Visions, Dreams, and Mathematics

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1 Introduction

Mathematicians can hardly avoid making use of stories of various kinds, to say nothing of images, sketches, and diagrams, to help convey the meaning of their accomplishments, and of their aims. As Peter Galison has pointed out, we mathematicians often are nevertheless silent—or perhaps even uneasy—about the role that stories and images play in our work.

If someone asks us *What is X?* where X is some mathematical concept, we boldly answer, for we have been well trained in the art of definitions. All the fine articulations of logical structure are at our fingertips. If, however, someone asks us *What does X mean?* we respond as any human must respond when explaining the *meaning* of something: we are thrust into the whirlwind of interpretations, intentions, aims, expectations, desires, and shades of significance that, in effect, depend largely upon the story we have woven around the concept. Consider, for example, the innocuous question:

What does it mean to find X in the polynomial equation $X^2 = 2$?

We frame a narrative the minute we open our mouths to answer this question.

If we say "$X = \pm\sqrt{2}$" *without* realizing that all we've done is just to give a cipher-like name "$\sqrt{2}$" to whatever is a solution of the problem, and have done hardly more than register that there are two solutions, we will have—in essence—reenacted the following joke posted by some high school student on the internet:

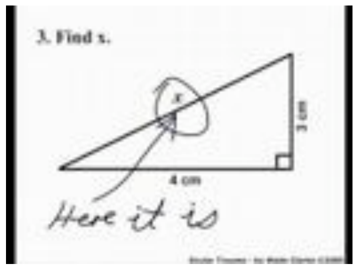

Figure 1:

If we *say* exactly the same thing, "$X = \pm\sqrt{2}$," but *fully* realizing that we've just given a cipher-like name "$\sqrt{2}$" to whatever is the solution of the problem, thereby christening an entity about which *all we know, and possibly, all we need to know* is that it behaves like any other number and that its square is two, then we will have—in essence—reenacted one of the great advances in early modern algebra that gives us extraordinary power in our dealings with algebraic numbers. This is a viewpoint to which the name Leopoldt Kronecker is often attached.

If we say $X = \pm 1.414\ldots$ we will be thrusting our problem into yet another context, with its own interpretations, and narrative.

Our story will be about Kronecker's desire—his *dream,* I will sometimes call it—-to find solutions of a large and interesting collection of polynomial equations. But, as we have just seen, what it means to find solutions—even for a single equation—requires framing. In fact, I will be less interested in Kronecker, and more in the disembodied desire, the dream, the *frame,* and especially how it changes as it is shaped by generations of mathematicians: I want to think about the voyage, if I can use that language, of the dream.

2 Voyages

The hero sets out... And then, if the story is like most good ones, the tale will make us passionately concerned about the hero's moments of elation and disappointment; love and death. For the voyager setting out with ambitious dreams, yes,

> L'univers est égal à son vaste appétit.
> Ah! que le monde est grand à la clarté des lampes!

But things—happily—don't always end up with the tragic disillusionment of Baudelaire's line:

Aux yeux du souvenir que le monde est petit!

A philosopher, and friend of mine, David Lachterman—who wrote a surprising book, *The Ethics of Geometry*—once said, with a hint of superiority, as I tried to explain some mathematics to him:

> in dark contrast to philosophy, there is no *tragedy* in mathematics ...

He meant, of course, no tragic ideas—no tragedy treated in the substance of the "ideas"—that form the staple of mathematics[1].

Real voyages, or fictional ones, are often resonant with impending loss, and accounts of them need only give the barest clues for us to detect a tragic timbre, as when a depressed schoolteacher opens his narrative asking to be referred to as Ishmael, or even as in the seemingly liberating opening lines of Kawabata's *Snow Country*

> The train came out of the long tunnel into the snow country.

Mathematics also has its voyages, of a sort[2], that begin with some *idea*, a vision of some mathematician who—because of the energy and urgency of the idea— is goaded on to try to achieve some grand project—a prophetic dream of some future theory to be developed. A *Dream* in short[3].

