Jumps in Mordell-Weil Rank and Arithmetic Surjectivity.

T. Graber, J. Harris, B. Mazur, J. Starr

In our work establishing a "converse" to the theorem of Graber-Harris-Starr (see [G-H-S], [G-H-M-S 1], [G-H-M-S 2]) the four of us had contemplated arithmetic issues related to our converse theorem. One of us (B. M.) presented a bit of this material (some linked *questions* together with a tiny piece of evidence in support of an affirmative answer to them) at the AIM workshop in Problems on Rational Points on Algebraic Varieties held in Palo Alto in December 2002. We record this in the text below. We are most thankful to the organizers for setting up such a wonderful engaging conference–an ideal format for discussion of these, and more general, topics.

1. Arithmetic surjectivity. Recall that a (smooth, proper) morphism $f : X \to B$ over a number field K is said to be **arithmetically surjective** if for all number field extensions L/K, the mapping on L-rational points, $f(L) : X(L) \to B(L)$, is surjective.

Denote by $C(f, L) \subset B(L)$ the complement of the image of $f(L)$, i..e., the set of L-valued points in the base whose f-fibers possess *no* L-rational points. So, f is arithmetically surjective if and only if for any number field extension L/K the complement of the image of L-valued points, $C(f, L)$, is empty.

We begin with a qualitative question.

Question 1. Let $f : X \to B$ be a pencil of (projective, smooth) curves of genus one over a number field K. Are the following two conditions equivalent?

(1) The morphism f is arithmetically surjective.

(2) The morphism f admits a section over K.

Here is a convenient way of restating this question.

Consider B an open subscheme of $\mathbf{P}^1$ over K and $X \to B$ a (flat) family of (projective, smooth) curves of genus 1 defined over K. Letting $E \to B$ be the jacobian (i.e., Pic^0) of the family, we have that $X \to B$ is a torsor over $E \to B$ and is represented therefore by some class $h \in H^1(B; E)$. This cohomology group is torsion, so let n denote the order of the element h. Invoking the Kummer sequence of group schemes over B,

$$0 \to E[n] \to E \to E \to 0,$$

where the mapping $E \to E$ is multiplication by n, we see that there is an element $\tilde{h} \in H^1(B; E[n])$ which maps to $h \in H^1(B; E)$. Choose such an $\tilde{h}$. The family $X \to B$ is determined up to isomorphism by the element $h \in H^1(B; E)$ and hence also by $\tilde{h} \in H^1(B; E[n])$; to reflect this fact we may refer to the family as $X_h \to B$, or $X_{\tilde{h}} \to B$.

Observation. For $A \to S$, an abelian scheme over a $\mathbf{Q}$-scheme S and a class $\tilde{h} \in H^1(S; A[n])$, for some integer n, let $h \in H^1(B; A)$ be the image of $\tilde{h}$ under the natural mapping induced from the inclusion $A[n] \hookrightarrow A$, and let $f : X_h \to S$ be the associated A-torsor. The long exact sequence on cohomology obtained from the Kummer sequence

$$0 \to A[n] \to A \to A \to 0$$

immediately gives the equivalence of the following statements:

- The family $X_h \to S$ admits no section (over S).

- The cohomology class $h \in H^1(S; A)$ is not zero.

- The cohomology class $\tilde{h} \in H^1(S; A[n])$ is *not* in the image of $H^0(S; A) = A(S)$ under the connecting homomorphism of the long exact sequence on cohomology induced by the Kummer sequence.

Question 2 (for a given natural number n). For $E \to B$ an abelian scheme of dimension one over B an open subscheme of $\mathbf{P}^1$ defined over a number field K, is it the case that if a class $\tilde{h} \in H^1(B; E[n])$ has the property that the associated E-torsor $X_h \to B$ admits no section, then there exists a number field extension L/K and an L-valued point of B, $\beta = \text{Spec } L \to B$, such that the fiber over β of the family $X_{\tilde{h}} \to B$ possesses no L-rational point?

An affirmative answer to **Question 2** for all $n > 1$ and all $E \to B$ is the same as an affirmative answer to **Question 1.**

As for quantitative questions, fix the number field K, and for real numbers C, let b run through the K-rational points of height $< C$ in the base B, and ask for upper bounds, as C tends to $+\infty$, of the number of fibers X_b in the family $X \to B$ that have K-rational points. Here is one formulation of such a question, keeping the notation $\tilde{h}$ and h of **Question 2.**

Question 3 (for a given natural number n). Let $E \to B$ be a smooth family of elliptic curves over B, an open nonempty subscheme in $\mathbf{P}^1$ over K. Is there a positive number e such that for any $\tilde{h} \in H^1(B; E[n])$ *not* in the image of $H^0(B; E) = E(B)$

$$\text{card}\{b \in B(K) \mid \text{height}(b) < C \text{ and } b^*(h) \neq 0\} \;>\; C^e$$

for sufficiently large C?

