OUR SEMINAR-COURSE *TRUTH*

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Contents

> *. . . came I into the world, that I should bear witness unto the truth.*

proclaimed Jesus to Pontius Pilate,[1] whose response was the taut unanswerable question:

"What is Truth?"

If Pilate went to the dictionary he would not have been enlightened:

- *truth* =the quality or state of being *true*;
- *true* = *accurate* or exact;
- *accurate*= *correct* in all details; exact;
- *correct* = in accordance with fact or *truth*.

But don't worry, all the most important words have circular definitions. And there are no satisfactory synonyms for *true*: try replacing the adjective in the phrase "*true* friend" by any of the putative synonyms above.

Deliberation, Reasoning, Assessment, Estimation (and in any of the many forms these activities take) all have a mission that could be labelled by that peculiar word *Truth*—a word that seems to take on a different shade of meaning each time it is used.

[1]according to the gospel of John

For example, *Truth* in law proceedings is a 'means' to achieve a sense of fairness, of justice; it's not exactly the 'end.' In the quotation above, Jesus used the phrase *bear witness to the truth*–which is natural enough since he was after all, in a judicial setting. Witnesses in courtrooms since the 13-th century are asked to take the oath that they will tell "the truth, the whole truth, and nothing but the truth." But such "wholeness" might well be compromised by the architecture of legal format—where certain specific facts–no matter how true—may be disallowed from being introduced into the proceedings; facts witnessed as hearsay are not allowed; and, in contrast, certain other issues may well be, by prior agreement, simply 'stipulated' (to be accepted as evidence… with no further argument for or against them). The nature of evidence, of course, is the key issue[2].

And then there is the overarching structure of legal fiction.

The aim of the course "PHIL 248: Truth" that I taught with Amartya Sen and Eric Maskin in the Fall semester of 2021 was to *live with* that word *Truth* this semester, to get a better understanding of its shades of meaning in our various areas of interest and experience and thought. By "our" I mean: the students and auditors, as well as my co-teachers and myself. This is the fifth seminar-course I've taught with Amartya Sen and Eric Maskin[3]. As described in the Syllabus, each of us dealt with specific aspects of this immense subject, with emphasis weighted by the interests, background, and preferences of the participants. These notes consist of a slightly edited version of the handouts and slides I presented at our seminar.

The three sessions that I chaired focused on philosophical, statistical and mathematical aspects as in sections 1,2,3 below.

[2]Regarding this, *Shadows of Evidence* `http://people.math.harvard.edu/~mazur/papers/Framing.comments.pdf` is a handout I distributed as 'introductory notes' to a seminar-course *The Nature of Evidence* that I taught at the Harvard Law School with Noah Feldman.

[3]Example: Here is my introductory write-up for the course we taught that was focused on '*Axiomatic Reasoning* `http://people.math.harvard.edu/~mazur/papers/Axiomatic-Reasoning.pdf`.

Part 1. A brief description of my three sessions

- Truth in the context of ancient Greek philosophy: Section 1
- Scientific Experiment and the Language of Statistics: Section 2
- Mathematics: Section 3

1. Truth in the context of ancient Greek philosophy

One shouldn't go overboard making etymological implications but... the Greek word for Truth is a-lethia (*un*-hidden)[4] with its distinctive *via negativa* feel. Many of the verbs that describe 'truth-seeking'' hint at some kind of backstory: *de*-liberate, *dis*-cover, *re*-cognize—all resonant with the sentiment expressed in Plato's *Phaedo*, that knowledge is *re*-collection (anamnesis)[5]

Aristotle in his books *Topics*, *Prior Analytics*, and *Posterior Analytics* (and also *Rhetoric*) establishes a setting, and sketches a format that is a basis for our quotidian activities—deliberation, discussion, argument, and communication of ideas—as we attempt to find some truth for ourselves, and persuade others of it.[6].

[4]For an engaging—easy to read—discussion of its use by the pre-Socratics—and by Plato and Aristotle (as well as by some moderns) read *Aletheia in Greek thought until Aristotle* by Jan Wolenski, Annals of Pure and Applied Logic **127** (2004) 339-360; *note:* You can get the article on line by Googling the title and author; I don't cite its url here since it's long; it would take up half a page.

[5]an-amnesis $\sim$ *un*-forgetting.

[6]Compare this discussion in Aristotle with the notion of *sensus communis.* This Latin phrase, taking over from Aristotle's *aisthesis koine* seems to have meant different things in different times, but in more contemporary sources it refers, vaguely, to the bedrock of common opinion, common judgment. It is what allows for a basis of discussion (e.g., as is what Descartes means by *le bon sens* in his 'Discourse on Method') or for what is commonly agreed upon (e.g., *taste* as in Section 40 of Kant's *Critique of Judgment*).

More to the point, a feeling that some viewpoint or opinion enjoys corroboration by the 'sensus communis' strongly reinforces our sense that it is true. So this is a fundamental notion to discuss, and to understand. But we won't have time to deal with the broad literature about it ; e.g.:

- Hans Gadamer's *Truth and Method*: Gadamer interprets G.B. Vico's *On the Study Methods of Our Time* appeal to the sensus communis as being a claim that 'abstract universality of reason' is not what "gives the human will its direction" but rather it is the "concrete universality represented by the community of a group, a people, a nation, or the while human race."—the point being that sensus communis is the protective bulwark of Truth for "the human sciences."

Logic as in the *Prior Analytics* and *Posterior Analytics* offers a framework within which we formulate our thoughts justifying statements we argue are true. It's, at the very least, the scaffolding for building such arguments and expressing such statements For example, in Book I of the *Prior Analytics*[7] Aristotle defines what he refers to as a *syllogism*:

> A syllogism is an argument (*logos*[8]) in which, certain things being posited, something other than what was laid down results by necessity because these things are so. (24b19-20)

Mathematical logic in its more contemporary dress is an offshoot of this.

There is not much hint in these particular books of Aristotle regarding some *transcendental element* to truth captured by these day-to-day activities. (Except for his discussion of *cause* [9] in the *Posterior Analytics* (Book II, Part 11).)

Aristotle's mission is naturally quite different, though, in the *Metaphysics* where he takes on the notion of *being,* of *substance*, and of *essence*[10]. But often, a business-like approach prevails in his writing, as in Book VI of the *Nichomachean Ethics*:

- Thomas Reid's *An Inquiry into the Human Mind: On the Principles of Common Sense* (1764)(Derek R. Brookes ed., Pennsylvania State Univ. Press 1997).
- The Third Earl of Shaftsbury's *Sensus Communis; An Essay on the Freedom of Wit and Humour* (1709) in CHARACTERISTICKS OF MEN, MANNERS, OPINIONS, TIMES 37 (Liberty Fund 2001).

[7]Aristotle's *Prior Analytics* Book I: Translated with an introduction and commentary by Gisela Striker (Clarendon Aristotle Series) 1st Edition.

[8]But see Stephen Read's commentary on the translation of the word *logos* as'argument' in this quotation: `https://www.st-andrews.ac.uk/~slr/The_Syllogism.pdf`

[9](aitia:) sometimes translated as simply: "explanation."