Some years ago, a certain mathematician—call him or her X—in commenting on the huge talent displayed by another mathematician Y—made a trenchant after-remark: "Y is an extraordinary mathematician, but he *has no dreams*." The expectation, then, is that good mathematicians have them. What does it mean to have—in the sense implied by that remark—-dreams? The old Delmore Schwartz short story *In Dreams Begin Responsibilities*[4] gets its energy from the urgency of a different genre of dream. But all dreams of vision—be it Martin Luther King's where it is a call to action—or Kronecker's, the particular focus of this essay, where it is a call to contemplation—come with responsibilities.

There are many examples where the artist, the scientist, or the mathematician has a vision of some way—as yet unformed—of thinking. And I don't mean merely of some *thing*

[1] He couldn't possibly have meant that there is none in the lives of the practitioners, on whom the fates have proportioned almost as much misfortune as on the rest of humanity.

[2] See Apostolos Doxiadis's essay *Euclid's Poetics: An examination of the similarity between narrative and proof*, where—among other things—*construction of a narrative* is compared with *construction of a proof* and where both are metaphorically voyages from one place to another, and the places "visited" can be laid out as on a map

[3] a wide-awake dream therefore; as distinct from sleeping dreams that contain mathematical ideas that can be transported to our waking life, such as is one of the themes of Michael Harris's great essay in this volume

[4] The protagonist is dreaming about watching a movie of his parents' courtship, and screams things at the screen. The *responsibility* for the character in that story, is to break away from his parents, to become an artist.

never before thought but rather, more wrenchingly, of some entire *way of thinking* never before thought. The *responsibility* is then clear: to follow it where it leads.

There is one striking difference between a straight story of a voyage[5] and any voyage of ideas in mathematics or in any of the sciences. Although the initial "traveller" is a person, a lone mathematician perhaps, if the arc of mathematical discovery and enlightenment provided by the dream is large enough, it is the disembodied dream that takes over; it is the idea that (or who) is the protagonist[6] and who continues the voyage.

The "story" aspect of this article is a prophetic vision of Kronecker—where I will take the vision itself (rather than the man Kronecker) as the only protagonist—to muse about its birth, its development, and the ingredients of its character. I don't mean to be taking a German Romantic stance and insisting on "idea" as "character," with a life of its own; just a storyteller's stance, with the view that this may be the best organization of a narrative that vividly brings home the manner in which Kronecker's ideas arose, unfolded, and even now envelope the goals of current mathematicians. I learned in conversation with some of the contributors to this volume how problematic it is to employ the word *character* in this somewhat disembodied setting, but I feel that it should be harmless if, instead of *character,* I view Kronecker's vision as something of an *agent* in the tale that I will recount.

3 Biographies of ideas

People sometimes say "that idea X took on a life of its own" and this brand of anthropomorphization often signals that it is the type of idea that can be most fully understood only by a narrative where the idea itself, X,—rather than the multitude of personalities who gave birth to it, developed it, extended it—occupies center stage. A quarter of a century ago, I.R. Shavarevich expressed a related thought, musing about a—fictional, to be sure—single nonhuman protagonist orchestrating mathematics as a whole.

> Viewed superficially, mathematics is the result of centuries of effort by many thousands of largely unconnected individuals scattered across continents, centuries and millennia. However the internal logic of its development much more resembles the work of a single intellect developing its thought in a continuous and systematic way, and only using as a means a multiplicity of human

[5] such as Rory Stewart's illuminating *The Places in Between* where the narrative trajectory has an elegant simplicity: walking in a straight line across Afghanistan, while the telling of it has an obsessive vivacity

[6] A (quite short) story of Chekhov has this type of arc, where the ostensible protagonist *Gusev* somehow, only once dead and summarily buried at sea, "covered with foam and for a moment [he] looked as though he were wrapped in lace,"—only then—does some non-living sense of Gusev soar, suffused into the water below and sky above.

> individualities—much as in an orchestra playing a symphony written by some composer the theme moves from one instrument to another, so that as soon as one performer is forced to cut short his part, it is taken up by another player, who continues it with due attention to the score[7].

An idea may begin as the passionate and precise goal of a single person, and then diffuse into something less tangible and more persuasive and pervasive, taken up by many. The felt experience (by people contemplating mathematics) that some of these multiply-shared ideas seem to have an uncanny unity—as if orchestrated by a single intelligence, as Shafarevich put it—-deserves, I believe, to be discussed along with the more common discussions regarding the felt experience of (what is often called) platonism in mathematics, i.e., that mathematical concepts are getting close to Plato's *eidoi,* those joists and pinions in the architecture of the cosmos; or more briefly—and in the standard peculiar way of saying it—that mathematical concepts are "out there.")

