The Language of Explanation

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Abstract

This essay was written for the University of Utah *Symposium in Science and Literature* (November, 2009) and in anticipation of a panel discussion on *Mathematics, Language, and Imagination* with Alice Fulton and Fred Lerdahl.

The arts and sciences and everything else we humans do depend upon our intuitions and sensibilities. They also depend upon one grand public resource: language, which—taken broadly—is more than a medium: it is both a primal soup whose nutrients contain all that is needed to guarantee the emergence and smooth continuation of our culture, and it is also the repository of that culture. The languages of music, mathematics, and poetry are completely different, just as the sensibilities developed by them are different. But if we reflect on each other's language we may learn things. For this hour I'll try to do a bit of this type of reflection, contrasting some aspects of the language of poetry to that of mathematics.

1 Languages and Beauty

How strange it is that our various languages—initially in the service of understanding and communicating—contrive, strive, to be beautiful; even if they are dealing with grim and tragic things[1]. So many

[1]

Since brass, nor stone, nor earth, nor boundless sea, sad yet wonderful human utterances reveal this to us that we needn't go to poetry to see this. But to take an example, how disturbingly beautiful is King Lear's

> Howl, howl, howl, howl! O, you are men of stones:

where *Howl* is an exhortation and a cry[2]. How fully we enjoy—if that's the right word—the strange consonance of despair-and-beauty as the trochaic pentameter line

> Never, never, never, never, never!

abruptly everts the iambic pentameter of

> Why should a dog, a horse, a rat, have life,
> And thou no breath at all? Thou'lt come no more,
> *Never, never, never, never, never!*

It is strange to think how beauty lurks amidst tragedy—or, at least, amidst the tragedy of others.

I've long been puzzled by the question of *why there is so much beauty in mathematics?* Aesthetics is simply front-and-center in the intended goals of the languages of music and poetry. But *beauty of intellectual constructions* often is achieved in mathematical work as a happy by-product, it would seem, of its intended aim. This certainly needs to be examined, and is one of the issues I hope will come up in discussions.

The language of mathematics, has as its primary mission to *explain things*—it is a tool of *understanding*. It is natural, of course,

But sad mortality o'er-sways their power,
How with this rage shall beauty hold a plea,
...

[2]Or take Alan Ginsberg's agonizing evocation where he used same word as both label of a cry, and a cry.

that this language feeds back on our intellect, enriching the very underpinnings of our imagination by sharpening our intuitions, and extending the reach of our thinking in breath-taking directions—for example, so that we can almost comprehend *infinity*. How wonderful that the language of mathematics helps us conjure up, and talk about ideas, concepts, structures, that happen to be of great beauty, even if the first intended use of that language is to help us measure wheat-fields or build bridges, or do whatever the early reckoners had in mind to do.

Perhaps beauty is a universal of language: we're beasts who wish to communicate beautifully and that's all that there's to it. All languages I've ever had any contact with, even the small-gauge languages thrown together for a particular purpose, or setting (I'm thinking, e.g., of the Morse code I tried to tap out when I was a teenage radio ham or, some dock jargon when I worked as a stevedore on a pier in NYC etc.) cajole their speakers towards beauty of expression. Let's call it the *ubiquity of the poetic drive.*

2 Resonances and Definitions

We're all introduced to the culture of our native language by being thrown screaming into *the middle of things* and somehow we pick up the intentions and meanings of words well enough. Actual *definitions* of new words (and perhaps of those old words we already use) are taught to us only much later.

Nevertheless we usually think that mathematical languages come to us differently—that is, definition-first so that everything else in the discourse of mathematics gets built up logically from those first definitions and first principles. But even a cursory look at "beginnings" (in ancient mathematics, as well as modern mathematics) makes us scratch our heads wondering exactly what people are actually getting from those "first definitions."

The famous *Definition 1* of Book I of Euclid's *Elements*[3]*:*

[3] that ancient mathematical sourcebook from which high school geometry is—or used to be—fashioned.

A point is that which has no part,

or *Definition 2*:

A line is breadthless length.

are hardly logically usable. Nevertheless, these definitions do extremely important work, and later writers (Proclus, Hilbert) made them more logical for us, but in Euclid's formulation they seem more like guides helping us focus on the kind of thing points and lines are, rather than definitions that can be worked with.

In fact, much of our primary mathematical intuition seems to come to us just as definition-less as anything else—meanings are picked up, and transmitted by common practice (I once heard a child toting things up on the chant

$$\ldots, 10, 11, 12, \textit{dig'n'delve}, 13, \ldots,$$

where *dig'n'delve* has wedged itself—for this child—as an ordinal number between 12 and 13. Only later in school is all this type of common practice sorted out—we hope).