> "We think we have scientific knowledge when we know the cause,
> and there are four causes:
> (1) the definable form,
> (2) an antecedent which necessitates a consequent,
> (3) the efficient cause,
> (4) the final cause."

[10]which is a nice enough English term standing for the Latin *essentia*, a made-up word dating from the middle ages meant to translate the (wonderful, in my opinion) phrase that Aristotle himself used: *to ti esti*: *"the what is it?"*

> Let it be assumed that the states by virtue of which the soul possesses truth by way of affirmation or denial are five in number, i.e.,
> *technê, epistêmê, phronêsis, sophia, and nous;*
> we do not include judgement and opinion because in these we may be mistaken.

The Platonic and Aristotelian terms *episteme* and *techne*[11] on the one hand; *doxa*, *phronesis*[12] on the other—and *sophia* hovering above all—constitute a range of temperaments of thought.

Palimpsested onto this gamut of vocabulary are the four levels of imagination and thought[13]: *noesis, dianoia*: and then *pistis, eikasia* corresponding to the "divided line" in Book VI of Plato's *Republic*.

2. Scientific Experiment and the Language of Statistics

From the early 17th century (and Francis Bacon's *Novum Organum*) until now there has been lively debate as to what constitutes a scientific experiment, and how to derive conclusions from such experiments[14]. Bacon moves away from Aristotelian terminology (e.g., *syllogism*) and leans on the word *induction*:

> The syllogism consists of propositions; propositions of words; words are the signs of notions. If, therefore, the notions (which form the basis of the whole) be confused and carelessly abstracted from things, there is no solidity in the superstructure. Our only hope, then, is in genuine induction.

2.1. **Sensus communis.** This Latin phrase, taking over from Aristotle's *aisthesis koine* seems to mean different things each time anyone uses it, but in more contemporary sources it refers, vaguely, to the bedrock of common opinion, common judgment. It is what allows for a basis of discussion (e.g., as is what Descartes means by *le bon sens* in his 'Discourse on Method'). More to the point, a feeling that some

[11] Roughly: knowledge, theoretical and practical

[12] Roughly: common belief and practical wisdom

[13] Roughly, in descending order—*comprehension of principle* and *discursive reflection* (in the upper realm) and then: *confidential conjecture* and finally: *not particularly substantiated conjecture* (in the lower realm)

[14] For background see `https://plato.stanford.edu/entries/francis-bacon/#SciMetNovOrgTheInd`.

viewpoint or opinion enjoys corroboration by the 'sensus communis' strongly reinforces our sense that it is true. So this is a fundamental notion to discuss, and to understand. What's more, it brings together lots of interesting—I'm hoping— reading; among other things:

- Hans Gadamer's *Truth and Method*[15].
- Thomas Reid's *An Inquiry into the Human Mind: On the Principles of Common Sense* (1764)(Derek R. Brookes ed., Pennsylvania State Univ. Press 1997).
- The Third Earl of Shaftsbury's *Sensus Communis; An Essay on the Freedom of Wit and Humour* (1709) in CHARACTERISTICKS OF MEN, MANNERS, OPINIONS, TIMES 37 (Liberty Fund 2001).

More importantly, the concept of sensus communis ties in with Immanuel Kant's notion of the universal subjective—(in the *Critique of Judgment*); and slightly less explicitly with his *synthetic a priori*—(in the *Critique of Pure Reason*).
Which brings us to:

2.2. **The Language of Statistics.** There is a vast contemporary literature about Scientific Experiment and it's various formats—such as *natural experiment* and *randomized control experiment.* Discussion of this is central to our topic.

When we are given some numerical statement labeled a 'statistic' (from a source we trust) we feel—naturally—that this statement offers some truth about the underlying substance that gave rise to that statistic. But how to describe this type of truth?

The two distinct viewpoints presented by the "Frequentist" and the "Bayesians" offer a choice of attitudes toward the nature truth in statistics—and they provide us with distinct vocabulary to describe these attitudes.

How well we can understand the *truth* that specific statistics from a database conveys depends on how well informed we are about how

[15]Gadamer interprets G.B. Vico's *On the Study Methods of Our Time* appeal to the sensus communis as being a claim that 'abstract universality of reason' is not what "gives the human will its direction" but rather it is the "concrete universality represented by the community of a group, a people, a nation, or the while human race."—the point being that sensus communis is the protective bulwark of Truth for "the human sciences."

the data was initially collected. If it is a sample taken from a larger collection of instances, what exactly were the choices involved in that sample?

Even if it is intended to be the total collection, how accurate are the records —e.g., see the CDC discussion of this regarding the total number of deaths due to Covid-19 (they cautiously refer to these numbers as "latest provisional death counts") `https://www.cdc.gov/nchs/nvss/covid-19.htm#understanding-the-numbers`.

Then there is the question of how much *noise* there is is mixed in with the data (e.g., as in Nate Silver's *The Signal and the Noise*[16]) and how to deal with such noise.

And finally, the eternal question regarding *correlation* versus *causation.*

All these important questions deserve exploring.

3. Mathematics

We can discuss the nature of Truth in Mathematics without requiring that much technical background. The architecture of Truth, for mathematical thought, is *Proof* but there are issues that transcend proof —and this can be vividly seen in the history of the subject that surrounds, and depends upon, the simple question:

What is a set?

a question that has not been, in any way, definitely settled.[17] It will come as no surprise that the act of *appropriately defining* a concept plays a powerful role in setting up a usable arena within which mathematical truth might emerge.

And there are the various (traditional) attitudes regarding the connection of Mathematics to Truth—these come with the labels *formalism, constructivism, intuitionism,* and *mathematical platonism.*

[16] *The Signal and the Noise: why so many predictions fail—but some don't* The Penguin Press, New York (2012)

[17] Here is a recent interesting popular article about this: `https://www.quantamagazine.org/how-many-numbers-exist-infinity-proof-moves-math-closer-to-an-answer-20210715/`.

A chief requirement in mathematical statements is non-ambiguity[18], utterly clear labeling[19] (i.e., what is proved and based on what assumptions has it been proved; what is expected to be true but not yet proved, etc.) and, of course, consistency.

The question *What are 'axioms' and how can they be most effectively used to organize thought—to lead us to 'truth'?* is one that is most naturally dealt with in the context of Mathematics, even though this is so important an issue for so many other disciplines.