Contemporary mathematics is rich in its broad horizon—with magnificent programs pointing to future large understandings. But one doesn't have to go too far into the subject to get a sense of traces of mighty illuminations that must have sparked visions.

Was there, for example, some ancient, somewhere, who realized that *five cows, five days,* and *five fingers* have something in common, and that if one—by a strange twist of thought, and by fiat—expresses that *something* as a noun, i.e., as the concept *five,* one will be setting off on a worthwhile path of thought?

Some more modern path-setters are quite conscious of the "setting out on a new path of thought," and at the same time humble in reflecting on the hardship their predecessors may have encountered pursuing the early visions in the subject. Here is Alexander Grothendieck (in the introduction of his masterpiece *le Langage des Schémas*) reflecting on the difficulty of grasping his new vision—and on the difficulty that future mathematicians will have to appreciate this "difficulty of grasping":

> Il sera sans doute difficule au mathématicien, dans l'avenir, de se dérober à ce nouvel effort d'abstraction, peut-être assez minime, somme toute, en comparaision de celui fourni par nos péres, se familarisant avec la Théorie des Ensembles.

The mathematical visions that I am currently fascinated by are those that begin with the mission of explaining something precise, and then—because of their extreme success—expand as a template refashioned and reshaped to explain, and to unify, larger and larger constellations of mathematical or scientific issues—this refashioning done by whole generations of mathematicians or scientists, as if a single orchestra. Things become particularly interesting, not when these templates fit perfectly, but rather when they don't quite fit, and yet despite this, their explanatory force, their unifying force, is so intense that we are impelled to reorganize the very constellation they are supposed to explain, so as to make them fit. A clean example of such a vision is *conservation of energy* in Physics, where

[7] I.R. Shafarevich: *On Certain Tendencies in the Development of Mathematics* Poetics Today **3** No. 1. (Winter 1982) pp 5-9. Transl.: A. Shenitzer

the clarity of such a principle is so unifying a template that one perfectly happily has the instinct of *preservation of conservation laws* by simply expecting, and possibly positing, new, as yet unconsidered, agents—if it comes to that—to balance the books, and thereby retain the principle of conservation of energy. Such visions become organizing principles, so useful in determining the *phenomena* to be explained, and at the same time in shaping what it means to *explain* the phenomena. There is a curious non-falsifiable element to such principles, for they get to organize our thoughts-about-explanation on a level higher than the notion of *falsifiability* can reach.

I will be telling—actually, just talking about—the story of one such vision, which has a much much smaller imprint that *conservation laws in physics*; nevertheless I love it for many reasons not the least of which is that it begins, as I will tell it, with one of the sparks that set off Greek mathematics, namely the formula for the length of the diagonal of a square whose sides have unit length (in the story this will have an algebraic disguise). Transformed and extended, the vision—initially referred to as Kronecker's *liebster Jugendtraum*—continues to shape the hopes of a certain branch of mathematics, today. I'll describe a piece of this in elementary terms and discuss the role it has played, and is continuing to play, and its potency as it has suffused into the broad goals of modern number theorists.

4 What are our aims when we tell stories about mathematics?

We should be clear about whether the stories we will be considering are *ends* or *means*. In fiction, telling the story is the ultimate goal, and everything else is a means toward that goal. I suspect that even Sheherezade, despite her dangerous situation, and the immediate mortal purpose for her storytelling, would agree to this. In mathematical expositions most story elements are usually intended to serve the mathematical ideas: story is a means, the ideas are the end.

If, then, stories in mathematical exposition are a means, and not an end, *to what* are they a means: what do they accomplish? Let us try to throw together a provisional taxonomy of "kinds of storytelling" in mathematics, by the various possible answers to this question. I feel that there are three standard forms, and also a fourth form—the one I am interested in—that has to do with the arc of a mathematical vision, the character being the vision itself. My names for the standard ones are

- *Origin-stories* explaining some original motivation for studying the mathematics being described, this motivation being external to the development of mathematical ideas themselves,

- *Purpose-stories* describing some purpose to the mathematical narrative, a purpose external to the context of mathematics itself, and

- *Raisins in the pudding* which are ornamental bits of story meant to provide anecdotal digressions or perhaps a certain amount of relief from the toils of the exposition. At the least they are intended to add extra color. But the primary relationship of the stories or story-fragments in this category to the mathematical subject is ornament: they are not required to help in furthering–in any direct way–the reader's comprehension of the material, nor do they fit in as a part of the structure of the argument presented.

