Reading Bombelli (April 27, 2001)

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Rafael Bombelli's *L'Algebra*, originally written in the middle of the sixteenth century, is one of the founding texts of the title subject, so if you are an algebraist, it isn't unnatural to want to read it. We are currently trying to do so.

Now, much of the secondary literature on this treatise concurs with the simple view found in Bourbaki's *Éléments d'Histoire des Mathématiques* :

"Bombelli ... takes care to give explictly the rules for calculation of complex numbers in a manner very close to modern expositions."

This may be true, but is of limited help in understanding the issues that the text is grappling with: if you open Bombelli's treatise you discover nothing resembling complex numbers until page 133[1] at which point certain mathematical objects (that might be regarded by a modern as "complex numbers") burst onto the scene, in full battle array, in the middle of an on-going discussion. Here is how Bombelli introduces these mathematical objects. He writes: "*I have found another sort of cubic radical which behaves in a very different way than the others.*"

"Ho trovato un'altra sorte di R.c.legate molto differenti dall'altre ....

The cubic radicals that Bombelli is contemplating here are the radicals that occur in the general solution of a cubic polynomial. Bombelli has come to the point in his treatise where he is working with Dal Ferro's formula for the general solution to cubic polynomial equations and considers (to resort to modern language) cubic polynomials with "three real roots" [2]. He produces the formula (a sum of cube roots of conjugate quadratic imaginary expressions) which yields ("formally", as we would say) a solution to the cubic polynomial under examination.

Complex numbers– which, when they occur in Cardano's earlier treatise *Ars Magna*, occur neatly as quantities like $2+\sqrt{-15}$. But they appear initially in Bombelli's treatise as cubic radicals of the type of quantities discussed by Cardano; a somewhat complicated way for them to arise in a treatise that is thought of as an organized exposition of the formal properties of complex numbers! Why doesn't Bombelli cite Cardano here? Why does he not mention his predecessor's discussion of imaginary numbers? Bombelli is not

[1] Our page numbers refer to Bortolotti and Forti's 1929 edition of "l'Algebra". For an account of the history of the publication of this treatise, see below. We have also listed some of the secondary literature in the bibliography below.

[2] this is what Bombelli's contemporaries called the "irreducible case" (a term still used by Italian mathematicians today).

shy elsewhere of praising the work of Cardano. Why, at this point, does Bombelli rather seem to be announcing a discovery of his own ("*I have found...*")?

Here is a glib suggestion of an answer: Bombelli has no way of knowing, given what is available to him, that his cubic radicals are even of the same *species* as the complex numbers of Cardano. How, after all, would Bombelli know that the cube root of a complex number is again a complex number? Of course one can go in the opposite direction with ease: that is, one can take a complex number z and cube it to get a number $y = z^3$ with known cube root, and one might be lucky in guessing z, given y. Bombelli, for example tells us that the cube root of $2+11\sqrt{-1}$ is $2+\sqrt{-1}$ and thereby gets the solution $x = 2+\sqrt{-1}+2-\sqrt{-1} = 4$ to the cubic equation $x^3 = 15x+4$. But the general problem of extracting cube roots is of a different order, for how you would go about solving the equation

$$(X+iY)^3 = A+iB,$$

or equivalently, the simultaneous (cubic, of course) equations

$$X^3 - 3XY^2 = A; \quad 3X^2Y - Y^3 = B,$$

without having various eighteenth century insights at your disposal? There is surely the smell of circularity here, despite the fact that a "modern" can derive some simple pleasure in analyzing the 0-cycle of degree 9 in complex projective 2-space given by the intersection of those two cubics. To Bombelli, his cubic radicals were indeed *new* kinds of radicals.

Can we be content with this answer?

A few paragraphs later Bombelli makes it clear that he was quite dubious, at first, about the legitimacy of his discovery and only slowly accustomed himself to it; he writes:

"*[This radical] will seem to most people more sophistic than real. That was the opinion even I held, until I found demonstration [of its existence] ...* "[3]

What, then, does Bombelli mean by *demonstration*? What does he mean by *existence*? As we shall see, Bombelli only ascribes *existence*, whatever this means, to the yoked sum of two cubic radicals (the radicands being, in effect, conjugate complex numbers). As he puts it,

"*It has never happened to me to find one of these kinds of cubic root without its conjugate.*"[4]

Let us add a further element to this stew of questions: In the "irreducible case", i.e., the case where the cubic polynomial has three real roots, does Bombelli believe that the solution given by his "new kind of cubic radicals" *correspond* to any, or all, of the

[3] Bombelli (1966), p. 133.
[4] Bombelli (1966), p. 134.