The subject "Mathematics" has always been the springboard for philosophical thought,

- from the ancients,
  as in the legendary sign at the portal of Plato's Academy: *mèdeis ageômetrètos eisitô mou tèn stegèn* —-i.e.: "let no one ignorant of geometry come under my roof."
- to the moderns,
  as in David Hume's wry comment that he observed, on examining the foundations of mathematics that:

  > the imagination, when set into any train of thinking, is apt to continue even when its object fails it, and, like a galley put into motion by the oars, carries on its course without any new impulse.
  > (*On Human Nature and the Understanding* IV.2)

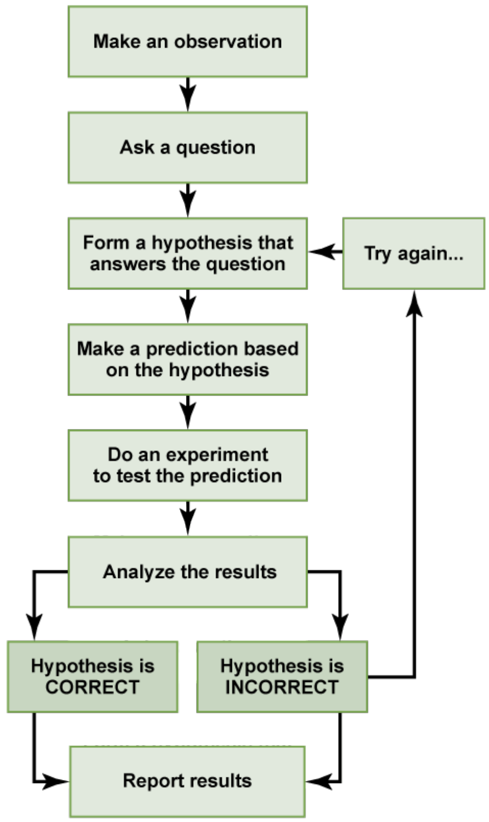
Or as in Kant's opening question in the *Critique of Pure Reason*: "How is Mathematics Possible?"—where, assuming that it is possible, he derives philosophical consequences. And in his notion of the *synthetic a priori* he examines the manner in which (what he calls) the intuitions are yoked to mathematical truth.

Part 2. A longer description of the three sessions

For the first session, *Truth in the context of ancient Greek philosophy* see the quotations from the assigned readings given in the file *Readings regarding Truth in the context of ancient Greek philosophy*

[18] or maybe one should more realistically say: a high level of non-ambiguity
[19] 'Truth in advertising'

4. Second Session: *Scientific Experiment and the Language of Statistics*



- We will continue one thread of Professor Maskin's discussion last week about the "conventional story about scientific progress:
- first, phenomenon to be explained is closely observed with many observations/experiments made
- on basis of observations—a hypothesis/model explaining observations proposed induction; then
- hypothesis makes additional predictions these are then checked
- *if they don't check out,* hypothesis modified continuing interplay between observation and theoretical work
- *if predictions are confirmed,* hypothesis elevated to status of theory, and stands as accepted explanation until perhaps anomalies arise and theory replaced. Like most conventional wisdom, story is too simple: actual science more complicated than that
- *but this is not terrible description of how a lot of science takes place...*"

4.1. **The initiation of the modern era of scientific experiment.** From the early 17th century (and Francis Bacon's *Novum Organum*; [1]) until now there has been lively debate as to what constitutes a scientific experiment, and how to derive conclusions from such experiments[20]. Bacon moves away from Aristotelian terminology (e.g., *syllogism*) and leans on the word *induction*:

> The syllogism consists of propositions; propositions of words; words are the signs of notions. If, therefore, the notions (which form the basis of the whole) be confused and carelessly abstracted from things, there is no solidity in the superstructure. Our only hope, then, is in genuine induction.

Readings 1. Francis Bacon—the short essay *Of Truth* [2].

This essay is dripping with Moral Schadenfreude. It begins, though, innocently enough:

> What is truth? said jesting Pilate, and would not stay for an answer. Certainly there be, that delight in giddiness, and count it a bondage to fix a belief; affecting free-will in thinking, as well as in acting. And though the sects of philosophers of that kind be gone, yet there remain certain discoursing wits, which are of the same veins, though there be not so much blood in them, as was in those of the ancients.

Readings 2. Francis Bacon–Novum Organum [1] (Book 1 XXXIX to LXII)—in preparation for a discussion about Bacon's four "idols": Idols of

- the *Tribe* (a sense of the truth common to humanity),
- the *Den* (the truth, given the comprehension, or sentiments of particular individuals—or groups of individuals)
- the *Market* (the truth as [a consequence of] our way of communicating with one another)
- the *Theatre* (the truth arising from ... "theory.")

4.2. **Types of Experiments.** There is a vast contemporary literature about Scientific Experiment and it's various formats—such as *natural experiment* and *randomized control experiment*. Discussion of this is central to our topic:

[20] For background see `https://plato.stanford.edu/entries/francis-bacon/#SciMetNovOrgTheInd`.

4.3. **The organization of the data base.** How well we can understand the *truth* that specific statistics from a database conveys depends—among other things—on how well informed we are about how the data was initially collected. If it is a sample taken from a larger collection of instances, what exactly were the choices involved in that sample?

Even if it is intended to be the total collection, how accurate are the records?

Consider, for example, the official disclaimers about COVID-19 Provisional Death Counts. . .

- Provisional death counts may not match counts from other sources, such as media reports or numbers from county health departments. Counts by NCHS often track 1-2 weeks behind other data.
- Death certificates take time to be completed. It takes extra time to code COVID-19 deaths. Most deaths from COVID-19 must be coded by a person, which takes an average of 7 days.
- States report at different rates. Currently, 63% of all U.S. deaths are reported within 10 days of the date of death, but there is significant variation between states.
- Other reporting systems use different definitions or methods for counting deaths. Death counts should not be compared across states. Some states report deaths on a daily basis, while other states report deaths weekly or monthlyState vital record reporting may also be affected or delayed by COVID-19 related response activities.

4.4. **Noise.** Then there is the question of how much *noise* there is mixed in with the data (e.g., as in Nate Silver's: *The Signal and the Noise: why so many predictions fail—but some don't.*)

Also take a look at: *Why Most Published Research Findings Are False* (by J. Ioannides)

And the eternal question regarding *correlation* versus *causation.*

—Mention Norvig-Chomsky debate—

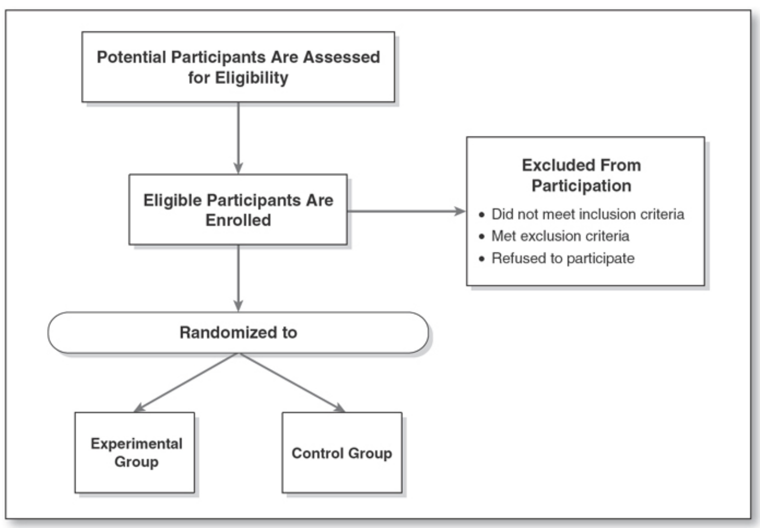
—discuss—

4.5. **Statistics and 'numbers'— e.g., $P \leq 0.05$.** *The American Statistical Association* offered a list of misconceptions among which are:

- $P > 0.05$ is the probability that the *null hypothesis* is true.
- $1 - P$ is the probability that the *alternative hypothesis* is true.
- A statistically significant test result ($P \leq 0.05$) means that the *test hypothesis* is false or should be rejected.
- A $P-$value greater than 0.05 means that no effect was observed.

