MODULAR SYMBOLS (CONTINUED)

Math 258 February 7, 2019

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Part 1. Modular symbols, L-values, and θ-elements

As I mentioned in the first session, Karl Rubin and I—inspired by the random matrix heuristics predictions of David-Fearnley-Kisilevsky [1]—had the idea of developing a somewhat more naive heuristic (motivated by the general statistics coming from the combinatorics of modular symbols) that might complement (and concur with—or not) the random matrix predictions.

Our aim is to give (heuristic!) support for conjectures asserting that elliptic curves have finitely generated Mordell-Weil groups over certain large abelian number fields. I won't get to the general formulation of these conjectures, nor of the precise heuristics that connect to the conjectures in this session. But I do want to give a general sense of what is known (and what is conjectured) concerning modular symbols, θ-elements and their statistics. It seems to me that there are many computational projects connected to these objects that might be interesting to consider. Much of this handout consists of pieces of text that I took (with Karl Rubin's permission) from a draft of a paper that Karl and I are writing.

1. **Recall:** Modular symbols

Lemma 1.1. *In the discussion below, fix E an elliptic curve over $\mathbb{Q}$ and let N be the conductor of E. Let $\delta = \delta_E \in \mathbb{Z}_{>0}$ be the lcm of the orders of the torsion points in the Mordell-Weil group $E(\mathbb{Q})$.*

For $r \in \mathbb{Q} \sqcup \{\infty\}$ we have:

(i) $[r]^{\pm} \in (2\delta)^{-1}\mathbb{Z}$,

(ii) $[\infty]^{\pm} = 0$,

(iii) $[r]^{\pm} = [r+1]^{\pm}$,

(iv) $[r]^{\pm} = \pm[-r]^{\pm}$,

(v) **Invariance:**
If
$$A := \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N) \subset \mathrm{SL})2(\mathbb{Z}),$$
so that for $r \in \mathbb{Q} \sqcup \{\infty\}$,
$$A(r) = (ar+b)/(cNr+d) \in \mathbb{Q} \sqcup \{\infty\},$$
we have the following relation in modular symbols:
$$[r]^{\pm} = [A(r)]^{\pm} - [A(\infty)]^{\pm},$$
and if $A \in \Gamma_0(N)$,as automorphism of $\mathbf{H}$, has a complex (quadratic) fixed point, then $[A(\infty)]^{\pm} = 0$, and therefore:
$$[A(r)]^{\pm} = [r]^{\pm}$$
for all $r \in \mathbb{Q} \sqcup \{\infty\}$,

(vi) **Atkin-Lehner relation:** *Suppose $m \geq 1$ and write $N = ef$ where $f := \gcd(m, N)$. If $a, d \in \mathbb{Z}$, and $ade \equiv 1 \pmod{m}$, and w_e is the eigenvalue of the Atkin-Lehner operator W_e on f_E, then*
$$[d/m]^{\pm} = -w_e \cdot [a/m]^{\pm},$$

(vii) **Hecke relations:** *Suppose ℓ is a prime, and a_ℓ is the ℓ-th Fourier coefficient of f_E.*
 (a) *If $\ell \nmid N$, then $a_\ell \cdot [r]^{\pm} = [\ell r]^{\pm} + \sum_{i=0}^{\ell-1} [(r+i)/\ell]^{\pm}$.*

(b) *If* $\ell \mid N$*, then* $a_\ell \cdot [r]^\pm = \sum_{i=0}^{\ell-1} [(r+i)/\ell]^\pm$.

Proof. The proofs of (i)—(v) are evident. For (vi), here is a construction of the Atkin-Lehner operator W_e. Let $f = gcd(m, N)$ and $N = ef$. The W_e operator is given by (any) matrix of the following form:

$$W_e := \begin{pmatrix} ae & b \\ cN & de \end{pmatrix},$$

with $a, b, c, d \in \mathbf{Z}$ and $\det(W_e) = e$.

Let $c = m/f$. Then (since e and f are relatively prime) we can find a and b to make a matrix of the desired form, and then

$$W_e(\infty) = ae/cN = a/cf = a/m,$$

and (computing)

$$W_e(d/m) = \infty$$

Thus W_e takes the path $\{\infty, d/m\}$ to the path $\{a/m, \infty\}$. It follows that $[d/m] = -w_E[a/m]$ where w_E is the eigenvalue of W_e acting on the newform uniformizing E, and $ade \equiv 1 (\text{mod } f)$ (the latter because $\det(W_e) = e$).