three solutions? (He seems to.) In what sense does Bombelli's general solution lead to a numerical determination of one, or more, of the three roots of the polynomial? If you do not have de Moivre's insight, or anything equivalent to that insight, you may be stymied by the problem of "using" the general solution by cubic radicals to help you find, or even approximate, any of the three real numbers which are roots of the cubic polynomial that the "general solution" purports to solve.[5]

An evolving theme in Bombelli's thought is the idea of connecting the ancient problem of angle trisection to the problem of finding roots of cubic polynomials. Of course, the modern viewpoint makes this connection quite clear. Bombelli also develops a method (as he says, "in the plane") for finding a real number solution to a cubic polynomial equation. His method involves making a construction in plane geometry dependent upon a parameter (the parameter being the angle that two specific lines in the construction subtend) and then "rotating" one of those lines (this "rotation" affects other changes in his construction) until the lengths of two line segments in the construction are equal ; these (equal) lengths then provide the answer he seeks (we will refer to this type of construction as a *neusis* construction: see section 5 below). To what extent do these discussions (trisection of angle – and *neusis* construction) play a role in providing a "demonstration" to Bombelli of the *existence* of his yoked cubic radicals? (We will discuss this in detail, in sections 4-6 below).

Tempering any answer that we might offer to any of these questions is the fact that the incubation period for Bombelli's text, and its writing, spanned more than two decades. Bombelli's treatise records the evolution of his thought, and the answers that Bombelli entertains for some of these questions change with time. Reading him may perhaps give us a portrait of an early father of algebra grappling with what it means for a concept to *exist*. We feel that this portrait deserves to be more fully drawn than has been done.

We are not yet ready to do this, and are only in mid-journey in our reading of Bombelli. Nevertheless we have put together the present article in hopes that what we have learned so far may be useful to other readers. We wish to thank David Cox and Chandler Davis for their helpful comments and questions regarding our earlier drafts.

1. Bombelli's writing. Bombelli wrote in Italian (which, according to Dante, is the language of the people). To our knowledge, his is the first long treatise on mathematics

[5] As De Moivre put it in his article published in 1738: "There have been several authors, and among them Dr. Wallis, who have thought that those cubic equations, which are referred to the circle, may be solved by the extraction of the cube root of an imaginary quantity, as of $81 + \sqrt{-2700}$, without any regard to the table of sines: but that is a mere fiction; and a begging of the question; for on attempting it, the result always recurs back again to the same equation as that first proposed. And the thing cannot be done directly, without the help of the table of sines, specially when the roots are irrational; as has been observed by many others." (Abraham De Moivre, "Of the Reduction of Radicals to more Simple Terms," *The Philosophical Transactions of the the Royal Society of London*, abridged by C. Hutton, G. Shaw, and R. Pearson, volume VIII (London: 1809) 276.

written in Italian. He was faced, therefore, with something of a Dante-esque project: to choose words for existing terms (generally from Latin) and to invent Italian words for the various concepts that came along. That his book is in Italian has a mild disadvantage, and a great advantage for a reader. On the one hand many of Bombelli's neologisms never caught on, and they may seem quite strange to a modern. These terms therefore must be carefully deciphered (we give a partial glossary in Appendix B below). On the other hand his style is quite personal (putting aside the lengthy computations about cubic irrationalities that are spelled out in prose!). At times the text reads as if it were a private journal. To get a sense of this see Appendix A below for a translation of his introductory remarks. What we know of Bombelli's life (see section 2 below) comes, it seems, entirely from this treatise. More importantly, as we have already mentioned, Bombelli's informality allowed him to keep in the text some of his earlier attitudes along with the changes that they went through over the twenty year period during which he worked on l'Algebra.

2. Bombelli and his *Algebra*

We do not know precisely where Bombelli was born. In his work, *L'Algebra*, he calls himself "citizen of Bologna." Bombelli was a member of a noble family from the countryside around Bologna. They came to Bologna at the beginning of the XIII century. At the end of the same century they, being "ghibellini," were forced to leave the city and only returned in the XVI century.

Bombelli was a civil engineer, and in *L'Algebra* he mentions his involvement in the project of draining the Chiana swamp in Tuscany. He recounts that during periods of interruption of this project he wrote his book. The treatise *L'Algebra* as edited and published by Bortolotti in 1929, consists of two "parts"[6] which were, it seems, initially written in 1550[7]. After having first written his manuscript, Bombelli came to know Diophantus' *Arithmetic* which was in a codex of the Vatican Library[8]. Bombelli, then, made a general revision of his manuscript and, among other things, included Diophantus' problems in his text. He published none of it until 1572. At that time Bombelli published only the first part. He apologized, saying that he could not also publish the other part because it had not yet been "brought to the level of perfection required by mathematics". However, it was surely circulating among scholars since in Bologna's libraries we still find two copies of the manuscript. The second part of the book was not published and was believed lost until the 1920's when Bortolotti found the complete manuscript (not just the last part, but also the first in an unrevised version) of the work in codex B 1560 of the "Biblioteca dell'Archiginnasio di Bologna."