In music and poetry—and I suspect that this is in slight contrast to what happens in mathematics—a resonance that certain words, or musical themes have developed—depending on their use in the musical and poetic tradition—unfolds and enriches meaning. Here is one–perhaps too simple—example:

Recalling that swallows are among the first birds to migrate south for the winter, it is not surprising to find mention of a gathering of swallows in these lines of the A.R. Ammons poem *Corson's Inlet*

the news to my left over the dunes and
reeds and bayberry clumps was
fall: thousands of tree swallows
gathering for flight:

But poems being poems, I don't think it is farfetched to say that

Ammons' swallows resonate with the "gathering" image in the last line of the *Ode to Autumn* of Keats:

> And gathering swallows twitter in the skies.

the wrench, here, being that the Autumn that Keats was writing about was to be his last, and he knew it; whereas, in contrast, Ammons ends *Corson's Inlet* with the expectation

> that tomorrow a new walk is a new walk.

Everything we tell each other has resonances, where much "shared past" echoes in the telling. Even brittle dictionary-definitions concede this, and offer us archaic usages that may still be faintly flavoring the modern use of the word. That, as Webster's tells us, *trust* is somehow related to *tree* sharpens our sensitivity to some otherwise unheard bourdon that comes along with each of these words.

But sometimes a definition throws our way a brand new word, a brand new phrase, or illuminates things, focusses our viewpoint. Everyone, I'm sure, has their favorite example where, say, a single perspicuous definition, or explanation of usage, clarifies things in a startling way[4].

People acquainted with mathematics have lists of them too. The nineteenth century mathematician Dedekind had an elegant way of characterizing sets that are *infinite*—elegant, stark, and enlightening. To appreciate Dedekind's definition, you should know that the word *set* refers to a collection of objects, a *subset* of a given set S is a collection of objects taken from S; and a subset T of S is called

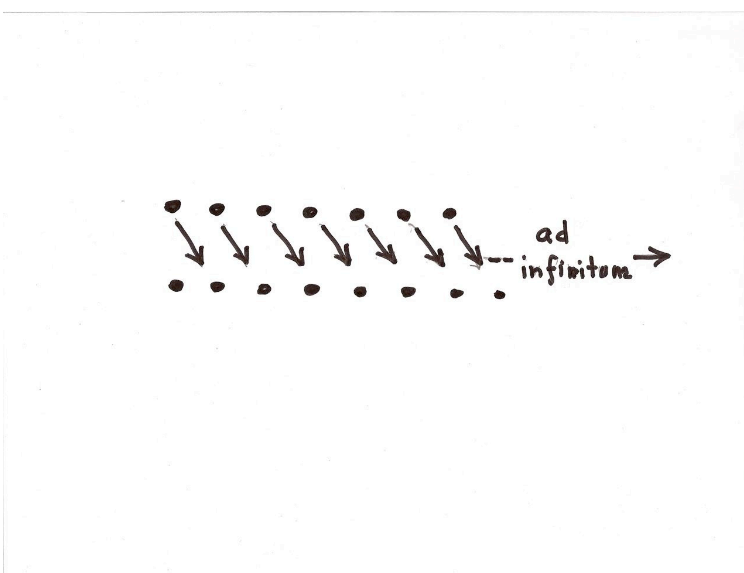
[4] There are also times when a definition of usage—meant to clarify—goes in the other direction, i.e., mystifies. In a letter to Robert Bridges, Gerard Manley Hopkins tries to explain the intention of the first word *Have* in his poem *Henry Purcell* describing it as *3rd person, imperative, in the past tense.* Hopkins notes that it would be no problem to imagine an imperative format for the word *have* in various tenses if used, say, in the 2nd person, as in the sentence "Have eaten" (e.g., before you come to the theater). This *3rd person, imperative, in the past tense* format for *have*, however, is certainly a startling usage, and if I only understood it better I'd discuss it more ...

proper if T is NOT all of S (not equal to the entire set S) so that there is at least one element of the collection S that is not in T. Finally, a *one:one correspondence* between two sets A and B is a rule that associates to every element of A an element of B so that every element of B is associated to a *unique* element of A. For example, the set of positive whole numbers $\{1, 2, 3, \ldots\}$ is in one:one correspondence with the set of even numbers $\{2, 4, 6, \ldots\}$ by the rule that simply doubles each number.

Now we're ready for Dedekind's definition:

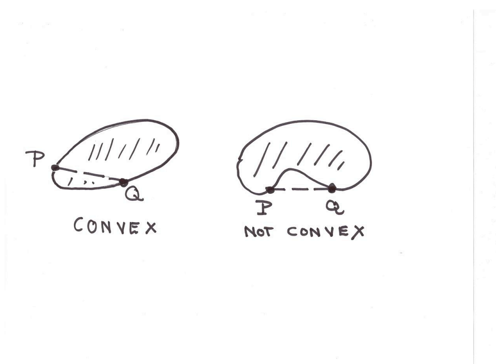
> A set is *infinite* if it can be put in one:one correspondence with a *proper* subset of itself.