The proof of (vii) is straightforward. $\square$

Remarks 1. (i) *A random example: For the elliptic curve* $E :=$ *"11a" (aka:* $X_0(11)$*) here are the values of* $[a/13]^+_E$.

$$[0]^+ = \mathbf{1/5},$$

$$[\frac{1}{13}]^+ = -4/5,$$

$$[\frac{2}{13}]^+ = \quad [\frac{3}{13}]^+ = 17/10,$$

$$[\frac{4}{13}]^+ = \quad [\frac{5}{13}]^+ = \quad [\frac{6}{13}]^+ = -4/5.$$

Note that $X_0(11)$ *has a rational point of order* 5*, so* $\delta = 5$*; hence the denominators of the values are multiples of* $2\delta = 10$*. If you believe BSD and the Shafarevitch-Tate conjecture the* $\mathbf{1/5}$ *already tells us (the true fact!) that the Mordell-Weil* *group of $X_0(11)$ is of order 5 and the Shafarevitch-Tate group is trivial.*

(ii) *SAGE conveniently computes modular symbols data (for a hint of this, see W. Stein's* `http://doc.sagemath.org/html/en/reference/modsym/sage/modular/modsym/modular_symbols.html`*).*

(iii) *Note that as in Chi-Yun's lecture, the computation of modular symbols via "Manin symbols'" follows the route of a continued fractions (i.e., Euclidean algorithm-type) reduction so one gets a qualitative ("log") upper bound for the size of modular symbols:*

$$|[a/m]^{\pm}| \ll \log(m).$$

(iv) *The Hecke relation vii(a) above applied to a prime $\ell \nmid N$, and $r = 0$ gives us*

$$(a_\ell - 1)\cdot[0]^+ = \sum_{i=0}^{\ell-1}[i/\ell]^+ = L(E,1)/\Omega^+ \tag{1.2}$$

and moving one "$[0]^+$" from right to left in 1.2 and switching sides we get:

$$\sum_{i=1}^{\ell-1}[i/\ell]^+ = (a_\ell - 2)\cdot[0]^+ \tag{1.3}$$

Proceeding similarly by induction, we have:

Proposition 1.4. *Let $m = \prod_{j=1}^{\nu}\ell_j$ be square free (the ℓ_j being distinct primes) and prime to N, then:*

$$\sum_{(i,m)=1;\ i\le m}[i/m]^+ = \prod_{j=1}^{\nu}(a_{\ell_j} - 2)\cdot[0]^+ \tag{1.5}$$

2. **Recall:** Modular symbols and L-values

Definition 2.1. Suppose χ is a primitive Dirichlet character of conductor m. Define the Gauss sum

$$\tau(\chi) := \sum_{a=1}^{m} \chi(a) e^{2\pi i a/m}$$

and, if $L(E, s) = \sum a_n n^{-s}$, the twisted L-function

$$L(E, \chi, s) := \sum_{n=1}^{\infty} \chi(n) a_n n^{-s}.$$

If $F/\mathbb{Q}$ is a finite abelian extension of conductor m, we will identify characters of $\mathrm{Gal}(F/\mathbb{Q})$ with primitive Dirichlet characters of conductor dividing m in the usual way.

Proposition 2.2. *If $F/\mathbb{Q}$ is a finite abelian extension, then*

$$L(E_{/F}, s) = \prod_{\chi:\mathrm{Gal}(F/\mathbb{Q})\to\mathbb{C}^\times} L(E, \chi, s).$$

Corollary 2.3. *If the Birch and Swinnerton-Dyer conjecture holds for $E_{/\mathbb{Q}}$ and $E_{/F}$, then*

$$\mathrm{rank}(E(F)) = \mathrm{rank}(E(\mathbb{Q})) + \sum_{\substack{\chi:\mathrm{Gal}(F/\mathbb{Q})\to\mathbb{C}^\times \\ \chi\neq 1}} \mathrm{ord}_{s=1} L(E, \chi, s).$$

Theorem 2.4 (Birch-Stevens)**.** *If χ is a primitive Dirichlet character of conductor m, then*

$$\sum_{a=1}^{m} \chi(a)[a/m]^{\epsilon} = \frac{\tau(\chi) L(E, \bar{\chi}, 1)}{\Omega_E^{\epsilon}}.$$

where the sign ϵ is equal to the sign of the character χ, i.e., $\epsilon = \chi(-1)$.