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[6] part I comprised of three "books"; part II of two.

[7] Bortolotti reached the conclusion that the manuscript he found in the Library of the Archiginnasio in Bologna (containing the entirety Bombelli's work, with both parts, the algebraic and the geometrical, in the first, unrevised version) went back to that date.

[8] In the introduction of the printed work, Bombelli tells us that he and Pazzi had translated the first five chapters of Diophantus while Pazzi was lector at Rome, i.e., sometime after 1567.

Here is a run-down of the contents of Bombelli's five books. As we have already mentioned, Bombelli's great innovation was to have "solved" the "irreducible case" of the general cubic polynomial; i.e., the case when the root of Dal Ferro's formula for solving cubic equations involves the square root of a negative number, a thing that at the time was considered a monstrous absurdity (Cardano called the expression containing square root of negative numbers "sophistic and far from the nature of numbers" and also "wild").

Bombelli gives a definition of *variable* and notation for exponents. He studies monomials, polynomials, and rules for calculating with them. He treats the equations from the first to the fourth degree, and solves, among other things, all "42" possible cases of quartic equations (improving on the work of Ferraro and Cardano). Following the practice of the time he also gives a *solid* geometric demonstration of the solution of cubic equations in terms of how a cube can be decomposed into two cubes and six parallelepipeds. Moreover, noticing the analogy between this problem and the classic problem of the insertion of two middle proportionals, he also offers his *plane* geometrical construction of the root of a cubic equation which we will be discussing below. This construction is, perhaps, superfluous for a cubic equation with only one real root, but necessary in the irreducible case where the decomposition of the cube is impossible. In doing this Bombelli developed a geometric algebra (he refers to this as "algebra linearia", that is to say *linear algebra*) which has a distinctly cartesian flavor. For, at times, Bombelli seems to be making the claim that geometry is not necessarily the only way to prove things: rather, certain geometric constructions are grounded in the underlying algebra that represents these constructions. Bombelli addresses the question of the relationship between the problem of the trisection of the angle and that of the solution of the cubic equation in the irreducible case. In his published treatise he expresses his intention to *use* the solution of the cubic equation in the irreducible case to solve the angle-trisection problem.[9] This represents a change of viewpoint from the earlier version of his manuscript where Bombelli simply maintained that angle-trisection leads to cubic equations that cannot be solved.[10]

His treatise contains a collection of problems which include all the problems of the first four books of Diophantus. *L'Algebra* remained for more than a century the fundamental text of advanced algebra. It was studied, for example, by Huygens and Leibniz.

3. "Ho trovato un'altra sorte di R.c.legate molto differenti dall'altre ...."

Here is how the text[11] continues. (We have shortened it a bit by putting the algebraic formulae in modern notation.)

... I have found another kind of cubic root of a polynomial which is very different from the others. This [cubic root] arises in the chapter dealing with the equation of the kind $x^3 = px + q$, *when* $p^3/27 > q^2/4$, *as we will show in that chapter. This kind of square root has in its calculation [algorismo] different operations than the others and has a*

[9] Bombelli (1966), p. 245.

[10] Bombelli (1966), pp. 639-641.

[11] Translation of pp 133-134 (in the Chapter *On the division of a trinomial made by cubic roots of polynomials and number*).

different name. Since when $p^3/27 > q^2/4$, the square root of their difference can be called neither positive nor negative, therefore I will call it 'more than minus' when it should be added and 'less than minus' when it should be subtracted. This operation is extremely necessary, more than the other cubic roots of polynomials, which comes up when we treat the equations of the kind $x^4 + ax^3 + b$ or $x^4 + ax + b$ or $x^4 + ax^3 + ax + b$. Because, in solving these equations, the cases in which we obtain this [new] kind of root many more than the cases in which we obtain the other kind. [This new kind of root] will seem to most people more sophistic than real. This was the opinion I held too, until I found its geometrical proof (as it will be shown in the proof given in the above-mentioned chapter on the plane). I will first treat multiplication, giving the law of plus and minus:[12]

$$\begin{aligned}
&(+)(+i) = +i\\
&(-)(+i) = -i\\
&(+)(-i) = -i\\
&(-)(-i) = +i\\
&(+i)(+i) = -\\
&(+i)(-i) = +\\
&(-i)(+i) = +\\
&(-i)(-i) = -
\end{aligned}$$

Notice that this kind of root of polynomials cannot be obtained if not together with their conjugate. For instance, the conjugate of $\sqrt[3]{2 + i\sqrt{2}}$ will be $\sqrt[3]{2 - i\sqrt{2}}$. It has never happened to me to find one of these kinds of cubic root without its conjugate. It can also happen that the second quantity [inside the cubic root] is a number and not a root (as we will see in solving equations). Yet, [even if the second quantity is a number], an expression like $\sqrt[3]{2 + 2i}$ cannot be reduced to only one monomial, despite the fact that both 2 and $2i$ are numbers.