(In the figure below the dots signify an infinite sequence of objects and the one:one correspondence sends the nth object of this set to the $(n+1)$st. This is a one:one correspondence between the set of all these objects and the *proper* subset consisting in all but the first object.)



3 Explicit versus Implicit Definitions

Consider the words *convex, limit, continuous, connected, smooth.* These have their standard ordinary meanings, but they have also all been given—by mathematicians—precise technical meanings referring to concepts within the formal vocabulary of mathematics. Their formal mathematical definitions capture, in a pleasing and pragmatic way, some essence of our prior worldly intuitions regarding these concepts. Now, each of these mathematical concepts have straightforward formal definitions expressed in terms of explicit criteria that—at least in favorable situations—can be checked. For example, a region R in the plane is *convex* if for any two points P and Q in that region the entire straight line segment between P and Q also lies in the region.



But this format (Something is an X if and only if Y holds) is only one of many formats for a "definition of X." Also pleasing are *indirect* or *implicit* definitions, such as definitions via a collection of "postulates," where the whole collection taken together happens to pinpoint a concept unequivocally. My current favorite for that sort of implicit-definition-by–a-collection-of-postulates comes from ancient mathematics and occurs in the treatise Archimedes where he discusses his "law of the lever." This is in the opening of Book I *On the equilibrium of planes.* Archimedes defines *equilibrium* and *center of mass* by beaming in on those concepts—cornering them, so to speak—by a series of seven postulates, the first and fourth of these postulates being:

- Equal weights at equal distances are *in equilibrium*, and equal weights at unequal distances are not in equilibrium but incline toward the weight at the greater distance.
- When equal and similar plane figures coincide if applied to one another, their *centers of gravity* similarly coincide.

A curiosity here is that any Calculus student, nowadays, could come up with straight definitions of *equilibrium* and *center of mass*. But—given the vocabulary of the time— *only* an implicit definition was readily available to Archimedes[5] who used them to establish his astounding discoveries of the ratios of areas (of segments) and of volumes (of spheres, cones, and prisms).

Perhaps the most famous ancient implicit definition is the definition of *Euclidean plane geometry* is given by the collection of "common notions" (the axioms of this geometry) set down by Euclid.

4 Characterizations and Definitions

I said above that Dedekind's elegant way of characterizing *infinite* sets offers a definition (an "alternate definition" if you wish) of the notion of *infinite* sets. There are, though, subtle differences between a property that merely *characterizes* a concept, and a property that you actually want to put forward as a reasonable *basic definition* of the concept. This *difference* I'm referring to is like the difference that philosophers make between between definitions via *intention* and definitions via *extension*, a distinction that is elegantly illustrated by the tale told by Diogenes Laertius about one of the hijinks played by Diogenes of Sinope (alias: "Diogenes the Cynic") in Plato's academy[6] (pucking the feathers of a chicken to produce a

[5] There is no stigma to implicit definition, though!

[6]

"man," i.e., a creature that fits Plato's characterization of man as *featherless biped*).

But here are two examples in mathematics. In the first, the "characterization" would be ludicrous if viewed as a primary definition. In the second, the situation is much more complicated. In this discussion, by the term *number* I will mean *positive whole number*.

1. By a *perfect power* I mean a number that is a square, or a cube, or a higher power of a number. The numbers 8 and 9 are the only two *consecutive* perfect powers:

   $$8 = 2^3 \text{ and } 9 = 3^2.$$

   This fact, very difficult to establish, was known as the Catalan Problem (posed in 1844) and was only recently proved (by Preda Mihailescu in 2002). So the property of being *consecutive perfect powers* characterizes the pair of numbers $\{8, 9\}$ but it would be a bit of a stretch to hope to view this property as a useful *definition* of that pair of numbers.

2. Contrast the above example with the following *definition* and *characterization* of a single concept, each developing into genuinely usable *primary* definitions, although of *different* concepts in the more general context. In **(a)** below you'll see a "standard definition" of prime numbers, and (in **(b)** below you'll see a "famous characterization" of prime numbers. Something funny happens in the history of the subject, related to these two ways of defining prime numbers.

   (a) The usual *definition* of prime number:

      A prime number p is a number > 1 that is not the product of two smaller numbers.

   (b) A *characterization* of prime number:

Plato had defined Man as an animal, biped and featherless, and was applauded. Diogenes plucked a fowl and brought it into the lecture-room with the words, "Here is Plato's man."