Remark: This also works for the *trivial primitive* character applied to the element $r = 0 \in \mathbf{P}^1(\mathbb{Q})$:

$$[0]^+ = L(E, 1)/\Omega_E^+. \tag{2.5}$$

(so the vanishing of $L(E, s)$ at $s = 1$ is equivalent to $[0]^+ = 0$).

3. **Recall:** θ-ELEMENTS AND θ-COEFFICIENTS

Definition 3.1. Suppose $m \geq 1$, and let $G_m = \mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_m)/\mathbb{Q})$. Identify G_m with $(\mathbb{Z}/m\mathbb{Z})^\times$ in the usual way, and let $\sigma_{a,m} \in G_m$ be the Galois automorphism corresponding to $a \in (\mathbb{Z}/m\mathbb{Z})^\times$ (i.e., $\sigma_{a,m}$ acts on $\boldsymbol{\mu}_m$ as raising to the a-th power). Define

$$\theta_m^\pm := 2\delta \sum_{a \in (\mathbb{Z}/m\mathbb{Z})^\times} [a/m]^\pm \, \sigma_{a,m} \quad \in \quad \mathbb{Z}[G_m].$$

If $F/\mathbb{Q}$ is a finite abelian extension of conductor m, so $F \subset \mathbb{Q}(\boldsymbol{\mu}_m)$, define the **$\theta$-element** (over F, associated to E) to be:

$$\theta_F^\pm := \theta_m^\pm|_F \;\in\; \mathbb{Z}[\mathrm{Gal}(F/\mathbb{Q})]$$

where $\theta_m^\pm|_F$ is the image of $\theta_m^\pm$ under the natural restriction homomorphism

$$\mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_m)/\mathbb{Q})] \to \mathbb{Z}[\mathrm{Gal}(F/\mathbb{Q})].$$

Note: We probably should denote θ_F as $\theta_{F/\mathbb{Q}}$ to emphasize that the base field here is $\mathbb{Q}$. An interesting project is to develop and possibly find algorithms for computing the analogous 'θ-elements,' $\theta_{F/K}$ for cyclic Galois extensions F/K where K is a more general number field. There is such a theory of 'θ-elements' (gotten by reverse-engineering the appropriate generalization of Theorem 2.4); and *conjecturally* $\theta_{F/K} \in \mathbb{Z}[\mathrm{Gal}(F/K)] \subset \mathbb{C}[\mathrm{Gal}(F/K)]$.

By Lemma 1.1(i) we have

$$\theta_F^\pm = \sum_{\gamma \in \mathrm{Gal}(F/\mathbb{Q})} c_{F,\gamma}^\pm \cdot \gamma \in \mathbb{Z}[\mathrm{Gal}(F/\mathbb{Q})] \tag{3.2}$$

where

$$c_{F,\gamma}^\pm = 2\delta \cdot \sum_{\substack{a \pmod m \\ \sigma_{a,m}|_F = \gamma}} [a/m]^\pm. \tag{3.3}$$

We will refer to the $c_{F,\gamma}^\pm \in \mathbb{Z}$ as *θ-coefficients*. Since we will most often be dealing with the 'plus'-θ-elements, we will simplify notation by letting $\theta_F := \theta_F^+$, $c_{F,\gamma} := c_{F,\gamma}^+$, and $\Omega := \Omega^+$. If F is a real field, then $\sigma_{-1,m}|_F = 1$, so

$$c_{F,\gamma} = 2\delta \cdot \sum_{\substack{a \in (\mathbb{Z}/m\mathbb{Z})^\times/\{\pm 1\} \\ \sigma_{a,m}|_F = \gamma}} [a/m]. \tag{3.4}$$

Remark: From the definition 3.3 we have

$$\sum_{\gamma\in\mathrm{Gal}(F/\mathbb{Q})} c_{F,\gamma} = 4\delta \cdot \sum_{i\in(\mathbb{Z}/m\mathbb{Z})^\times/\{\pm 1\}} [i/m]. \tag{3.5}$$

and therefore by Proposition 1.4, if $m = \prod_{j=1}^{\nu} \ell_j$ is squarefree and prime to N, we have:

$$\sum_{\gamma\in\mathrm{Gal}(F/\mathbb{Q})} c_{F,\gamma} = 4\delta \cdot \prod_{j=1}^{\nu}(a_{\ell_j} - 2) \cdot [0]^+. \tag{3.6}$$