Commentary: *The cube equal to a coefficient times the unknown plus a number* refers to the equation which in modern dress is:

$$x^3 = px + q.$$

Here, p is *the coefficient* and q is *the number*. Bombelli prefers to think of his equations having only positive numbers as coefficients, so will treat separately (in different chapters)

[12] In a more literal translation of Bombelli's words:

Plus times more than minus makes more than minus.
Minus times more than minus makes less than minus.
Plus times less than minus makes less than minus.
Minus times less than minus makes more than minus.
More than minus times more than minus makes minus.
More than minus times less than minus makes plus.
Less than minus times more than minus makes plus.
Less than minus times less than minus makes minus.

equations of the form $x^3 + px = q$ etc., terms being assembled to the left or right of the equality sign to arrange that p and q are positive. For efficiency, let us cheat, and peek at the modern, but still pre-Galois, treatment of the general cubic equation

$$x^3 = px + q :$$

If we formally factor the polynomial

$$x^3 - px - q = (x - \theta_1)(x - \theta_2)(x - \theta_3)$$

as a product of linear factors, we have that

$$\theta_1 + \theta_2 + \theta_3 = 0,$$

and Δ, the discriminant of the polynomial, i.e., the square of

$$(\theta_1 - \theta_2)(\theta_2 - \theta_3)(\theta_3 - \theta_1),$$

is equal to

$$\Delta = 4p^3 - 27q^2,$$

which is positive if all three roots $\theta_1, \theta_2, \theta_3$ are real, and is negative if precisely one of them is real. In any event, a "formula" for the real solution(s) to this polynomial is given by:

$$x = \sqrt[3]{q/2 + \frac{1}{6}\sqrt{-\Delta/3}} \;+\; \sqrt[3]{q/2 - \frac{1}{6}\sqrt{-\Delta/3}},$$

where if Δ is negative (and we are looking for the unique real solution) the above formula has an unambiguous interpretation as a real number and gives the solution.

If, however, Δ is positive (which is what Bombelli is encountering when he considers the case where the cube of 'the third of the coefficient' is greater than the square of 'half the number', or equivalently, where $\frac{q^2}{4} - \frac{p^3}{27}$ is negative and $\sqrt{\frac{q^2}{4} - \frac{p^3}{27}}$ is imaginary), the above solution, i.e.,

$$x = \sqrt[3]{q/2 + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} \;+\; \sqrt[3]{q/2 - \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}$$

involves imaginaries. To a modern eye, this expression is dangerously ambiguous, there being three possible values for each of the cubic radicals in it: to have it "work", of course, you have to coordinate the cube roots involved. That is, to interpret the expression correctly you must "yoke together" the two radicals in the above formula by taking them to be complex conjugates of each other, and then, running through each of the three complex cube roots of $q/2 - \frac{1}{6}\sqrt{-\Delta/3}$ you get the three real solutions.

4. Geometrical "demonstration". Bombelli knows that any cubic polynomial has a root. The (post-cartesian) argument (that a cubic polynomial $p(x)$ takes on positive and negative values, is a continuous function of x and therefore, as x ranges through all real numbers, $p(x)$ must traverse the value 0, at least once) is not in Bombelli's vocabulary, but as the reader will see, there remains a shade of this argument in Bombelli's geometrical "demonstration". Bombelli convinces himself that cubic polynomials have roots by *two* distinct methods– the first by consideration of volumes in space, a method which does *not* work in the irreducible case; and the second by consideration of areas in the plane, being a method which does work in the irreducible case.[13]