(from Diogenes Laertius. *Lives of Eminent Philosophers.* Vol. 2. Trans. R. D. Hicks. London, William Heinemann, 1925)

A number $n > 1$ is **prime** if and only if it has the following property: whenever n divides a product of two numbers $a \cdot b$ then n either divides a, or n divides b.

Now, as is shown in Euclid's *Elements*, the *definition* **(a)** and the *characterization* **(b)** describe exactly the same (infinite) collection of numbers—the usual primes

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$$

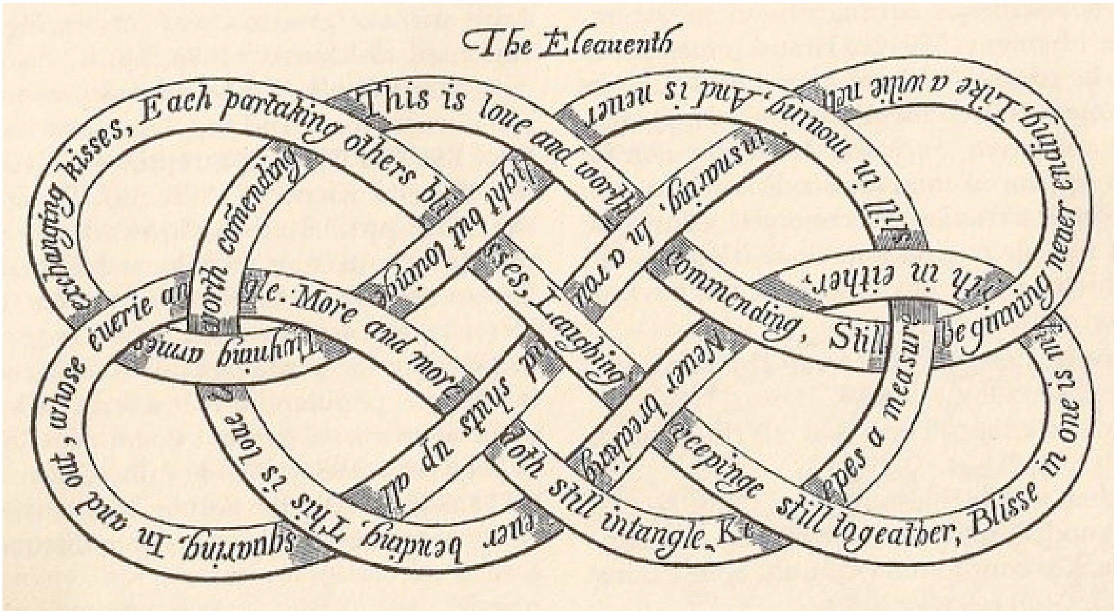
Since the tradition in mathematics is to always try to find the appropriate general formulation of any concept so as to make the most use of the concept, and to achieve the best comprehension of it, it should not be a surprise that both *definition* **(a)** and *characterization* **(b)** have been extended to broader and broader contexts. In these larger arenas they diverge significantly: *definition* **(a)** gives rise to the general notion called **irreducibility** while *characterization* **(b)** becomes the forerunner of the fundamental notion that is known in the trade as **prime ideal**, a concept equally fundamental, but different[7].

5 How it looks on the page

The visual aspect of a poem, as in the spacing between words, the indentations, the line breaks, the squeeze of text between two large margins, and—simply– the look of it, is sometimes an important part of our experience of the poem. I don't only mean those marvelous visual poems[8] such as

[7]The question of whether—in a given number system—these two concepts coincide—i.e., whether every prime ideal is generated by a single irreducible element or are distinct, is one of the subtle questions in arithmetic about which even today there remain unresolved questions, one such conjecture due to Karl Friedrich Gauss (1777-1855).

[8]For a thrilling compendium of ancient and medieval visual poems, see (*)



But the visual aspect is important for almost any written poem, for example, the poem we have just discussed, *Corson's Inlet*, which crags visually on the page just as the coastline along which the poet is walking is fractally crenellated. Here I quote a longer swath of that poem (containing the lines given earlier):

> the news to my left over the dunes and
> reeds and bayberry clumps was
> fall: thousands of tree swallows
> gathering for flight:
> an order held
> in constant change: a congregation
> rich with entropy: nevertheless, separable, noticeable
> as one event,
> not chaos: preparations for
> flight from winter,
> cheet, cheet, cheet, cheet, wings rifling the green clumps
> beaks
> at the bayberries

Punctuation is sometimes exquisitely exact in a poem. Sometimes it is fashioned by the poet into an idiolect, a private sign language so to speak, as in Emily Dickinson's work (discussed in Heather McHugh's essay *What Emily Dickinson makes a dash for*[9]). And sometimes—like the chain of colons cascading down the portion of *Corson's Inlet* we've just quoted—it plays the dual role of logical connective and visual effect, illustrating, perhaps, what T.S. Eliot meant when he said that *poetry is a form of punctuation.*