Proposition 2.4 can be rephrased as follows:

Corollary 3.7. *Suppose $F/\mathbb{Q}$ is a finite real cyclic extension of conductor m and $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \twoheadrightarrow \mathrm{Gal}(F/\mathbb{Q}) \hookrightarrow \mathbb{C}^\times$ is a character that cuts out F. Then*

$$\bar\chi(\theta_F) = 2\delta \cdot \frac{\tau(\bar\chi)L(E,\chi,1)}{\Omega_E}. \tag{3.8}$$

4. θ-elements for cyclic field extensions of prime order

(i) Let χ be a character of order a prime number $p > 2$ and of squarefree conductor $m = \prod_{j=1}^{\nu} \ell_j$ relatively prime to N, so $p\,|\phi(m)$ and then (by Equation 3.2)

$$\bar\chi(\theta_F) = \sum_{\gamma\in\mathrm{Gal}(F/\mathbb{Q})} c^+_{F,\gamma} \cdot \bar\chi(\gamma) \in \mathbb{Z}[e^{2\pi i/p}].$$

If $\gamma_0 \in \mathrm{Gal}(F/\mathbb{Q})$ is a generator of the group $\mathrm{Gal}(F/\mathbb{Q})$, putting $\zeta_p := \bar\chi(\gamma_0)$ we can write the above equation as:

$$\bar\chi(\theta_F) \;=\; \sum_{i=0}^{p-1} c^+_{F,\gamma_o^i} \cdot \zeta_p^i \in \mathbb{Z}[e^{2\pi i/p}].$$

So, the vanishing of $\bar\chi(\theta_F)$ (equivalently: of $L(E,\chi,1)$) occurs if and only if all the $c^+_{F,\gamma}$ are equal. Recalling:

$$\sum_{\gamma\in\mathrm{Gal}(F/\mathbb{Q})} c_{F,\gamma} = 4\delta \cdot \prod_{j=1}^{\nu}(a_{\ell_j} - 2) \cdot [0]^+. \tag{4.1}$$

we get:

Proposition 4.2. *Let χ be a character of order a prime number $p > 2$ and of squarefree conductor $m = \prod_{j=1}^{\nu} \ell_j$ relatively prime to N, cutting out the Galois cyclic field extension $F/\mathbb{Q}$ (of order p) then the following are equivalent:*

(a) $\chi(\theta_F) = 0$,

(b) $L(E, \chi, 1) = 0$,

(c) *For all* $\gamma \in \mathrm{Gal}(F/\mathbb{Q})$

$$c_{F,\gamma} = \frac{4\delta}{p} \cdot \prod_{j=1}^{\nu} (a_{\ell_j} - 2) \cdot [0]^+. \tag{4.3}$$

(ii) **Example:** $E := X_0(11)$ **over** $\mathbb{Q}$

The Mordell-Weil group of E) is cyclic of order 5. So $\delta = 5$. Also (see Remark 1 above) we have $[0]^+_E = \frac{1}{5}$.

So if χ satisfies the hypotheses of Proposition 4.2, and if $L(E, \chi, 1) = 0$, Equation 4.3 above would read:

$$c_{F,\gamma} = \frac{4}{p} \cdot \prod_{j=1}^{\nu} (a_{\ell_j} - 2). \tag{4.4}$$

and since $c_{F,\gamma} \in \mathbb{Z}$ this would force

$$a_{\ell_j} \equiv 2 \mod p$$

for at least one $j = 1, 2, \ldots, \nu$. That is, we have the converse:

If χ satisfies the hypotheses of Proposition 4.2 then

$$a_{\ell_j} \not\equiv 2 \mod p, \text{ for all } j \leq \nu \implies L(E, \chi, 1) \neq 0.$$

Discuss the corresponding issues with Selmer.