The method by consideration of volumes. Bombelli starts with a cube whose linear dimension let us denote by t. He then decomposes it into a sum of two cubes nesting in opposite corners of the big cube, these being of linear dimensions, say, u and $t-u$, and three parallelopipeds, following the algebraic formula:

$$(t-u)^3 + 3tu(t-u) = t^3 - u^3.$$

Stripping the rest of Bombelli's demonstration of its geometric language, here is how it proceeds. Put $p := 3tu$ and $q := t^3 - u^3$, and note that the quantity $x := t-u$ is a solution of the cubic equation

$$x^3 + px = q.$$

Of course, if we have such an equation with given constants $p, q > 0$ we wish to solve, we would first have to arrange to find the t and the u that worked, but ignore this, and let us proceed. Substituting

$$u = \frac{p}{3t}$$

in the equation $t^3 - u^3 = q$, we get

$$t^3 - \frac{p^3}{27t^3} = q,$$

or

$$t^6 - qt^3 - \frac{p^3}{27} = 0,$$

which we think of as a quadratic equation in t^3:

$$(t^3)^2 - q(t^3) - \frac{p^3}{27} = 0,$$

and applying the quadratic formula (available, of course, in Bombelli's time) to get

$$t^3 = \frac{q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2},$$

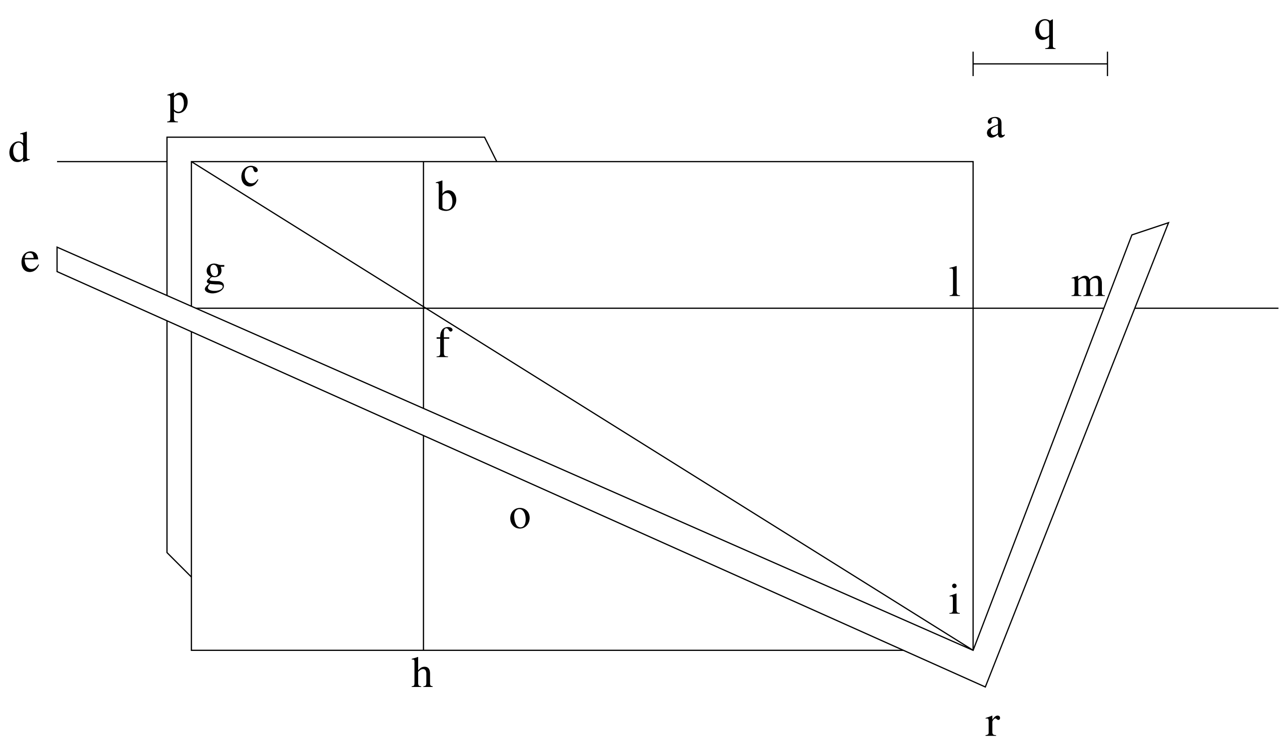
[13] For the first method see Bombelli (1966), pp. 226-228. For the second method see Bombelli (1966), pp. 228-229.

i.e., Cardano's formula for the solution $x = t - \frac{p}{3t}$ of the cubic equations of the form $x^3 + px = q$. All this is performable geometrically to actually produce the x only if t^3 is a real number. That is, this geometric demonstration doesn't work in the irreducible case[14].

The method in the plane.

Bombelli's second method resembles some of the *neusis*-constructions in ancient Greek geometry– used in questions of angle-trisection (see **5** below) and indeed does work in the irreducible case. Bombelli promotes this method (invoking of the august authority of the ancient authors, who used similar methods) because, he claims, it provides a "geometric demonstration" that his cubic radicals "exist".