The way in which mathematics is displayed on the page is hugely important for clear communication of meaning. Mathematicians have devised manners of representation of information that elegantly do the job—and sometimes this requires a departure from the standard word-after-word approach of ordinary language. Graphs and histograms abound. As do matrices, which are rectangular arrays of data, such as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ & & & \dots & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}.$$

But my favorite pieces of mathematical terminology that are wonderfully visually self-explanatory were dreamt up by Leibniz and have to do, respectively, with the differential and integral Calculus; namely, the compact and elegant notation:

$$\frac{df}{dt} \text{ and } \int_a^b f(x)dx.$$

If you have never taken Calculus, here is a crash course in these two examples of brilliant notation.

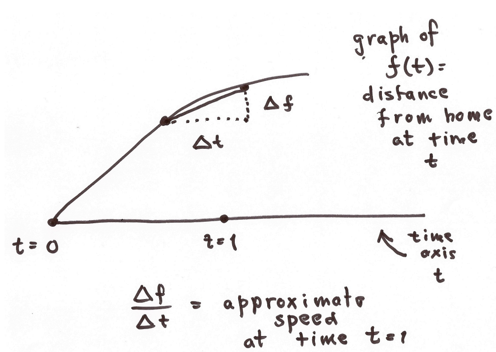
9

> [Emily Dickinson's] richest work is precisely what critics since Higginson have called "elusive," and its signature is the sign of the dash—that suspense of punctuation, that undecidability, which is not an indecision.

(from Heather McHugh, *Broken English: poetry and partiality*, Wesleyan University Press (1993))

$\frac{df}{dt}$

Let t be a variable that represents time (so, let us say that $t = 0$ is Yogi Berra's *now* and that $t = 1$ is one hour in the future). Imagine that we start, now, from home, and that we're driving (say, along a straight road) in a car. For each moment of time t, let $f(t)$ be the total distance we have driven from home up to that time. So $f(0) = 0$ since we've started from home. If, say, we decide (in advance) that we want to compute a rough approximation to "how fast" we are going at time $t = 1$ (i.e., one hour into our journey) we might naturally do something like this: fix a time "just before" $t = 1$ and a time "just after" $t = 1$ and call the difference between those two times Δt (you should think of the symbol Δ as standing for "increment" so Δt is the increment of time between the *before* and the *after*). Your task is then to measure the distance you've traveled between the "before moment" and the "after moment." Call that distance Δf (which you can think of as the "increment of distance" travelled during the chosen incremental time interval). Then, a rough approximation to the *speed* that you were traveling around the time $t = 1$ is the ratio $\frac{\Delta f}{\Delta t}$. (See the figure below)



But this is a mere approximation to the infinitesimal speed at $t = 1$ and to get an "exact reading" (perhaps something closer to the reading of the speedometer at time $t = 1$) we would have had to do something that is humanly impossible: to make a succession of finer and finer readings, choosing our "before" and "after" moments closer and closer to $t = 1$ and pass to the limit of these approximate

$$\frac{\Delta f}{\Delta t}$$

readings[10]. Leibniz's notation for this limit is elegantly

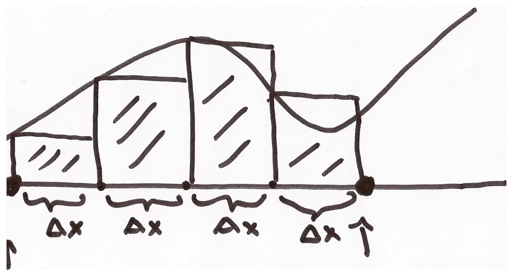
$$\frac{df}{dt}$$

or—since we're dealing only with time $t = 1$ and we might be looking ahead to deal with more general times, we might call it $\frac{df}{dt}(1)$. This symbol which has the "look" of a fraction $\frac{df}{dt}$ is not to be thought of as a fraction, but rather as a *single* symbol: you're not allowed—at least in the context that Leibniz was working—to separate the numerator from the denominator; nevertheless this symbol, faithfully mimicking the notation for the approximations as it does, provides a mnemonic for the structure of this concept and also the symbol offers us a guide for many of the formal manipulations that can be legitimately done with the concept (in certain senses it *does* behave like a fraction).

