Foreword

May 18, 2020

What a wonderful book this is! How generous it is in its tempo, its discussions, and in the details it offers. It heads off—rather than merely clears up—the standard confusions and difficulties that beginners often have. It is perceptive in its understanding of exactly which points might prove to be less than smooth sailing for a student, and how it prepares for those obstacles. The substance that it teaches represents a unified arc; nothing extraneous, nothing radically digressive, and everything you need to learn, to have a good grounding in the subject: *Number Fields.*

What makes Marcus' book particularly unusual and compelling is the deft choice of approach to the subject that it takes, requiring such minimal prerequisites; and also the clever balance between text-in-each-chapter and exercises-at-the-end-of-the-chapter: whole themes are developed in the exercises that fit neatly into the exposition of the book; as if text and exercise are in conversation with each other—the effect being that the student who engages with this text and these exercises is seamlessly drawn into being a collaborator with the author in the exposition of the material.

When I teach this subject, I tend to use Marcus' book as my principal text for all of the above reasons. But, after all, the subject is vast, there are many essentially different approaches to it[1]. A student—even while learning it from one point of view—might profit by being, at the very least, aware of some of the other ways of becoming at home with Number Fields.

There is, for example, the great historical volume *Hilbert's Zahlbericht* published originally in 1897[2]. This hugely influential treatise introduced generations of mathematicians to Number Fields, and was studied by many of major historical contributors to the subject, but also has been the target of criticism by André Weil[3]:"More than half of his [i.e., Hilbert's] famous Zahlbericht (viz., parts IV and V) is little more than an account of Kummer's number-theoretical work, with inessential improvements..." One could take this as a suggestion to go back to the works of Kummer—and that would be an enormously illuminating and enjoyable thing to do but not suitable as a *first* introduction to the material. Marcus, however, opens his text (Chapter 1) by visiting Kummer's

[1] I sometimes ask students to also look at Pierre Samuel's book *Algebraic Theory of Numbers*, Dover (2008), since it does have pretty much the same prerequisites and coverage that Marcus's book has but with a slightly different tone—a tiny bit more formal. No Exercises, though!

[2] The English translation: "The Theory of Algebraic Number Fields." Springer 1998; see especially the introduction in it written by F. Lemmermeyer and N. Schappacher: `http://www.fen.bilkent.edu.tr/~franz/publ/hil.pdf`)

[3] This is in Weil's Introduction to the works of Kummer. See the review published in BAMS: `http://www.ams.org/journals/bull/1977-83-05/S0002-9904-1977-14343-7/S0002-9904-1977-14343-7.pdf`

approach to Fermat's Last Theorem[4] as a way of giving the reader a taste of some of the themes that have served as inspiration of generations of mathematicians engaged in number theory, and that will be developed further in the book.

André Weil, so critical of Hilbert, had nothing but praise for another classical text—a text quite accessible to students, possibly more so than Hilbert's Zahlbericht—namely, "Lectures on the Theory of Algebraic Numbers" by Erich Hecke (The most recent publication of it: Springer 1981). This is the book from which I learned the subject (although there are no exercises in it). To mention André Weil again: "To improve on Hecke in a treatise along classical lines of the theory of algebraic numbers, would be a futile and impossible task."

The flavor of any text in this subject strongly depends on the balance of emphasis it places on *local concepts* in connection with the global objects of study. Does one, for example, treat—or even begin with—local rings and local fields, as they arise as completions and localizations rings of integers in global fields? Marcus' text takes a clear position here: it simply focuses on the global. This has advantages: students who are less equipped with algebraic pre-requisites can approach the text more easily; and the instructor can, at appropriate moments, insert some local theory at a level dependent on the background of the students[5].

There is the staggeringly important computational side of our subject, which is nowadays readily available to anyone who might want to explore and get closer to large pools of data of number fields, their discriminants, rings of integers, generators of the group of units, class groups. To have this capability of exploration is enormously helpful to students who wish to be fully at home with the actual phenomena[6]. And there are texts that deal with our subject directly from a computational point of view[7].

This book offers such a fine approach to our subject, and is such a marvelous guide to it. It also is an inviting introduction to many of the modern issues of extreme interest in number theory. For example:

- *Cyclotomic fields*[8], which Serge Lang once labeled "the backbone of number theory," is a continuing thread throughout the book, used throughout as a rich source of theory and examples including: Kummer's results (in the exercises for chapters 1 & 2), the Kronecker-Weber the-

[4]This famous theorem was proved by Andrew Wiles in 1994.

[5]For a text that deals with the local considerations at the outset, and that covers roughly the same material with slightly steeper, but still comparable, pre-requisites, see A. Fröhlich & M.J. Taylor *Algebraic Number Theory*, Cambridge University Press (1991). Excellent 'classic' texts having even more focus, at the beginning, on the local aspects—but requiring *much* more background—are

1. Cassels & Fröhlich, *Algebraic Number Theory*, Thompson (1967)
2. Serre, J.-P. *Local Fields*, Springer (1979).

[6]See the web-site `http://www.lmfdb.org/NumberField`.

[7]E.g.:

1. *A Course in Computational Algebraic Number Theory* by Henri Cohen (Springer 1993)
2. *Algebraic Number Theory: A Computational Approach* by William Stein (2012) `\http://wstein.org/books/ant/ant.pdf`.

[8]A *cyclotomic field* is a field generated over the field of rational numbers by a root of unity.

orem (in the exercises to chapter 4) and Stickelberger's criterion (in exercises for chapter 2). The Kronecker-Weber theorem (proved at the turn of the nineteenth century) asserts that any number field that is an *abelian*[9] Galois extension of **Q**, the field of rational numbers, is contained in a cyclotomic field. This theorem is a precursor of Class Field Theory (a theory that gives a description and construction of abelian extensions of number fields). And Class Field Theory is itself a precursor of a program—the 'Langlands Program'—intensely pursued nowadays whose goal is to construct a very far-reaching but intimate relationship between algebraic number theory and the representation theory of reductive algebraic groups.

- *The Geometry of Numbers* as in the classical result of Minkowski is used in fundamental theorems such as Dirichlet's Unit Theorem, and the finiteness of the class number.
- There is a fine array of explicit exercises regarding class number, including an excursion into *Gauss's class number one problem* (chapter 5).
- There is the corresponding introduction to *Analytic Number Theory*, treating zeta functions, regulators and the class number formula (chapters 6-7),
- and a first view of *Class Field Theory* (chapter 8).

But I should end my foreword and let you begin reading.

[9]Meaning: a Galois extension with an abelian Galois group.