For a K-rational point $b \in B$ denote by $E(b)$ the Mordell-Weil group $E_b(K)$. By a **jump point** $b = \text{Spec } K \to B$ (for a family of elliptic curves $E \to B$, over the number field K) let us mean a K-rational point of B such that the induced mapping on Mordell-Weil groups $E(B) \to E(b) := E_b(K)$ has cokernel of positive rank. Denote by $\mathcal{J}(E/B, K)$ the set of jump points of $E \to B$, over the number field K.

The remainder of this article consists of a fragmentary result related to Question 3, connecting it to the rarity of jump points. More specifically, for the E-torsors $f : X_h \to B$ to be examined in the proposition below, the set of jump points $\mathcal{J}(E/B, K) \subset B(K)$ will shown to be "small" and the union of the sets $\mathcal{J}(E/B, K)$ and $C(f, K)$ will shown to be all of $B(K)$. This will lead us to a query about a specific pencil of elliptic curves, constructed by Cassels and Schinzel, [**C-S**], where every rational point in the base is a jump point.

2. Quadratic twist families. Let

$$E_1 : \quad y^2 \ = \ x^3 + ax + b$$

be an elliptic curve over the number field K. Let

$$B := \mathbf{P}^1 \ - \ \{0, \infty\} = \mathrm{Spec}K\ [t, t^{-1}].$$

By the **quadratic twist family** $E \to B$ attached to E_1 over K we mean the pencil of elliptic curves

$$E_t : \quad ty^2 \ = \ x^3 + ax + b.$$

Proposition. Let $K = \mathbf{Q}$. Let E_1 be any elliptic curve over $\mathbf{Q}$ with none of its points of order two rational over $\mathbf{Q}$. Let $E \to B$ be its associated quadratic twist family. Then **Question 3** for $n = 2$ has an affirmative answer and we can take $e = 1 - \epsilon$ for any positive ϵ.

Corollary. Under the same hypotheses, **Question 2** for $n = 2$ has an affirmative answer.

Proof of the proposition. Let $\bar{\mathbf{Q}}$ be an algebraic closure of $\mathbf{Q}$. For X any $\mathbf{Q}$-scheme put

$$\bar{X} := X \otimes_{\mathrm{Spec}\mathbf{Q}} \mathrm{Spec}\bar{\mathbf{Q}},$$

the scheme obtained by extending scalars to $\bar{\mathbf{Q}}$. Put $G := \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$.

We will be dealing with cohomology of the Kummer sequence

$$0 \to E[2] \to E \to E \to 0$$

over the $\mathbf{Q}$-scheme B and the restriction of that cohomology to rational points $b \in B$. We are in a particularly nice situation because the group scheme $E[2]$ over B is isomorphic to the pullback via $B \to \mathrm{Spec}(\mathbf{Q})$ of the group scheme $E_1[2]$ over $\mathrm{Spec}(\mathbf{Q})$.

The following result surely is somewhere in the literature; we give its proof at the end of this section.

Lemma. $E(\bar{B}) = E[2](\bar{B})$.

By the Hochschild-Serre Spectral Sequence we have an exact sequence

$$0 \to H^1(G; H^0(\bar{B}; E[2])) \to H^1(B; E[2]) \to H^1(\bar{B}; E[2])^G.$$

Since the group scheme $E[2]$ over B comes by pullback from the group scheme $E_1[2]$ over $\mathbf{Q}$ we see that

$$H^1(G; H^0(\bar{B}; E[2])) = H^1(G; E_1[2]).$$

Also,

$$H^1(\bar{B}; E[2])^G = \mathrm{Hom}(H_1(\bar{B}, \mathbf{Z}/2\mathbf{Z}), E[2])^G = \mathrm{Hom}(\mathbf{Z}/2\mathbf{Z}, E[2])^G = E_1[2]^G = 0,$$

the first equality coming from the Universal Coefficient Theorem, the second equality a consequence of the fact that $H_1(\bar{B}; \mathbf{Z}/2\mathbf{Z})$ is canonically $\mu_2 \cong \mathbf{Z}/2\mathbf{Z}$, and the last equality following from our hypothesis that there is no nontrivial 2-torsion in the Mordell-Weil group $E_1(\mathbf{Q})$.

Therefore we get that the natural mapping

$$H^1(G; E_1[2]) \to H^1(B; E[2])$$

is an isomorphism, which tells us that for any $\mathbf{Q}$-rational point $b \in B$, the induced morphism on cohomology,

$$H^1(B; E[2]) \to H^1(b; E[2])$$

is an isomorphism.