[10] This limit seems to be something that our speedometer's reading is close enough to, for any practical consideration we might have.

$$\int_a^b f(x)dx$$

(See the figure below)



If you want to compute the area under the graph along the x-axis from a to b, which is the shaded area in the figure above, you first might think of approximating it by taking the sums of the areas of a bunch of rectangles that form the roller-coaster scaffold of the above figure. This area–in the figure as drawn—is the sum of four rectangles all of the same width—which I will call Δx (going from rectangle to rectangle with left-bottom vertices $a = x_1, x_2, x_3, x_4$, respectively, you are creeping by "increments" Δx along the x-axis) and heights $f(x_1), f(x_2), f(x_3), f(x_4)$.

The total area, then, is

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

which in mathematical shorthand is

$$\sum_{i=1}^{4} f(x_i)\Delta x.$$

Since this sum of four rectangular areas is only a crude approximation to the area under the graph of f, happily—in good cases—one can get closer and closer to the area by refining the scaffolding (making Δx smaller, and using more, but thinner, rectangles). If you do this and pass to the limit you no longer are dealing with a ratchety finite sum $\sum_{i=1}^{n} f(x_i)\Delta x$.

So, how then should we *denote* this limit, which at least in our figure will be approximating more and more accurately areas related to a smooth curve? Leibniz's great idea is to keep a strong hint of the shorthand notation displayed above by "sandpapering" the angular edges of the $\sum$ to turn it into a smooth $\int$ and replacing, again, the Δx by dx, and finally recording that we are going from a to b; and voilá: $\int_a^b f(x)dx$. The brilliance of this notation is that many of its formal properties of it can be guessed—or if not quite guessed, at least easily remembered—given the fact that this symbol is *visually analogous* to the finite sum $\sum_{i=1}^{n} f(x_i)\Delta x$.

In both of these symbols, $\frac{df}{dt}$ and $\int_a^b f(x)dx$, we witness the marvelously compact expression of much mathematical insight, rather like what I'm told happens in some Chinese and Japanese poems where the visual characters themselves tell a story that enhances the story they tell as (can I say?) "textual characters." In mathematics, certain ideas are best understood when expressed in compactly abbreviated form, just as in poetry certain emotions are most powerfully evoked by means of a startling economy-of-expression.

6 Unequivocal and Metaphorical Language

Before we get started here let me say that *unequivocal* and *metaphorical*, strange as it may seem, are not in enmity.

The language of mathematics requires unequivocality of the most uncompromising level. A concept given—say—by a symbol must have a crystal clear unambiguous meaning. This is not to say that the symbol need refer to only one "thing." To take a random example, consider the sentence "Let X be a (whole) number between 1 and 10." This is a completely unequivocal definition of the symbol X even though we don't have a "value" that we can assign to it: the symbol X has been appropriately labelled and has a clear meaning. Truth and precision in labeling is key to the conscience of a mathematician.

But mathematics also thrives on metaphor[11]. Since *metaphor* is quite a loaded word and–I think—amorphous, I find it more helpful just to cut off a few bite-size portions of the concept and view each of them separately. Without being specific about what *metaphor* means, I'll list below—in order of complexity— some *metaphorical or meaning-stretching* activities:

- **choosing just the right resonant vocabulary** that calls to mind—by means of metaphorical resonances—the concepts being discussed,
- **noun-ification,**
- **extension of the domain of a concept** guided by the urge to find an appropriately general setting,
- **unification of two or more concepts,** guided by their "similarities" and bridging their differences.

Just the right words. The first of these items, *choosing resonant vocabulary,* is evidently a good thing, and is so often well-practiced—and so noticed when badly practiced—that we may lose sight of the psychological importance it has. When an innovator "chooses" a word or phrase to be given a technical definition there are these possibilities to choose from: either the word has a common everyday use and is being deputized as a technical term; or it is derived from the prior technical vocabulary of the field; or it is purely made-up word; or it is constructed out of a proper noun, or nouns.

We all can produce examples of each of these four categories; for me the following four terms from physics go neatly in order through these categories: *work*, *electro-magnetic, gas,* and *Planck's constant.* The stroke of real genius among these four labels seems to be *gas*, which—as I understand it—was consciously made up from nothing to be a general serviceable term used exactly as widely and as effectively as the word *gas* is used today.

In mathematics it is useful to have fully resonant words to label concepts that attempt to capture basically intuitive concepts, so the

[11] and this theme is developed in various essays and books; cf. Yuri Manin's wonderful collection of essays *Mathematics as Metaphor* A.M.S. (2007) or the work of George Lakoff and Rafael Nuñez (e.g., *Where Mathematics Comes from* published by Basic Books).

list of words given above, *limit, continuous, connected, smooth,* all come to mathematics resounding with the music of their real-world resonances; this is psychologically very helpful insofar as their mathematical definitions are meant to give precise mathematical meaning to these intuitive notions. But there is no lack of importation of evocative words to mathematics as labels of precise technical terms. *Sheaves, fiber bundles, cross-section* come immediately to my mind, as does the French word *étale* which, in my field, is used as noun or adjective and has international significance as a technical term. In French its adjectival meaning is:

1. without movement, immobile (une mer étale)
2. without agitation (une journée étale)

and its nominal use is, for example, to denote the moment of slack tide.