5. The effect of the Atkin-Lehner involution on θ-coefficients

Definition 5.1. Suppose F is a finite real cyclic extension of $\mathbb{Q}$, let m be its conductor, and let $f = \gcd(m, N)$ where N is the conductor of E and assume that f is relatively prime to $e := N/f$. Let γ_F be the image of e under the map

$$(\mathbb{Z}/m\mathbb{Z})^\times \twoheadrightarrow \mathrm{Gal}(F/\mathbb{Q}).$$

Define an involution ι_F of the set $\mathrm{Gal}(F/\mathbb{Q})$ by

$$\iota_F(\gamma) = \gamma^{-1}\gamma_F^{-1}.$$

Recall that $\theta_F = \sum_{\gamma\in\mathrm{Gal}(F/\mathbb{Q})} c_{F,\gamma}\gamma$.

Lemma 5.2. *Suppose F is a finite real cyclic extension of $\mathbb{Q}$.*

(i) *We have*

$$c_{F,\gamma} = -w_e c_{F,\gamma'}$$

where $\gamma' := \iota_F(\gamma) = \gamma^{-1}\gamma_F^{-1}$ and where w_e is the eigenvalue of the Atkin-Lehner operator W_e acting on f_E.

(ii) *The fixed points of ι_F are the square roots of γ_F^{-1} in* $\mathrm{Gal}(F/\mathbb{Q})$, *so the number of fixed points is:*
- *one if $[F:\mathbb{Q}]$ is odd,*
- *zero if γ_F is not a square in* $\mathrm{Gal}(F/\mathbb{Q})$,
- *two if $[F:\mathbb{Q}]$ is even and γ_F is a square in* $\mathrm{Gal}(F/\mathbb{Q})$.

(iii) *If $\gamma = \iota_F(\gamma)$ and $w_e = 1$, then $c_{F,\gamma} = 0$.*

Proof. Assertion (a) follows from the Atkin-Lehner relations satisfied by the modular symbols (Lemma 1.1(iv)). Assertion (b) is immediate from the definition, and (c) follows directly from (a). □

Definition 5.3. If $F/\mathbb{Q}$ is a real cyclic extension, we say that $\gamma \in \mathrm{Gal}(F/K)$ is *generic*, (resp., *special*$^+$, resp., *special*$^-$) if $\gamma \neq \iota_F(\gamma)$ (resp., $\gamma = \iota_F(\gamma)$ and $w_e = -1$, resp., $\gamma = \iota_F(\gamma)$ and $w_e = 1$).

By Lemma 5.2(iii), if γ is special$^-$ then $c_{F,\gamma} = 0$.

Part 2. **Statistics of modular symbols, theta-elements, and L-values**

6. Distribution of modular symbols

The following fundamental result about the distribution of modular symbols was proved by Petridis and Risager (cf. (8.6) of [4]). For simplicity, we will formulate these results only if the conductor N of E is squarefree (even though their results are more general).

Definition 6.1. Let $\mathcal{C}_E := 6/\pi^2 \prod_{p\mid m}(1+p^{-1})^{-1}\cdot L(\mathrm{Sym}^2(E),2)$.

Theorem 6.2 (Petridis & Risager [4]). *As X goes to infinity the values*

$$\left\{ \frac{[a/m]^+}{\sqrt{\log(m)}} : m \leq X, a \in (\mathbb{Z}/m\mathbb{Z})^\times \right\}$$

approach a normal distribution with variance $\mathcal{C}_E$.

Numerical experiments led to the following conjecture. Denote by $\mathrm{Var}(m)$ the variance

$$\mathrm{Var}(m) := \frac{1}{\varphi(m)} \sum_{a \in (\mathbb{Z}/m\mathbb{Z})^\times} ([a/m]^+)^2$$

Conjecture 6.3. (i) *As m goes to infinity, the distribution of the sets*

$$\left\{ \frac{[a/m]^+}{\sqrt{\log(m)}} : a \in (\mathbb{Z}/m\mathbb{Z})^\times \right\}$$

converge to a normal distribution with mean zero and variance $\mathcal{C}_E$.

(ii) *For every divisor κ of the conductor N, there is a constant $\mathcal{D}_{E,\kappa} \in \mathbb{R}$ such that*

$$\lim_{\substack{m \to \infty \\ (m,N)=\kappa}} (\mathrm{Var}(m) - \mathcal{C}_E \log(m)) = \mathcal{D}_{E,\kappa}.$$

Note that Theorem 6.2 is an "averaged" version of Conjecture 6.3(i). Inspired by Conjecture 6.3, Petridis and Risager [5, Theorem 1.6] obtained the following result, which identifies the constant $\mathcal{D}_{E,\kappa}$ and proves an averaged version of Conjecture 6.3(ii).