5 Kronecker's Dream

No matter how one tells the story, to my mind, the seed for Kronecker's dream is in Gauss's expression for square roots of integers as trigonometric sums, i.e., as linear combinations of roots of unity. A *root of unity* is an algebraic number with the property that a power of it is equal to 1; so $i = \sqrt{-1}$ is a fourth root of unity, and $e^{\frac{2\pi i}{n}}$ is an n-th root of unity. The ur-example of an expression of a square root of an integer as a trigonometric sum is

$$\sqrt{2} = |1 + i|$$

(more generally, see this footnote[8]). From gazing at this formula to envisioning Kronecker's grand hope is a giant step, and we will proceed slowly. (For one thing, we need wrestle with the question: what does the right-hand side of the formula gain for you in dealing with the left hand-side, and more generally: why is it a good thing to express square roots explicitly as weighted sums of roots of unity?)

Kronecker's Jugendtraum was cryptically expressed as "Hilbert's 12th Problem," and people who wish to follow the narrative of Kronecker's dream with the Hilbert Problems as a backdrop, should consult Norbert Schappacher's *On the History of Hilbert's Twelfth Problem: A Comedy of Errors* which offers both a majestic view of the mathematical climate of the times, and a sensitive close reading of the textual evidence available to us; remnants of this climate. For people with a more technical background who wish to have a full exposition of the mathematics involved, there is the treatise *Kronecker's Jugendtraum and Modular Functions* by S.G. Vladut (Gordon and Breach, New York, 1991).

There are many ways of telling the tale, and in recent epochs Kronecker's Jugendtraum has been folded into one of the grand goals of modern number theory. I will try—at the very end of this essay—to give the briefest indication of what is involved.

[8]If p is an odd prime number, we can—following Gauss—express $\sqrt{p}$ (decorated by a sign and a power of i) as a linear combination of powers of $e^{\frac{2\pi i}{p}}$ as follows:

$$\pm i^{\frac{p-1}{2}}\sqrt{p} = e^{2\pi i/p} + \binom{2}{p} e^{4\pi i/p} + \binom{3}{p} e^{6\pi i/p} + \cdots + \binom{-2}{p} e^{-4\pi i/p} + \binom{-1}{p} e^{-2\pi i/p},$$

where the coefficients in this linear combination are ± 1 and more specifically: $\binom{a}{p}$ is $+1$ if a is a *quadratic residue modulo* p; that is, if a is congruent to the square of an integer modulo p and -1 if not; and even the ambiguous $\pm$ in the formula can be pinned down in a closed form.

6 Boiling it down

But for now, let me successively peel away more and more of the technical context of Kronecker's idea so as to get to what may be thought of as its heart. The first thing to say about it (in slightly more modern vocabulary than Kronecker himself might express himself) is that:

> Kronecker's Jugendtraum is the vision that certain structures in *Algebraic Geometry* and/or *Analytic Geometry* [9] can be put to great service: to provide explicit and elegantly comprehensible expressions—in a uniform language—for an important large class of algebraic numbers.

Stripping away some of the particular technical language of the above description we get that Kronecker's Jugendtraum is of the very broad class of visions of the following kind:

> One mathematical field can be a source of *explanation* by providing *explicit* solutions to problems posed in another mathematical field.

Now, mathematicians who know the technical aspects of this development will, I hope, agree with me that the source of explanatory power in Kronecker's dream is the uniform *explicitness of the solution that he sought, as well as the economy of the vocabulary.*

Let us strip some more, to note that we are dealing here with the interplay of three notions:

- explicit,
- explanation,
- economy.

These notions will form the backbone of our story.

The word *explicit* is an exceedingly loaded (but informally used) word in mathematical literature. What is curious is how quintessentially inexplicit is its definition, for its meaning is very dependent upon context; it's an "I know it when I see it" sort of thing[10]. Often, but not always, to say "X is an explicit solution to Y" is meant to elicit a favorable affect on the part of the reader. On the whole, "explicit" is good. Except, of course, when it is not.