Noun-ification: Surely the first human use of the concept *five,* for example, must have been in some adjectival setting, such as *five cows* or *five fingers.* The act of realization that there is a "thing" to be reflected on, about the similarity between those various uses, and that one could focus on these reflections better if one re-ified, so to speak, the adjective by forming the noun *five* must have been one of the great leaps of early mathematical thought. I find that a lot of writing about metaphor in mathematics devotes too much attention on leaps of thought in more technical domains (e.g. about complex numbers, or in even more specific areas of mathematics) that are not as universally shared (i.e., by people not in math or the sciences). I also feel that focusing on the simplest concepts concentrates the discussion better, invites more people into that discussion, and—at bottom—may end up dealing with richer metaphorical content than any discussion having to do with specific technical metaphors.

Extension of the domain of a concept: One quite instructive example of this is the extension of the operation of *multiplication* to all the integers, I.e., the metaphorical act I have in mind here is the act of settling on an appropriate meaning for the operation of multiplication of whole numbers, be they positive or negative or zero— a meaning that faithfully and usefully extends the "old" sense of the term, that multiplication is just repeated addition of positive numbers. The act of extending this operation to a larger collection of "numbers" begs for our judgment regarding the suitability of this extension. If you wish to start thinking about this, you might ask yourself why we believe[12] that -1 times -1 equal to $+1$?

Unification of concepts, guided by their "similarities" and bridging their differences A metaphorical bridge, begun in ancient mathematics and going on—in more and more profound ways—in modern mathematics is to view *geometry as algebra* and *algebra as geometry*. Mathematicians have welded these great fields—each with their own distinct brand of intuition—into a grand synthesis where there is a combined intuitive power not matched by either alone. René Descartes, commenting about his merger of algebra and Euclidean geometry, said:

> I would borrow the best of geometry and of algebra and correct all the faults of the one by the other.

This synthesis that mathematicians have created by yoking geometry (with its vibrant visual intuition) with algebra (with its more verbal, symbolic, combinatorial intuition) is, perhaps one of the most venerable, but hardly the only grand unification of subjects, converting an elusive analogy to an illuminating unity. Mathematics is rife with these. In number theory, one of the great analogies that ties together two different fields (each with different fundamental guiding intuitions) is called the *Langlands Program* which would yoke algebraic number theory (which in the old days at times went under the name of "the higher arithmetic") with group representation theory (and specifically the part of that subject that combines analysis of continuous groups with algebraic geometry).

To pass to a more mundane example, I think that the way we *all* treat *time* as *distance* has lots to teach us.

7 Language restrained

Here is a dumb question. As I typed this section of my talk I watched through my study window the leafy tree waving in the breeze and partly obscuring the vertical fire escape ladder hugging

[12] and we do

the tan stucco side of my neighbor's house. If I didn't have language, how much of this scene could I have experienced? (I don't mean *described*, I mean *experienced*.) Probably not much.

I find this a tantalizing question, and feel, for example, that had I no word to express the concept of *tree* my experience of the scene would be significantly different. I certainly don't believe, with the Laputans in Jonathan Swift's *Gulliver's Travels*, that "in reality all things imaginable are but nouns," but I do believe that certain things that we deem imaginable would not be so if there were no nouns to pin them down. (And other parts of speech or forms of syntax!)

There is, of course, a hefty meta-literature of experiments in fiction and poetry that deprive language of certain strategies, allowing us to contemplate what might just happen to the imagination as a result. There are, for example, the resourceful Laputans alluded to above who felt that

> since words are only names for things, it would be more convenient for all men to carry about them such things as were necessary to express the particular business they are to discourse on.

Or there is the cry for a banishment of metaphor utterly, in the Wallace Stevens poem *The man on the dump*, which begins:

> Day creeps down. The moon is creeping up.
> The sun is a corbeil of flowers the moon Blanche
> Places there, a bouquet. Ho-ho ... The dump is full
> Of images. . . .

but things change in the course of the poem:

> Everything is shed; and the moon comes up as the moon
> (All its images are in the dump) and you see
> As a man (not like an image of a man),
> You see the moon rise in the empty sky.

and the poem ends starkly:

> Where was it one first heard of the truth? The the.

Something perplexing, and significant, is going on in this and other poems by Wallace Stevens. Why would you discard images? The simpleminded answer to this is that *images, metaphors* are getting in the way of achieving some level of direct experience, unimpeded by— as Wallace Stevens puts it in another poem— the "vital I." Another possibility is that depriving ourselves of metaphor somehow makes those things-in-themselves *unfamiliar* to us, but in a useful way.