The ASA panel (*after much discussion!*) defined the P value as something about the probability under a specified statistical model of a previously specified statistical summary of the data.

4.6. **Randomized Control Experiment.**



The discovery of (essentially) Vitamin C as being effective against scurvy was via an early (the first?) recorded Randomized Control Test by James Lind—in 1747— who was the surgeon to HMS Salisbury, a ship whose crew suffered an outbreak of the disease.

> On the 20th of May 1747, I selected twelve patients in the scurvy, on board the Salisbury at sea. Their cases were as similar as I could have them…

Lind then separated the twelve into six groups of two–giving different treatments to each group. One lucky group got oranges and lemons:

> The consequence was, that the most sudden and visible good effects were perceived from the use of oranges and lemons; one of those who had taken them, being at the end of six days fit for duty... The other was the best recovered of any in his condition; and... was appointed to attend the rest of the sick.

4.7. **Natural Experiment.** *Example taken from* [3]*:* In Helena, Montana a smoking ban was in effect in all public spaces, including bars and restaurants, during the six-month period from June 2002 to December 2002. Helena is geographically isolated and served by only one hospital. The investigators observed that the rate of heart attacks dropped by 40% while the smoking ban was in effect.

Opponents of the law prevailed in getting the enforcement of the law suspended after six months, after which the rate of heart attacks went back up. The study [4] was an example of a natural experiment, called a case-crossover experiment, where the exposure is removed for a time and then returned. The study also noted its own weaknesses which potentially suggest that the inability to control variables in natural experiments can impede investigators from drawing firm conclusions.

4.8. **Statistics and Experiment.** When we are given some numerical statement labeled a 'statistic' (from a source we trust) we feel—naturally—that this statement offers some truth about the underlying substance that gave rise to that statistic. But how to describe this type of truth?

The two distinct viewpoints presented by the "Frequentist" and the "Bayesians" offer a choice of attitudes toward the nature truth in statistics—and they provide us with distinct vocabulary to describe these attitudes.

How well we can understand the *truth* that specific statistics from a database conveys depends on how well informed we are about how the data was initially collected. If it is a sample taken from a larger collection of instances, what exactly were the choices involved in that sample?

Even if it is intended to be the total collection, how accurate are the records —e.g., see the CDC discussion of this regarding the total number of deaths due to Covid-19 (they cautiously refer to these numbers as "latest provisional death counts") `https://www.cdc.gov/nchs/nvss/covid-19.htm#understanding-the-numbers`.

Then there is the question of how much *noise* there is is mixed in with the data (e.g., as in Nate Silver's *The Signal and the Noise*[21]) and how to deal with such noise.

And finally, the eternal question regarding *correlation* versus *causation.*

4.9. 'Educating your beliefs' versus 'Testing your Hypotheses'.

'Bayesian intertwining'

The naive view of an empirical investigation which we might call the **straight Baconian model** for a scientific investigation has, as we have discussed, the simple recipe:

$$\textbf{Set-up and Hypotheses} \longrightarrow \textbf{Data Collecting} \longrightarrow \textbf{Processing Data and Conclusion.}$$

The manner in which one proceeds from data to conclusion is often understood to be a straight comparison of what the hypotheses would predict and what the data reveals[22], the comparison being (usually) quantitative with a pre-specified tolerance of discrepancy (between prediction and observation).

All this is significantly modified by the Bayesian viewpoint, which methodically intertwines the first two steps, and has a different take on each of these ingredients: hypothesis, data, conclusions. We'll discuss this below[23]. We'll look at the Bayesian viewpoint as offering a 'model' to help us understand, and deal with, the interplay between those ingredients. Let's call it the **Bayesian model** for a scientific investigation.

A further issue that complicates the contrast of *models of getting to scientific conclusions* alluded to above is the difference between the

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[21] *The Signal and the Noise: why so many predictions fail—but some don't* The Penguin Press, New York (2012)

[22] although it might be difficult to find this expressed in Bacon's writings as bluntly

[23] A disclaimer: I know very little statistics; I'm a total outsider to this field and especially to the extended conversation—and the somewhat sharp disagreements—that Bayesians and Frequentists have.

Bayesian's and the Frequentist's work; their methods are not the same, and they have slightly different *primary goals*.

We'll get to that, eventually.

4.10. **Prior information and the Birthday problem.** To introduce ourselves to this 'Bayesian intertwining' (taking as a **black box**—at least at first—some of the mathematical procedures involved) let's revisit a famous problem: the birthday problem. You have a class of fifth graders in an elementary school. Suppose there are 23 students in the class. What is the probability that two of them have the same birthday? Or, to seem more mathematical, suppose there are n students. What is the answer as a function of n?

Here is the simple naive analysis of this problem. We assume, of course, that the probability of anyone having a birthday at any specific day, e.g., April 22, is 1/365 (ignoring the leap year issue). Think of the teacher marking off—successively— on a calendar the birthdays of each student. We are going to gauge the possibility that in his class of n students there are no two birthdays on the same calendar day. The first student's birthday is duly marked. We can't possibly have a concurrence of birthdays (call it a *hit*) at this point, there being only one mark. So we can record "1" as the probability that we didn't get a *hit* at least so far[24].

As for the second student, the probability of him or her not having a birthday on the same day as student #1—i.e., that there not be a *hit*— is

$$1 - \frac{1}{365} \;=\; \frac{364}{365}.$$

Given this situation, and passing to the third student, in order for there not to be a hit, his or her birthday has to avoid two days, so that probability is

$$1 - \frac{2}{365} \;=\; \frac{363}{365}.$$

[24]We are going to write probabilities as numbers between 0 and 1. So if the probability of an event is $\frac{1}{2}$ that's the same as saying that it is *even odds of it happening or not happening* or that 50% *of the time it happens*, or one sometimes simply says that there's is a 50/50 chance of it occurring.

Putting the two probabilities together we get that–so far in our count—the probability that there isn't a hit with these three students is

$$(1 - \frac{1}{365})(1 - \frac{2}{365}) \quad = \quad (\frac{364}{365}) \cdot (\frac{363}{365}).$$

Working up (by mathematical induction) the probability that there's no hit, with n students is then:

$$(1 - \frac{1}{365})(1 - \frac{2}{365}) \cdots (1 - \frac{n-1}{365}),$$

which when $n = 23$ is close to $\frac{1}{2}$. That is, for a class of 23 students the chances are 50/50 that there's a concurrence of birthdays—given this analysis.

My Bayesian friend Susan Holmes tells me that she has actually tried this out a number of times in real live classes, and discovered that the odds seem to be much better than 50/50 for 23 students; you even seem to get 50/50 with classes of as low as 16 students.

There is something too naive in the analysis above, says Susan. We should, at least, make the following (initial) correction to our setting-up of the problem. We said above:

> We assume, of course, that the probability of anyone having a birthday at any specific day, e.g., April 22, is 1/365

BUT we actually *know* stuff about the structure of our problem that we haven't really registered in making that assumption.

