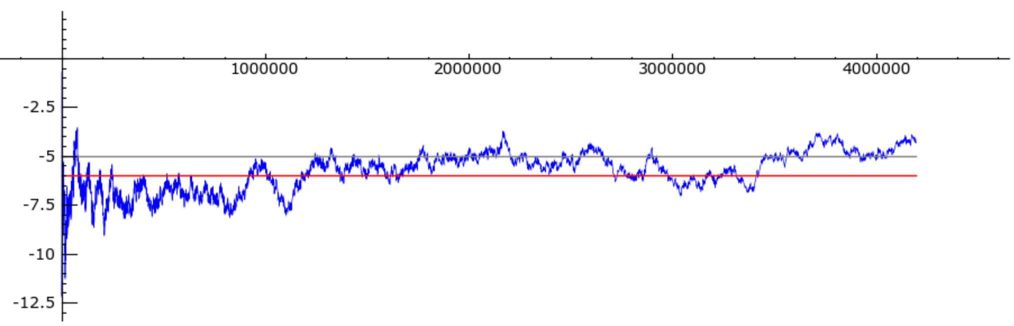


Figure 18.4:

Rank $r = 4$**.**

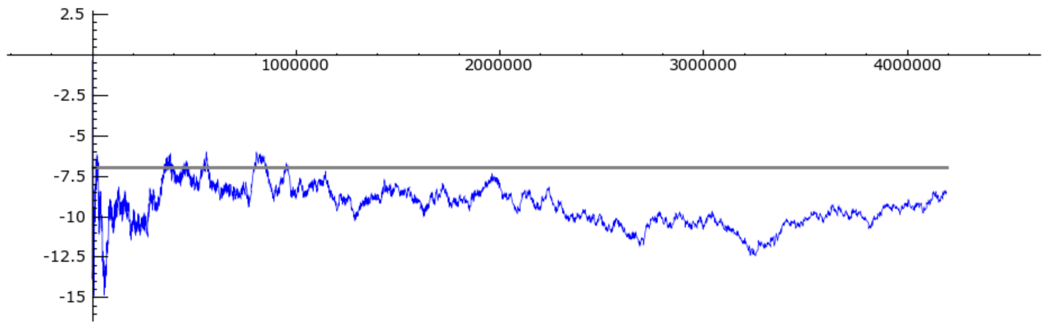


Figure 18.5:

Rank $r = 5$**.**

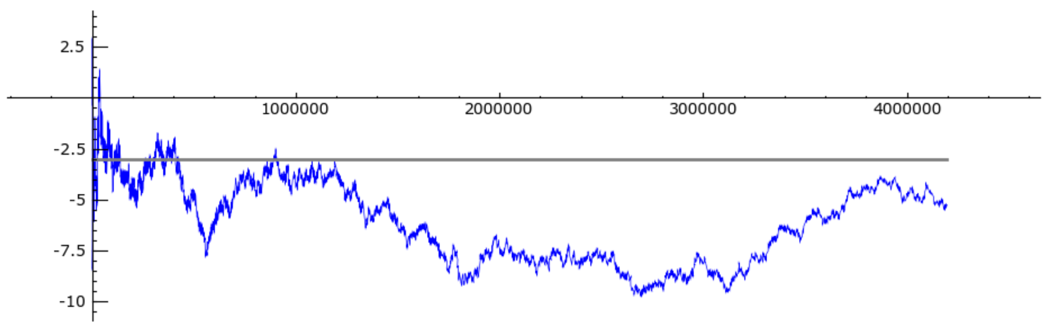


Figure 18.6:

Rank $r = 6$.

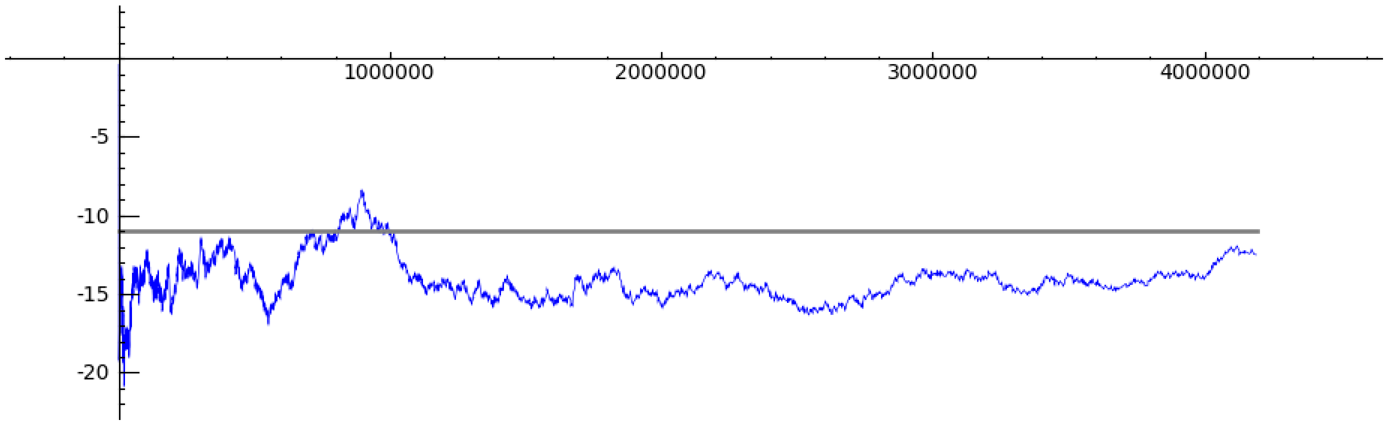


Figure 18.7:

19 The raw data: $\Delta_E(X)$

(Graphs of $X \mapsto \Delta_E(X) = \frac{\log X}{\sqrt{X}} \#\{p < X \mid a_E(p) > 0\} - \#\{p < X \mid a_E(p) < 0\}$)

Rank $r = 0$: $\mathcal{E}$ =11A.

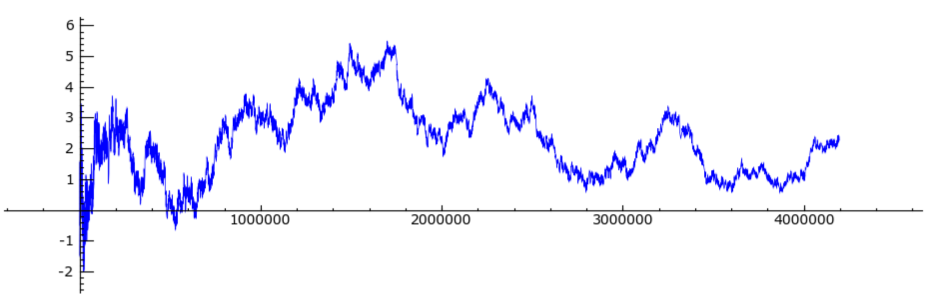


Figure 19.1:

Rank $r = 1$**:** $\mathcal{E} =$**37A.**

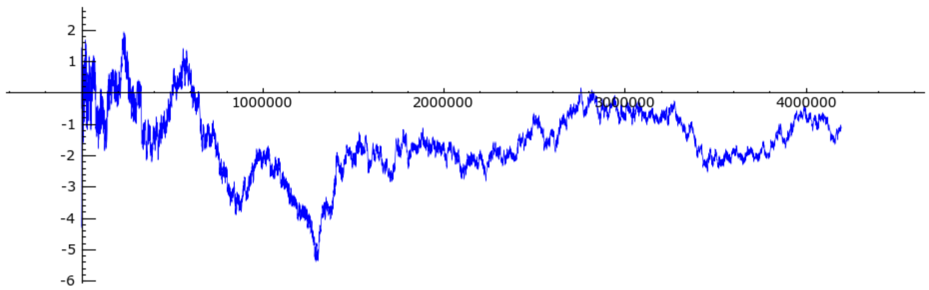


Figure 19.2:

Rank $r = 2$**:** $\mathcal{E} =$**389A.**

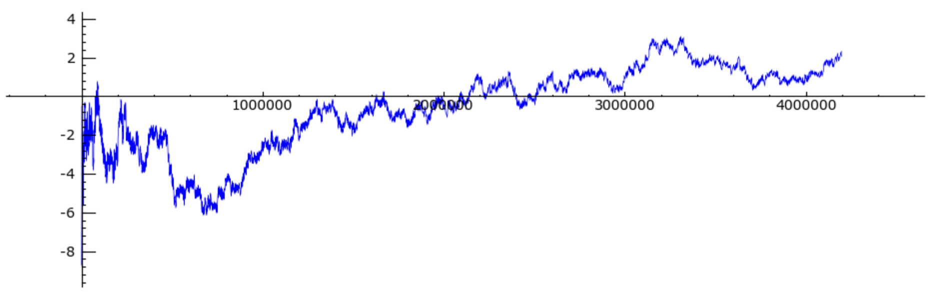


Figure 19.3:

Rank $r = 3$**:** $\mathcal{E}$ =**5077A.**

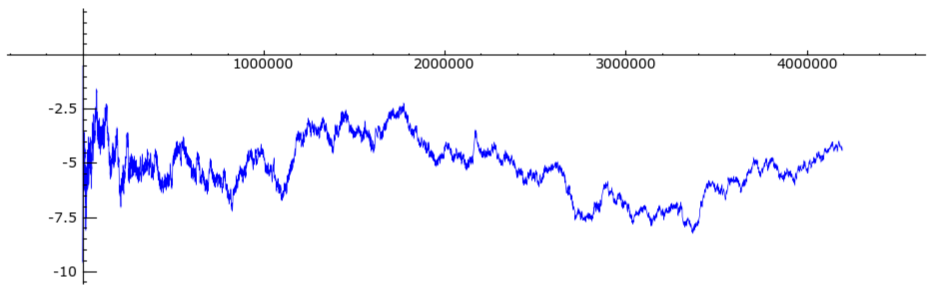


Figure 19.4:

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