By a *gnomon* let us mean an "L-shaped" figure; i.e., two closed line segments joined at a 90 degree angle at their common point (the *vertex*). Bombelli uses a construction with two gnomons (if that is the plural form of the word), one with vertex r and one with vertex unfortunately labeled p in the diagram (taken from his manuscript) below.



He will construct such a diagram from the data of his cubic equation $x^3 = px + q$, i.e., from the pair of real numbers p and q (from dimension considerations, we can expect p to appear as an area, and q/p as a linear measurement). Let us calibrate the diagram by putting

$$\overline{lm} = unity.$$

14 This type of "decomposition of the cube" argument had already been used by Cardano in the Ars Magna to explain how, for a particular equation ($x^6 + 6x = 20$) one can derive his formula; Cardano never considered the irreducible case.

Now by suitably moving the two gnomons, moving the first up and down and pivoting the second about its vertex, Bombelli shows that one can obtain a diagram with

$$\overline{la} = \frac{q}{p},$$

and the area of the rectangle $\overline{abfl}$ equal to p, and moroever, for such a diagram, the root x of his equation will appear as the length $\overline{li}$.

5. ***Neusis*****-constructions and the trisection of angles.**

The problem of trisecting a general angle with the aid of no more than an unmarked straight edge and compass as posed by the ancient Greek mathematicians is impossible. The fact that (the general solution of) this problem is impossible was established only in the beginning of the 19th century by Pierre Laurant Wantzel [15] who also made explicit the connection between trisection and solutions of cubic equations. But ancient mathematicians had an assortment of methods of angle-trisection that made use of "equipment" more powerful than mere compass and straight edge. One such method (referred to as *neusis*: verging, inclination) useful for solving certain problems involves making (as in the gnomon construction of Bombelli's that we have just sketched) a plane geometric construction, or more precisely a "family of constructions" dependent upon a single *parameter of variation.*[16] In general, the strategy is to show that by "varying the construction" one can arrange it so that two designated points on a specific line (of the construction) switch their order on the line, under the variation. This then allows one to argue, in the spirit of the modern intermediate value theorem, that there is a member of the family where the two designated points actually coincide. One then applies the features of this particular member of the family to help with the problem one wishes to solve.

In the *Book of Lemmas* Archimedes (III BC) trisects a general angle using a *neusis* construction. (We do not have the original Greek of this work; we have an Arabic translation which does not seem to be completely faithful to the original Archimedean text.) Hippias (end of the fifth century BC), instead, used a curve that he invented, the so-called Quadratix of Hippias. By means of this curve one can divide a general angle in any number of equal parts. Nicomedes (II BC) made his *conchoid curve* by means of a *neusis* construction and he used the conchoid to solve the problem of trisection. Apollonius (late III BC - early II BC) achieved angle-trisection using conics (the two cases we have, transmitted to us by Pappus in his *Mathematical Collection*, use a hyperbola).

6. Suggestions. We feel that there are three distinct elements that contribute to Bombelli's "faith" in cubic radicals.

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[15] more specifically in 1837

[16] For *neusis* see, for instance, Fowler (1987), 8.2; Heath (1921), 235-41, 65-68, 189-92, 412-13; Grattan-Guinness (1997), 85; Bunt, Jones and Bedient (1976), 103-106; Boyer and Merzbach (1989), 151 and 162.

First, Bombelli deals with the "inverse problem" and he does this in two ways: As we mentioned above Bombelli does explicitly tell us, on occasion, what the cube root of a specific number is (the cube root of $2 + 11\sqrt{-1}$ is $2 + \sqrt{-1}$) and thereby explicitly solves an equation (e.g., $x = 4$ is a solution of $x^3 = 15x + 4$) saying that if one follows his geometrical method for the solution of this problem one obtains that same solution. But he also may simply start with a sum of two yoked cubic radicals,

$$\sqrt[3]{a + i\sqrt{b}} + \sqrt[3]{a - i\sqrt{b}},$$

and discover the cubic equation of which this is a root.[17] Since he has proven by his geometric method that the cubic equation has a real solution, (in fact "three" of them) it follows that this sum of two yoked cubic radicals, in some sense *represents* such a solution (and, thus, in some sense, represents a number). But whether it represents one, or all three, of the solutions is not dealt with. It would be difficult, in any case, for us to say what it meant for Bombelli's yoked cubic radicals to *represent* numbers for him, since they don't lead to the determination, or approximation of the number that they represent.