From the above discussion and our hypotheses we have that $E(B) = 0$. It follows that for any nontrivial $\tilde{h} \in H^1(B; E[2])$ the specialization of its image $h \in H^1(B; E)$ to a $\mathbf{Q}$-rational point b of B, $b^*(h) \in H^1(b; E)$, is nontrivial if the Mordell-Weil group $E_b(\mathbf{Q}) = E(b)$ has rank zero; equivalently, if b is *not* a jump point for the quadratic twist family. We have shown, in other words, that

$$\mathcal{J}(E, B) \cup C(f, K) = B(K)$$

where $f : X_h \to B$ is the torsor associated to h.

To conclude the proof of our proposition, then, we must control the number of jump points; specifically it suffices to show that for any positive ϵ,

$$\mathrm{card}\{b \in B(\mathbf{Q}) \mid \mathrm{height}(b) < C \text{ and } E(b) \text{ finite}\} \ > \ C^{1-\epsilon}$$

for C sufficiently large.

For this, we use the fact that E_1 is modular. Let f be the newform of weight two which is attached to E_1. Ono and Skinner (**[O-S]**) have estimates on the number of nonzero Fourier coefficients of the modular form F of weight 3/2 associated to such a newform f via the Shimura lift. By classical results of Waldspurger [W], Ono and Skinner then get that for any positive ϵ

$$\text{card}\{b \in B(\mathbf{Q}) \mid \text{height}(b) < C \text{ and analytic rank of } E_b \text{ is zero}\} \; > \; C^{1-\epsilon}$$

for C sufficiently large, and they conclude the same estimate for arithmetic rank using the results of Kolyvagin [**K**]. I am thankful to Brian Conrey for pointing out the Ono-Skinner reference.

Proof of the lemma: Consider $\bar{B}' \to \bar{B}$ the double cover ramified at 0 and ∞ (i.e., extract a square root of the parameter t) and note that the pullback $\bar{E}' = \bar{E} \otimes_{\bar{B}} \bar{B}'$ is a product, $\bar{E}' = \bar{E}_1 \times \bar{B}'$. A section $\sigma : \bar{B} \to \bar{E}$ gives us a section $\sigma' : \bar{B}' \to \bar{E}'$ such that $\sigma' \cdot i = j \cdot \sigma'$ where $i : \bar{B}' \to \bar{B}'$ is the involution of $\bar{B}'$ as double cover of $\bar{B}$ and $j : \bar{E}' = \bar{E}_1 \times \bar{B}' \to \bar{E}_1 \times \bar{B}'$ is given by $j(e_1, b') = (-e_1, i(b'))$ for $e_1 \in \bar{E}_1$ and $b' \in \bar{B}'$. Since there are no nonconstant morphisms from $\bar{B}'$ to $\bar{E}_1$, σ' is a constant section, i.e., $\sigma'(b') = (e_1, b')$ for a fixed $e_1 \in \bar{E}_1$ and all $b' \in \bar{B}'$. Therefore e_1 is in $\bar{E}_1[2]$ proving the lemma.

3. The Cassels-Schinzel Example. Since jump points seem to be relevant to arithmetic surjectivity questions, it may be useful to recall a classic example due to Cassels-Schinzel ([**C-S**]) of a pencil of elliptic curves over $\mathbf{Q}$ where *every* $\mathbf{Q}$-rational point of the parameter space B is a jump point. Their example is a certain pencil E_t of twists of the elliptic curve $X_0(32)$ over $\mathbf{Q}$ where the Mordell-Weil group of its sections (over $\mathbf{Q}$) is equal to the Klein 4-group (consisting of four sections in the 2-torsion subgroup $E_t[2]$) and yet for each $t_0 \in \mathbf{Q}$ the elliptic curve E_{t_0} has odd, hence nonzero, (analytic) rank (the game in finding such examples is to "work the equations" to guarantee that odd parity happens for all rational t_0's). Of course, the "rank" computed is the analytic rank, but by a recent result of Nekovár, it is the same parity as the Selmer rank. To pass from this "parity of the Selmer rank" to the parity of Mordell-Weil rank we must invoke the Shafarevich-Tate conjecture (at least when the Mordell-Weil rank is > 1 for then we can't invoke Heegner points and Kolyvagin's theorem). Here is the explicit example:

$$E_t : \;\; y^2 = x(x^2 - (7 + 7t^4)^2).$$

It would be interesting to get asymptotic results (which are either like or unlike the estimates given in the proposition above) for torsors over such an example. It would also be interesting to get asymptotic results over $\mathbf{Q}$ for pencils of plane cubics possessing no sections.

4. References.

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