However we conceive it, this seems not to be a call for language to *bring experience more vitally to us* but rather, for language to *get out of the way* so that we can go about our business of "being," less encumbered.

A phenomenon, in some ways similar and in some ways not, happens in ordinary language all the time. Some words—maybe all words— can play the role of *protector* for us when it is efficacious for us *not to think* rather than to think. Euphemisms such as *collateral damage* do their job, sometimes frighteningly well. But more commonly there is the process dubbed by Victor Shklovsky[13] *algebrization.* Here is Charles Baxter's description of Shklovsky's idea[14]:

> Algebrization is the process of turning an event or familiar object into an automatic symbol. It's like saying *Oh, she's having another one of her crazy tantrums* or *Yeah, it's another goddamn Freeway gridlock.* We protect ourselves from the force of her tantrum by turning it into an algebraic equivalent: let x be the tantrum. Well, she's having another x. It's just one of those things she does.

Here's how Shklovsky describes it:

[13] in an essay entitled *Art as Technique* written in 1917

[14] Charles Baxter, *Burning Down the House: Essays on Fiction* Graywolf Press (1997)

> We see the object as though it were enveloped in a sack. We know what it is by its configuration, but we only see its silhouette.

Shklovsky's 'algebraic' method of thought derives from—and therefore has its counterpart in—the language of mathematics. But it occurs in mathematics with a special "extra": we can actually engage effectively with the symbolic avatars of the mathematical objects that are algebrized, and work with them, calculate them and calculate fluently *with* them, and combine them in larger constellations. There are simply times when the best thing that mathematical language can do for us is to allow us *not to think* rather than the reverse. It would take yet another hour to give you a catalogue—not of the examples of this, but just—of the types of examples of this. Sometimes certain mathematical theories have the virtue of *putting some aspects of the content in the shadows* so that one can focus on form unencumbered by content. Sometimes, on top of that, the theory will package content into *symbols* that compactly hide a complexity of content that would otherwise "get in the way" so that we can get to work with these symbols following the appropriate (pre-thought out) rules.

Here, then, is one example of this *algebrization* in mathematics that has always seemed striking to me: many historical accounts of mathematics have been devoted to delving into aspects of Georg Cantor's magnificent *Theory of Sets* and for good reason: there are wonderful things to relate about the profundity, the complexity, the strangeness, of the theory itself, and of the life stories of the mathematicians involved. Well before Cantor, mathematicians and philosophers[15] thought about 'collections of objects.' People also considered diagrams to depict sets and their inter-relations. For example, Euler in the eighteenth century and Venn in the late nineteenth century used diagrams like:

[15] Aristotle included

but given Cantor's aims it was very natural for him to turn to the algebra of symbols, rather than the geometry of diagrams—in short, to "algebrize" his theory—which required him, among other things, to actually assign letters to denote sets. Merely saying "Let S be a set" puts one in the frame of mind to deal with sets as manipulable objects with a salient structure of their own and amenable to a kind of algebra. This simple act of "algebrization" ("Let S be a set with such-and-such properties") is something that is second-nature to any modern practicing mathematician; we do this without thinking and it is a source of considerable power. It is so natural a strategy that, at times, I forget that it *is* a strategy: there were epochs during which much great mathematics was done, but no mathematician had thought of symbolizing sets in order to work with them the more fluently.

In summary:

- poets may–in certain cases— choose to constrain image-making-language in order to allow for either a more direct confrontation with *things-in-themselves* or a more telling slant confrontation, thereby intensifying *feeling*, and
- mathematicians may choose to encapsulate the content of their language in *symbols* so that—instead of being enmeshed in the particularity of content one can all the more naturally work with, and all the more intensely understand, the shape of things.

Here, then, are ways in which language is controlled exquisitely to pilot our imagination in the directions we choose: in poetry it is often to intensify feeling, and in mathematics it is often to intensify understanding.

8 Intuitions, sensibilities

At the beginning of this talk I mentioned two constituent elements of our imaginative apparatus. Namely:

- our *intuitions,* by which I mean the various resources available to us that allow us to "see" things— either in the literal sense of picturing things or comprehending them by means of some type of visualization; or in the metaphorical sense of grasping ideas and being "at home" with them.

And

- our *sensibilities* by which I mean the various resources available to us that allow us to "feel" things.

I don't know what these useful angels (intuitions and sensibilities) really are; but, at least in my personal experience, they are constantly wrestling with language, and are alternately trammeled and nurtured by language. This internal tug-of-. . . if not war, then art, must be quite a common experience and I very much look forward to our discussion about it.