We have put quotation-marks around "three" when we discussed the "three" solutions to the cubic equation in the irreducible case because Bombelli does not consider negative solutions. Nevertheless, by appropriately transforming the equation, Bombelli is able to turn negative solutions of an equation into positive solutions of the transformed equation. See page 230 where Bombelli transforms the equation $x^3 + 2 = 3x$ into the equation $y^3 = 3y + 2$, where $y = -x$, and pp. 230-231 where Bombelli divides $x^3 - 3x + 2$ by $x + 2$ ($y = 2$). In his discussion of reducible cases of cubic polynomials, however, Bombelli talked of their (single, real) root and was surely unaware of the possibility that there might be "complex" interpretations of the relevant "yoked cubic radical" so as to provide the two complex roots of the cubic polynomial.

Secondly, it seems to us that Bombelli gains confidence in the "existence" of his yoked cubic radicals through his ability to perform algebraic operations with them, and thirdly, by his increased understanding of the relationship between the solution of the general cubic equation and the classical problem of angle-trisection. But it would be good to pin this down more specifically than we have done so far.

APPENDICES:

A. Bombelli's Preface.

To the reader

I know that I would be wasting my time if I tried to use mere finite words to explain the infinite excellence of the mathematical disciplines. To be sure, the excellence of mathematics has been celebrated by many rare minds and honored authors. Nevertheless, despite

[17] cf. Bombelli (1966), p. 226 (the paragraph "Dimostrazione delle R.c. Legate con il +di- e -di- in linea".

my shortcomings, I feel obliged to speak of the supremacy, among all the mathematical disciplines, of the subject that is nowadays called algebra by the common people.

All the other mathematical disciplines must use algebra. In fact the arithmetician and the geometer could not solve their problems and establish their demonstrations without algebra; nor could the astronomer measure the heavens, and the degrees, and, together with the cosmographer, find the intersection of circles and straight lines without having been compelled to rely on tables made by others. These tables, having been printed over and over again and, furthermore, by people with little knowledge of mathematics, are extremely corrupted. Thus, anyone using them for calculation is certain to make an infinite number of errors.

The musician, without algebra, can have little or no understanding of musical proportion. And what about architecture? Only algebra gives us the way (by means of lines of force) to build fortresses, war machines, and everything that can be measured: solid, and proportions, as occurs when dealing with perspective and other aspects of architecture.

Algebra also allows us to understand the errors that can occur in architecture.

Setting all these (self-evident) things aside, I will say only this: either because of the inherent difficulty of algebra, or because of the confused way that people write about it, the more algebra is perfected the less I see people working on it. I have thought about this situation for a long time and have not been able to figure out why. Many have said that their hesitations with algebra stemmed from the distrust they had of it, not being able to learn it, and from the ignorance that people generally have of algebra and of its use. But I think rather that these people want only to protect themselves by making such excuses. If they were willing to tell the truth they should rather say that the real cause [of their lack of interest in algebra] is the weakness or roughness of their own minds. In fact, given that all mathematics is concerned with speculation, one who is not speculative works hard, and in vain, to learn mathematics. I do not deny that for the scholars of algebra a cause of enormous suffering and an obstacle to understanding is the confusion created by writers and by the lack of order that there is in this discipline.

Thus, to remove every obstacle to those who are speculative and who are in love with this science, and to take every excuse away from the cowardly and inept, I turned my mind to try to bring perfect order to algebra and to discuss everything about the subject not mentioned by others. Thus, I started to write this work both to allow this science to remain known and to be useful to everyone.

To accomplish this task more easily, I first set about to examine what most of the other authors had already written on the subject. My aim was to compensate for what they missed. There are many such authors, the Arab Muhammad ibn Musà being considered the first. Muhammad ibn Musà is the author of a minor work, not of great value. I believe that the name "algebra" came from him. For the friar Luca Pacioli of Borgo del San Sepolcro from the Minorite order, writing about algebra in both Latin and Italian, said that the name "algebra" came from the Arabic, that its translation in our language was "position" and that this science came from the Arabs. This, likewise, had been believed and said by those who wrote after him.

Yet, in these past years, a Greek work on this discipline was found in the library of our Lord in the Vatican. The author of this work is a certain Diophantus Alexandrine, a

Greek who lived in the time of Antoninus Pius. Antonio Maria Pazzi, from Reggio, public lector of mathematics in Rome, showed Diophantus' work to me. To enrich the world with such a work, we began to translate it. For we both judged Diophantus to be an author who was extremely intelligent with numbers (he does not deal with irrational numbers, but only in his calculations does one truly see perfect order). We translated five books of the seven that constitute his work. We could not finish the books that remained due to commitments we both had. In this work we found that Diophantus often cites Indian authors. Thus, I came to know that this discipline was known to the Indians before the Arabs. A good deal after this, Leonardo Fibonacci wrote about algebra in Latin. After him and up to the above mentioned Luca Pacioli there was no one who said anything of value. The friar Luca Pacioli, although he was a careless writer and, therefore, made some mistakes, nevertheless was the first to enlighten this science. This is so, despite the fact that there are those who pretend to be originators, and ascribe to themselves all the honor, wickedly accusing the few errors of the friar, and saying nothing about the parts of his work that are good. Coming to our time, both foreigners and Italians wrote about algebra, as the French Oronce Finé, Enrico Schreiber of Erfurt and "il Boglione",[18] *the German Michele Stifel and a certain Spaniard*[19] *who wrote a great deal about algebra in his language.*