For example, it is a class of fifth-graders so, chances are, they were all (or mostly) born in the same year. In particular, the years of their birth all (or mostly) had the same weekends and weekdays. In the era of possible c-sections and induced births—given that doctors and hospital staff would prefer to work on weekdays rather than weekends—one might imagine that the probability of being born on a weekday is somewhat skewed. We also know more that might make us think that fixing 1/365 at the rate is too naive.

Perhaps then, instead of sticking to the probability $p = 1/365$ per day hypothesis, allow a bit of freedom and a priori allow that there are different probabilities

$$p_1, p_2, p_3, \cdots, p_{365}$$

for each day of the year[25], about which we can make very very rough guesses. But let us not write this in stone yet. Make a mildly educated guess of these p_i; e.g., if "i" is a Saturday or Sunday (or a holiday), then p_i is probably slightly less than $1/365$; if a weekday, slightly more. This initial guess (of the values of $p_1, p_2, p_3, \cdots, p_{365}$) we'll call a **Prior**. From any prior we can deduce—essentially by a straight computation as we did above with the "constant prior: $1/365$"—all the expected odds and whatever statistics one wants.

BUT we have hardly gotten our best answer! All these p_i's constituted, after all, just our very very rough guess based on some intuitive hunch, prior to having any hard data. Computing with these p_i's gives us a "number" as output—perhaps more accurate than the 23 we started this discussion with, but how does this number compare with the actual numbers we're actually accumulating by sampling birthday statistics for classes of fifth-graders? The Bayesian will use this accumulating *Data* to "correct" the prior (guessed) probabilities p_i, to be more in tune with the data. This is what I mean by the Bayesian intertwining: the data–as it comes in–is used to "educate the prior." This educated-prior is called (naturally) a **posterior**. In some sense, the principal role of data in this Baysian model is *to be fed back into the prior to refine it* to produce successive posteriors rather than (with a straight up or down judgment) to verify or contradict an hypothesis.

Starting anew with the latest **posterior** rather than the original **prior** we can deduce—essentially as we did above with the "constant $1/365$" or any prior **prior** —all the expected odds and whatever statistics one wants.

In fact, there are no firm hypotheses within the Bayesian model, and no firm conclusions. I said, though: "*within* the Bayesian model." Nevertheless from this procedure one might extract a conclusion, but this is outside the format.

This is a preliminary move in the Bayesian direction, but we aren't quite there yet. Another–and better–way of viewing this move (reflecting our most up-to-date version of belief about the set-up) is that the

[25]these summing to 1

initial values

$$p_1, p_2, p_3, \cdots, p_{365}$$

should not be taken as hard unchangeable numbers but rather are to be viewed as "random variables" in their own right, and subject to their own distributions, which we are bent on determining, given enough **Data**. The grand function of the data is to be fed back to educate the prior but retaining its status as probabilities. The movement here is as follows:

$$\text{Prior (probabilities)} \quad \overset{\textbf{Data}}{\longrightarrow} \quad \text{Posterior (probabilities).}$$

The **black box**—so far—is that I have not yet said anything about the mathematical procedure Bayesians use to feed back (as an after-burner) information obtained by the Data into the prior assumptions, in order to effect the "education" of these prior assumptions and thereby produce the posterior. For the moment—in this discussion—it is more important for me simply to emphasize that *whatever this procedure is* it is, in fact, a *predetermined procedure.*

4.11. **Predesignation versus the self-corrective nature of inductive reasoning.** Now you might well worry that this Bayesian ploy is like curve-fitting various hypotheses[26] to the data—a kind of hypothesis-fishing expedition, if you want. You keep changing the entire format of the problem, based on accumulating data. The Bayesians have, as I understand it, a claim: that any two 'reasonable' priors, when "corrected" by enough data will give very close posteriors. That is, the initial rough-hewn nature of the prior will iron out with enough data. Their motto:

Enough data swamps the prior.

I've been playing around with another formulation of that motto:

[26]I want to use the word *hypothesis* loosely, for the moment; that is, the way we generally use the word; and not in the specific manner that statisticians use it.

> *Any data-set is, in fact, a 'data point' giving us information about the probability distribution of priors.*

In contrast, there is a motto that captures the sentiment of a Frequentist:

> *Fix hypotheses. This determines a probability distribution to be expected in the data. Compute data. If your hypotheses are good,* **in the limit** *the data should conform to that probability distribution.*

About the above, one of the early great theorizers in this subject (and specifically regarding probability, randomness, and the law of large numbers) was Jacob Bernoulli. He *also* was a theologian preaching a specifically Swiss version of Calvinism. You see the problem here! There is a strict vein of *predetermined* destiny or fatalism in his theology, someone who is the father of the theory of randomness. How does he reconcile these two opposites? Elegantly, is the answer! He concludes[27] his treatise *Ars Conjectandi*, commenting on his law of large numbers, this way:

> Whence at last this remarkable result is seen to follow, that if the observations of all events were continued for the whole of eternity (with the probability finally transformed into perfect certainty) then everything in the world would be observed to happen in fixed ratios and with a constant law of alternation. Thus in even the most accidental and fortuitous we would be bound to acknowledge a certain quasi necessity and, so to speak, fatality. I do not know whether or not Plato already wished to assert this result in his dogma of the universal return of things to their former positions [apokatastasis], in which he predicted that after the unrolling of innumerable centuries everything would return to its original state.

[27] It is, in fact, the conclusion of the *posthumously* published treatise (1713) but it isn't clear to me whether or not he had meant to keep working on the manuscript.

Apokatastasis is a theological term, referring to a return to a state before the fall (of Adam and Eve)[28].

4.12. **Priors as 'Meta-probabilities'.** Suppose you are a cancer specialist studying a specific kind of cancer and want to know if there is a gender difference: do more men than women get this type of cancer? Or more women than men?

Now suppose I asked you (cancer specialist) to make some kind of guess—when considering groups of people that get this cancer—about the proportion of men-to-women that get it. You might tabulate this as a probability P that a random choice of person in this group is male. So P is a number between 0 and 1. You might actually give me a number if you are very confident, but more likely, for a spread of possible values of P, you'll give me an estimate of greater or lesser levels of confidence you have that this P is indeed the sought-for-probability. Taking the question I asked more systematically, you might interpret it as follows:

> As P ranges through all of its possible values, from 0 (no males get it) to 1 (only males get it) tell me (your guess of) the probability that P is the ratio $\frac{M}{M+W}$ where M is the number of men and W the number of women in the group? In effect, draw me a graph telling your

[28]Noah Feldman once suggested to me that Calvinists might be perfectly at home with random processes leading to firm limiting fatalism, in that the fates of souls—in Calvinist dogma—are *randomly assigned* and not according to any of their virtues; i.e., to misquote someone else: "goodness had nothing to do with it."