Theorem 6.4 (Petridis & Risager [5]). *We continue to suppose that N is squarefree. For every divisor κ of N, there is an explicit (see* [5, (8.12)]*) constant $\mathcal{D}_{E,\kappa} \in \mathbb{R}$ such that*

$$\lim_{X \to \infty} \frac{1}{\sum_{\substack{m<X \\ (m,N_E)=\kappa}} \varphi(m)} \sum_{\substack{m<X \\ (m,N)=\kappa}} \varphi(m)(\mathrm{Var}(m) - \mathcal{C}_E \log(m)) = \mathcal{D}_{E,\kappa}.$$

(*Yesterday*) I also received a very new preprint by Junwong Lee and Hae-Sung Sun *Dynamics of Continued Fractions and Distribution of Modular Symbols* with a very different proof of these results. (I have the permission of the authors to put it on the course web-page—which I'll do.)

6.1. The 'irrelevant' nonrandomness of the modular symbols.

Remark 6.5. The modular symbols are not completely "random" subject to Conjecture 6.3. Specifically partial sums $\sum_{a=\alpha}^{\beta}[a/m]$ behave in a somewhat orderly way—even though it seems only to be the *full sum* that affects the statistics of θ-coefficients. Numerical experiments led the authors and William Stein to propose the following conjecture.

Conjecture 6.6. *If* $0 < x < 1$ *then*

$$\lim_{m\to\infty} \frac{1}{m}\sum_{a=1}^{mx}[a/m] = \sum_{n=1}^{\infty} \frac{a_n \sin(\pi n x)}{n^2 \Omega_E}$$

where $\sum_n a_n q^n$ *is the modular form* f_E *corresponding to* E.

This conjecture was recently proved for prime denominators by Kim and Sun [3, Theorem A].

Theorem 6.7 (Kim & Sun [3])**.** *If* $0 < x < 1$ *then*

$$\lim_{\substack{m\to\infty \\ m \text{ prime}}} \frac{1}{m}\sum_{a=1}^{mx}[a/m] = \sum_{n=1}^{\infty} \frac{a_n \sin(\pi n x)}{n^2 \Omega_E}.$$

7. Distribution of θ-coefficients

By (3.4), if γ is generic (resp., special$^+$) then the theta coefficient $c_{F,\gamma}$ is twice a sum of $\varphi(m)/(2[F:\mathbb{Q}])$ modular symbols (resp., four times a sum of $\varphi(m)/(4[F:\mathbb{Q}])$ modular symbols). If these were *randomly chosen* modular symbols, one would expect from Conjecture 6.3(i) that these coefficients would have a normal distribution with variance $2\mathcal{C}_E\varphi(m)\log(m)/[F:\mathbb{Q}]$ (resp., variance $4\mathcal{C}_E\varphi(m)\log(m)/[F:\mathbb{Q}]$).

However, calculations *do not* support this expectation. Instead, they support the following conjecture—which our hope is to eventually make a good deal more precise!

For every $d > 2$, let Σ_d denote the collection of data

$$\Sigma_d := \left\{ \frac{c_{F,\gamma}\sqrt{d}}{\sqrt{\varphi(m)\log(m)}} : F/\mathbb{Q} \text{ real, cyclic of degree } d, \right.$$
$$\left. m = \operatorname{cond}(F),\ \gamma \in \operatorname{Gal}(F/\mathbb{Q}) \text{ generic} \right\},$$

ordered by m. Let Σ_d^+ be defined in the same way, for γ special$^+$ instead of generic.

Conjecture 7.1. *For every $d \geq 2$, the collections of data Σ_d and Σ_d^+, ordered by m, have limiting distributions $\Lambda_{E,d}(t)$ and $\Lambda_{E,d}^+(t)$. As d grows, $\Lambda_{E,d}(t)$ (resp., $\Lambda_{E,d}^+(t)$) converges to a normal distribution with variance $2\mathcal{C}_E$ (resp., $4\mathcal{C}_E$).*

Question 7.2. *Is it the case that for $d \gg 0$ $\Lambda_{E,d}(t)$ and $\Lambda_{E,d}^+(t)$ are continuous bounded functions?*

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