However, truly, there had been no one who penetrated to the secret of the matter as much as Gerolamo Cardano of Pavia did, in his Ars Magna where he spoke at length about this science. Nevertheless, he did not speak clearly. Cardano treated this discipline also in the "cartelli" that he wrote together with Lodovico Ferrari from Bologna against Niccolò Tartaglia from Brescia. In these "cartelli" one sees extremely beautiful and ingenious algebraic problems but very little modesty on the part of Tartaglia. Tartaglia was by his own nature so accustomed to speaking ill that one might think he imagined that by doing so he was honoring himself. Tartaglia offended most of the noble and intelligent thinkers of our time, as he did Cardano and Ferrari both minds divine rather than human.

Others wrote about algebra and if I wanted to cite them all I would have to work a great deal. However, given that their works have brought little benefit, I will not speak about them. I only say (as I said) that having seen, thus, what of algebra had been treated by the authors already mentioned, I too continued putting together this work for the common benefit. This work is divided in three books. The first book includes the practical aspect of Euclid's tenth book, the way of operating with cube roots and square roots; this mode of operating with cube roots is useful when one deals with cubes, that is to say solids. In the second book, I treated all the ways of operating in algebra where there are unknown quantities, giving methods to solve their equations and geometrical proofs. In the third book I posed (as a test for this science) about three hundred problems, so that the scholar of this discipline {algebra} reading them could see how gently one may profit from this science. Accept, thus, oh reader, accept my work with a mind free of every passion, and try to understand it. In this way you will see how it will be of benefit to you. However, I warn you that if you are unfamiliar with the basics of arithmetic, do not engage in the

[18] Bortolotti, in a footnote on p. 9 of his edition of Bombelli's text, says that "il Boglione" is not identified.

[19] According to Bortolotti, the Spaniard, although not clearly identified, is perhaps the Portuguese Pietro Nunes. See Bombelli (1966) p 9.

enterprise of learning algebra because you will lose time. Do not condemn me if you find in the work some mistakes or corrections that do not come from me but from the printer. In fact, even when all possible care is used, it is still impossible to avoid typographical errors. Equally if you see some impropriety in the framing of my sentences, or a less than lovely style do not consider it [harshly]. ... My only purpose (as I said earlier) is to teach the theory and practice of the most important part of arithmetic (or algebra), which may God like, it being in his praise and for the benefit of living beings.

B. A glossary of terms.

Agguagliare {equating}: to solve an equation
Agguagliatione {the equating}: the solving of an equation
Algorismo {algorithm}: a method for calculating
Avenimento {what happens}: the quotient of a division
Cavare {to extract}: to subtract
Censo : name of x^2 (used in the manuscript, *censo* is substituted in the published book by *potenza* that is to say "power"
Creatore {creator}: root
Cubato {cubed}: the cube of a number or of x
Cuboquadrato {squared cube}: the sixth power
Dignitá {dignities}: the powers of numbers or of x from the second power on
Esimo {-th}: a word used to express a fraction
For instance 2/4 is 2 *esimo di* 4 that is "2 th of 4", or "two fourths".
Lato {side}: root
Nome {name}: monomial
Partire {to part}: to divide
Partitore {the one who parts}: divisor
Positione {position}: equation
Potenza {power}: x^2
Quadrocubico {square cubic}: sixth power
Quadroquadrato {square squared}: fourth power
R.c. : it is the sign for "radice cubica" that is to say cubic root
R.c.L. or *R.c. legata* {linked cubic root}: it is the cubic root of a polynomial
R.q.: square root
R.q. legata {linked square root}: square root of a polynomial
R.q.c. or *R.c.q.* : the signs for "radice quadrocubica" and "radice cuboquadrata" that is to say the sixth root
R.R.q.: the sign for "radice quadroquadrata" that is to say the fourth root
Residuo {residue}: it is a binomial made by the difference of two monomials. It thus used for the conjugate roots
Rotto {broken}: fraction
Salvare {to save}: to put a quantity aside for a moment for using it again later
Tanto {an unknown quantity}: x
Trasmutatione {transmutation}: linear transformation of an equation

Valuta {value}: the value of x
Via {by}: the sign for the multiplication

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