Also, we might connect the above with C.S. Peirce's 1883 paper "A Theory of Probable Inference." For a readable discussion of this paper, see: Len O'Neill's *Peirce and the Nature of Evidence* published in the Transaction of the Charles S. Peirce Society **29** Indiana Univ. Press (1993) pp. 211-224. Peirce makes a distinction between *statistical deduction* and *statistical induction* the first being thought of as reasoning from an entire population to a sample, and the second being reasoning from sample to population. As O'Neill says, in the first it is a matter of long run frequency (i.e, the Frequentist's motto) whereas the second is related to a Peircean conception of *the self-corrective nature of inductive reasoning* (and this sounds like the Bayesian protocol).

Peirce dwells on the issue of *predesignation* in the Frequentist's context (i.e., you fix a model and then collect evidence for or against it; you don't start changing the model midstream in view of the incoming evidence). As already mentioned, there is a curious type of *meta-predesignation* in the Bayesian context, in that the manner in which you change the model, given incoming evidence, is indeed pre-designated.

> probability-estimate for each of the P's in the range between 0 and 1.

Your initial guess, and initial graph, is the Prior (I privately call it the *meta-probability*). It *will* be educated by the data accumulating.

Let's imagine that you say "I have no idea! This probability P could–as far as I know–equally likely be any number between 0 and 1." If so, and if you had to draw a graph illustrating this noncommittal view, you'd draw the graph of a horizontal line over the interval $[0, 1]$. Or, you might have some reason to believe that P is close to $1/2$ but no really firm reason to believe this and you might have no idea whether gender differences enter at all. Then the graph describing your sense of the likelihood of the values of P would be humped symmetrically about $P = 1/2$. Or if you are essentially certain that it is $1/2$ you might draw it to be symmetrically spiked at $P = 1/2$.

What you are drawing is–in a sense–a *meta-probability density* since you are giving a portrait of your sense of how probable you think each value between 0 and 1 might be the actual probability-that men-get-this-type-of-cancer. Your portrait is the graph of some probability density function $f(t)$.

There are theoretical reasons to suggest, for some such problems, that you would do well to be drawing the graphs of a specific well-known family called **beta-distributions**. These beta-distributions come as a two parameter family[29] $\beta_{a,b}(t)$. That is, fix any two positive numbers a, b (these numbers a, b are called the *shape parameters* of the beta-distribution) and you get such a graph.

Here are some general ground-rules for choosing these βs: shape parameters that are equal give distributions symmetric about $1/2$; i.e., you choose such a β if you expect that gender plays no role in the probability of contracting this cancer. Choosing $a > b$ means that you are skewing things to the left; i.e., you believe that men get this type of cancer less frequently than women; choosing $b > a$ means the reverse. The larger these parameters, the sharper the peak of the curve; i.e., the more "sure" you are that the probability occurs at the peak.

Choose parameters, say, $a = 2, b = 5$; or, say, $a = 2, b = 2$ and you have probability distributions $\beta_{2,5}(t)$, or $\beta_{2,2}(t)$, these being the blue and the magenta graphs in the figure below (taken from a wonderful Wikipedia entry: `http://en.wikipedia.org/wiki/Beta_distribution`).

c

[29]These are distributions $t^{a-1}(1-t)^{b-1}dt$ normalized to have integral equal to 1 over the unit interval.

4.13. **Back to our three steps.**

(1) (**Choosing the Prior**) Now, Bayesian cancer doctor that you are, when you start doing your statistics, choose a Prior. For this type of question you might do well, as I said, to choose some beta-distribution. If you imagine that there might be a gender bias here, but have no idea in which direction, you might choose one that is symmetric about $t = 1/2$ (which, as it turns out, means that you'd be taking shape parameters a equal to b). But size up the situation as best as you can, taking into account everything that you think is important to the problem and come up with a choice of a Prior. Let us say that your Prior is $\beta_{a,b}(t)$.

(2) (**The Data**) Suppose you now get a data sample of 100 people with cancer—perhaps the result of some specific study of some particular population, and suppose that 60 of these cancer victims are men (so 40 are women).

(3) (**Passing to the Posterior**) The beauty of the family of beta-distributions is that when you appropriately *educate* a beta-distribution (the Prior) with new data, the new distribution (the Posterior) is again a beta-distribution. The only thing is that the shape parameters may change; say, from (a, b) to a new pair of numbers (a', b'):

$$\beta_{a,b}(t) \quad \stackrel{\text{new data}}{\longrightarrow} \quad \beta_{a',b'}(t)$$

I'm told that this change can be very easily computed. That is, in this example problem, the a', b' will depend on hardly more than the original a, b, the percentage of men with cancer, and the size of the study.

4.14. **A numerical example and a question.** For this example I'm normalizing things so the numbers work simply so we don't get bogged down in mere arithmetic. Imagine that your Prior is $\beta_{20,20}$ and you test a sample population (of just the right size for the normalizations to work out as I'm going to assume they do below) and in that population Men/ Women cancer ratio is 60/40. The Posterior is then (I'm told) $\beta_{20+60,20+40}$. And if you compute (based on that Posterior) the probability that men get this type of cancer more than women, that probability is:

$$0.955\ldots$$

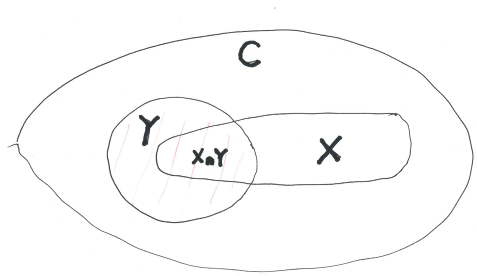
If you did the analogous thing with the Prior $\beta_{10,10}$, getting, as Posterior, $\beta_{10+60,10+40}$ you'd compute (based on that Posterior) the probability that men get this type of cancer more than women to be:

$$0.966\ldots$$

Question: Why is it *reasonable* that the second estimate of probability of gender-difference be bigger than the first?

4.15. **Bayes' Theorem.** I will assume that people have learned something of the background of Bayes himself from the other readings, and just concentrate on the statement and intent of his theorem. (A discussion of his theorem often is the *start* of expositions on Bayesian things, but it seems to me that one needs some of our prior discussion if we want (not merely to understand the theorem, but) to focus on its effect: a distinctive model of the role of statistical inference in the formation of scientific conclusions.

To begin, imagine that we have a region C of the Euclidean plane (say, an open compact subset) and two (open) subsets of C, $X, Y \subset C$, so that we also may consider their intersection $X \cap Y$ viewed as a subset of X and of Y.



Letting the absolute value sign $|\;|$ indicate area, we have the tautology:

$$(*) \qquad \frac{|X \cap Y|}{|Y|} \cdot \frac{|Y|}{|C|} \; = \; \frac{|X \cap Y|}{|X|} \cdot \frac{|X|}{|C|}.$$

This is evident.

Now, interpret C as a community of individuals or entitities, and X, Y as the sub-communities of C consisting of individuals that have specific traits (call them, respective, 'trait x' and 'trait y'). View 'area' as giving numbers of individuals.

Then:

- we can think of $\frac{|X \cap Y|}{|Y|}$ as the *probability that an individual in the community C has trait x, given that it has trait y.* This is usually abbreviated: $P(x \mid y)$.
- we can think of $\frac{|X \cap Y|}{|X|}$ as the *probability that an individual in the community C has trait y, given that it has trait x.* This is usually abbreviated: $P(y \mid x)$.
- we can think of $\frac{|X|}{|C|}$ as the *probability that an individual in the community C has trait x, abbreviated as $P(x)$,* and
- we can think of $\frac{|Y|}{|C|}$ as the *probability that an individual in the community C has trait y, abbreviated as $P(y)$.*

Rewriting the tautology **(*)** above in terms of these probabilities "P" we have:

$$(**) \quad P(x \mid y) \cdot P(y) \; = \; P(y \mid x) \cdot P(x).$$

interpreted as:

> The conditional probability that an individual in the community C has trait x, given that it has trait y *times* the probability that an individual of the community has trait y
>
> is equal to
>
> the conditional probability that an individual in the community C has trait y, given that it has trait x *times* the probability that an individual of the community has trait x.

This is Bayes' Theorem, which is sometimes written:

$$(***) \quad P(x \mid y) \;=\; \frac{P(y \mid x) \cdot P(x)}{P(y)}.$$

So, what does this theorem have to do with the discussion we've given in the previous sections of this handout? The answer (in my opinion) has two prongs.

- I think that the more important connection that Bayes' Theorem has to the general Bayesian viewpoint is that the theorem is a 'promissory note,' so to speak, that *conditional probabilities*—i.e., probabilities based on conditions that express what things we know about the situation—will be our vocabulary, and we have the beginnings of a way of dealing with conditionality.

- But it also has a 'straightforward' type of application. Here's one of enumerably many such simple examples (all below gotten from Wikipedia).

  Suppose you are an entymologist dealing with a species of beetles and there is a rare subspecies of beetle, usually identifiable because of a certain pattern on its back. I say, usually, but not always. Here is what you know:

  - the probability[30], given that you have an individual of the rare subspecies, that the pattern occurs on its back is 98%. I.e.,
  $$P(Pattern|Rare) = 98\%,$$
  and
  - the probability, given that you have a 'common' individual of the species, that the pattern occurs on its back is 5%. I.e.,
  $$P(Pattern|Common) = 5\%.$$

  Moreover, the rare subspecies accounts for only 0.1% of the entire population of this species of beetle.

  Now you capture a beetle with the pattern on its back. What is the probability that it is a member of the rare subspecies? That is, what is $P(Rare|Pattern)$? Bayes Theorem comes to the rescue.

[30] I'll give probabilities here in terms of percentages.

For further concrete applications, see http://en.wikipedia.org/wiki/Bayes'_theorem which is the first (and probably the most useful) of the many hits you get when you Google "Bayes' Theorem."

To discuss...

4.16. **The nature of Truth in the vocabulary of Statistics.**

References

[1] Francis Bacon, Novum Organum (ed. by Joseph Devey) M.A. (New York: P.F. Collier, 1902): https://oll.libertyfund.org/title/bacon-novum-organum
[2] Francis Bacon, Of Truth http://essays.quotidiana.org/bacon/truth/
[3] Wikipedia: Natural Experiment https://en.wikipedia.org/wiki/Natural_experiment
[4] Sargent, R. P.; Shepard, R. M.; Glantz, S. A. (2004). "Reduced incidence of admissions for myocardial infarction associated with public smoking ban: before and after study". British Medical Journal. 328 (7446): 977-980—the abstract: https://pubmed.ncbi.nlm.nih.gov/15066887/

5. For my third session: *Shades of Mathematical Truth*

I'm pretty sure that Immanuel Kant would put the mathematical statement

$$2 + 1 = 3 \tag{1}$$

in the category of assertions that he calls the *analytic a priori.* I know that he views the statement

$$5 + 7 = 12 \tag{2}$$

as quite different!—he refers to the latter equation in his *Critique of Pure Reason* as an example of his notion: the *synthetic a priori*[31].

[31]The word 'priori' signals that these categories are genres of thought *prior* exercised without (i.e., *prior to*) considering things other than the circle of the faculties of our mind—e.g. prior to input coming from perceptions, or more generally, from "the world"—all that being *a posteriori.*

The synthetic a priori, for Kant, requires some engagement with our *intuitions*—as he calls them—space and time, these being not "out there," as one might naturally think but rather, as Kant would have it, resources of our faculties of mind: resources in which we dress objects of thought so as to be able to properly think about them.

The former counts as *analytic a priori* in the sense that the left-hand side of Equation (1) is simply the definition of the right-hand side. In effect its a tautology.

But to understand Equation (2) you need to organize a structure of attack; e.g., in some form or other you will have incorporated (as part of your apparatus of comprehension) a version of an associative law. You need to go beyond the mere statement: you shape a mental strategy, satisfactorily go through the exercise of thought it leads you to do and... understand something.

So Equations (1) and (2) relate to different shades, if not grades, of truth.

An elementary example of *synthetic a priori*—that might capture the imagination more than Equation (2) above—is given by Proposition 32 of Book 1 of Euclid's Elements: the sum of the angles of a triangle is 180^o:

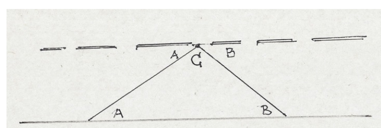


FIGURE 1

The difference with the previous example is that a *construction*—a 'synthetic act'— launches the standard proof of this theorem. You first choose a side of the triangle as base; then draw the line parallel to that base, through the opposite vertex; and then note that the three angles (A, C and B as in the above figure) at the vertex opposite the chosen base are, in fact, equal to the three angles of the triangle. (Here you use a previous result that guarantees that for a line L that cuts a pair of parallel lines, the opposite angles created at the points of intersection of L with each of the parallel lines are equal).

There is another major difference between this proposition and Equation (2). Namely, appropriately interpreted[32], this proposition actually *characterizes* the Euclidean plane (among all smooth not-necessarily-Euclidean 'planes'). In a sense, the proposition can be taken to offer a *definition* of the Euclidean plane. (The Pythagorean Theorem—appropriately understood— also characterizes the Euclidean plane.)

[32]e.g., 'straight lines' are taken to be geodesics

5.1. *Definitions* **as conveyors of truth.** The essential roles that 'definition' play for us are: to delineate the objects of interest to be studied; to encapsulate; to abbreviate; and to focus.

- Definitions, then, can range from lowly tautologies (e.g., Equation (1)) to illuminating examples of pieces of mathematics that Kant would call synthetic a priori (e.g., Proposition 32).

- *Defining Infinity:* Here is an exercise (for people with math backgrounds): what is the difference between these four possible definitions of *infinite set*? (The first is due to Dedekind and resonates with that corridor in *Hilbert's Hotel*):

  Definition ∞_1 : A set S is **infinite** if there exists an *injective* mapping $f : S \to S$ that is not surjective[33] (equivalently: is not a one:one correspondence between the set S and itself).

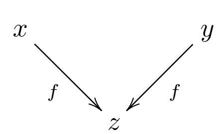
  .

  Definition ∞_2 : A set S is **infinite** if there exists a *surjective* mapping $f : S \to S$ that is not injective (equivalently: is not a one:one correspondence between the set S and itself).

  Definition ∞_3 : A set S is **infinite** if there exists an injective mapping of the set $\mathbf{N}$ of natural numbers into S.

  (The set of natural numbers is what you think it is: $\mathbf{N} := \{1, 2, 3, 4, \ldots\}$, even though the ancients were dubious about the number 1 as being in the same category as the other whole numbers. To actually *define*[34] this set $\mathbf{N}$ without making use of

[33]A mapping $(f : S \to T)$ from a set S to a set T is called **injective** (synonym: "**one-one into**") if for any two different elements $x \neq y \in S$ their images under the mapping f are also different; i.e., $f(x) \neq f(y) \in T$. That is, there is no collapsing:



A mapping $f : S \to T$ is **surjective** (synonym: "**onto**") if every element $z \in T$ is in the image of S under the mapping f; i.e., there exists an element $x \in S$ such that $f(x) = z$.

[34]And here, the late middle English sense of the word 'define' (to *bring to an end*) fits neatly.

the *dot-dot-dots* requires some apparatus—e.g., mathematical induction).

Definition ∞_4 : A set S is **infinite** if there exists an surjective mapping of the set S onto $\mathbf{N}$.

5.2. *Definition or Characterization?* Within the appropriate axiomatic set-theoretic context, the four definitions of "infinite set" are equivalent, so we have a choice:

- We can choose one of them as our primary definition, and the other three can be thought of as 'characterizations' of the then-defined concept—infinite set.
  item We can simply say: these are all equivalent and any one can serve as "the" definition.

The relationship between these choices depend on the ambient axiomatic context in which are working. For example, if you accept the 'Axiom of Choice' then if a set is infinite following Definition ∞_2 it is also infinite following Definition ∞_1.

The question, then, (*What is an infinite set?*) depends on the choice: definition versus characterization. The same holds for:

5.3. *What is a Prime Number?*

As for the power of definition to provide 'focus,' consider the two equivalent definitions of prime number (given by (1) and (2) below)—where one is left to make the choice of regarding one of these as 'definition' and the other as 'characterization':

A prime number p is a (whole) number greater than one

(1) that is not expressible as the product of two smaller numbers.

or

(2) having the property that if it divides a product of two numbers, it divides one of them.

If you choose (2) as the fundamental definition you are placing the notion of prime number in the broader context of 'prime'-ness as it applies to number systems more general than the ring of ordinary numbers—and more specifically in the context of *prime ideals* of a general ring. So choosing (2) as definition casts (1) as a specific feature that characterizes prime numbers, thanks to the theorem that guarantees the equivalence of these to formulations. Going the other route—i.e., focusing on (1), the unfactorable quality of prime number, would then cast (2) as a basic more general feature also *characterizing* prime-ness.

As for dependence on context, we might turn to the question—necessarily prior to the question *What is an infinite set?*—namely:

5.4. *What is a set?*

I'm not sure we have a definitive answer to this yet. A 'set' is a pretty lean mathematical object, evoked—if not captured—by the simple phrase *a collection of things*. Nevertheless *sets* provide the substrate for such a wide variety of mathematical objects. So, an axiom system that 'models' set theory is clearly of foundational importance in mathematics.

Discuss; Section 21 of [5]

5.5. **Shapes of Mathematical Proof.** The historical discussion has been laced with curious proclamations, such as:

> Pure mathematics is a branch of logic.
>
> (— *Max Black, The Nature of Mathematics* [1][35])

This is what is known as the *logicist's* view and we can discuss it.

We should also discuss the other (traditional) attitudes regarding the connection of Mathematics to 'Proof,—and hence to Truth—namely: *formalism, constructivism, intuitionism,* and *mathematical platonism.*

But I prefer to put less specific, but more directly comprehensible labels on the fundamental substrata of mathematical proof (and mathematical thought):

- Language
- Mind
- Some transcendental architecture

[35]who also wrote that we could see in Leibniz "the germ of the entire logistic conception."

where *formalism, intuitionism,* and *mathematical platonism* fall respectively into these three categories, and *constructivism* straddles the first two.

Language: The (extreme) formalist view has it that the essence of Mathematics is "the grammar of all symbolic systems . . . ," or again, as "the crystallized syntax of all systems of interrelated objects." (cf. [1]). That is: mathematics is given by its very language: the basic elements are—primitive symbols, and strings of symbols—together with recursive rules for manipulating them, and for determining which strings count as well-formed. (See [2].)

discuss

Mind: Intuitionism takes mathematics as a direct manifestation of our mental faculties, the result of our mental constructions, with emphasis on the word *construction*.

The Brouwerian View (L.E.J. Brouwer —1881-1966): A mathematical statement is 'true' if it is the (logical) consequence of an explicit mental construction. (See [3]).

The Kantian view would have the 'a priori' properties of mathematics arising from the mental faculties and intuitions (i.e., *space* and *time* in Kant's sense) operating within, and on, themselves without any necessary reference to perception or other connection to the world.

discuss

Transcendental architecture: The view labeled *mathematical platonism* would have mathematical concepts—e.g., "number, circle, etc"—exist, and these are to be investigated as a physicist investigates (e.g.) atoms. (See [4]. For a somewhat chatty discussion of this, see [6].)

discuss

5.6. **Mathematical Truth approached by Heuristics, Plausible Inference, Conjecture.** *How do we gain confidence in [mathematical] guesses, before we actually prove them?*

- *experiment, computation, accumulation of confirming data in special cases*
- *reasoning from consequence*,
- *reasoning from randomness*,

- *reasoning from analogy.*

discuss [7], [8]*; (see Section 5.7 below)*

5.7. **Specific Readings to be discussed. I.** Read the opening paragraph of each of these: [2], [3], and [4].

II. Read:

(1) Axiomatic Reasoning [5]:

- Section 3; pages 10,11
- Section 8, A, B; pages 20-23
- Sections 11,12; pages 29, 30.
- Section 21; page 47.

(2) Conjecture [8]:

Sections 2-5; pages 199-202

(3) Is it Plausible? [7]:

Section 2; page 8.

References

[1] Max Black, The Nature of Mathematics, New York, Harcourt-Brace (1934) ix+219 pp.
[2] Formalism in Mathematics, *Stanford Encyclopedia of Philosophy* `https://plato.stanford.edu/entries/formalism-mathematics/`
[3] Intuitionism in the Philosophy of Mathematics, *Stanford Encyclopedia of Philosophy* `https://plato.stanford.edu/entries/intuitionism/#Con`
[4] Platonism in the Philosophy of Mathematics, *Stanford Encyclopedia of Philosophy* `https://plato.stanford.edu/entries/platonism-mathematics/`

Four articles of mine that I'm posting as files on our course website:

[5] Axiomatic Reasoning
[6] Mathematical Platonism and its Opposites, European Mathematical Society Newsletter, **68** (2008) 19-21 `https://www.scribd.com/doc/219584271/Mathematical-Platonism-and-Its-Opposites`
[7] Is it Plausible? *The Mathematical Intelligencer*, **36** (2014) 24-33
[8] Conjecture, Synthèse **111** (2):197-210 (1997)