[9] The algebraic and/or Analytic Geometry enters into the story via commutative algebraic groups and structures related to these. For a further comment, see footnote 12.

[10] although Potter Stewart, the Supreme Court Justice who was just quoted, was describing pornography, a concept that only a decade later would be commonly referred to by the adjective *explicit*

To my knowledge, no one has agonized, in print, about the usage—in mathematical literature—of this word *explicit*[11] or asked questions about its evolution. So, one aim of this essay is to make a start here. A companion word to *explicit* is *constructive* with its own vast history, and perhaps a more expressive description of Kronecker's hope is that one might explicitly construct *algebraic number fields*[12] by making heavy use of certain specific, well-understood, algebraic geometric or analytic objects the virtue of this being that the construction would be

- transparently clear,
- uniform (in the sense that it constructs all the fields we want to construct in the same manner, and therefore is)
- strikingly economical,
- allowing us to directly see many of the important properties of the constructed fields, and finally would be:
- definitive in the sense that the construction constructs all the fields we want to construct and none other than them.

The word *explain* is perhaps even more important, but does have an immense literature surrounding it. Nevertheless, I will try out a homegrown discussion of it. And we shall see, I hope, how the notion *economical* plays into both of the other ones.

Let us begin with the mere words.

The word *explicit* is from the Latin *explicitus* related to the verb *explicare* meaning to "unfold, unravel, explain, explicate" (*plicare* means "to fold;" think of the English noun "ply").

The word *explanation* is from the Latin and is related to the verb *explanare* meaning "to make plain or clear, explain," or more literally "to make level, flatten," (planus means "flat" as in the English "plane.") It was originally spelled *explane* with its spelling altered by the influence of the word "plain." The marriage of these two spellings proclaim the "clarity of seeing" that a plane, such as Euclid's, provides us with. This being so, it is no wonder that Matthew Arnold's phrase "a darkling plain" is so trenchant.

A flat plane lets us "see things all in one shot," and the desirability for this "all-in-one-shot-ness" in our explanations is already hidden in the English word *explain*.

[11]But see footnote 15

[12]For a discussion of *algebraic number fields* see Section 8.

7 Three Truths about Explanation

I'm thankful that no one has ever put me up before a blackboard and asked me to explain what I mean when I use the word *explain.* I'm also puzzled that this tends not to happen to me, or to anyone else: we are courteous, and adapt ourselves to an impressionistic sense of each other's general usage of extremely important words—like *explain,*—happy enough that formal and semi-formal words like *prove, demonstrate, show* seem, at least, to have a clearer significance.

The first major truth about *explanation* as opposed to *proof* is that the supreme judge of what *does not* constitute an explanation is the *subject,* i.e., the person to whom things are being explained; in other words, you and me. If you or I feel that X does *not* explain Y to us, there is no appeal; it just doesn't. The explainer might try to rephrase things better, go slower, or even just start again from scratch; but the X didn't work. In a word, there are no "false negatives' in terms of the judgment of the person to whom the explanation is aimed; there are, however, "false positives:" we all have had occasions where we judged something as adequately explained to us, at the time, and later thought differently about the matter.

Things are quite different when it comes to *proof.* The general effect of formal systems, the natural language of proofs, is to de-subjectivize aspects of our science. What exactly constitutes a *proof* is generally thought to be—hoped to be—a pretty objective question. Tim Gowers wrote an absorbing essay entitled "Rough structure and classification,[13]"—which is partly an offering of a collection of open problems that itself paints a vivid personal portrait of one mathematician's approach to his art, and partly an exploratory futuristic vision of an imagined dialogue between mathematician-of the-future and machine-of the-future both conjoined in pursuit of the demonstration of mathematical problems. Apostolos Doxiadis refers to Gowers' description of *proof* in terms of the "equation"

$$\text{PROOF} = \text{EXPLANATION} + \text{GUARANTEE}$$

and this puts its finger on the basic question: what are the units here? I.e., how much weight are each of these two ingredients, *explanation* and *guarantee* given?[14]. For, the summand "guarantee" on the righthand side of this equation must have the imprimateur of *objectivity* (i.e., independence from the whims of any *subject*) before it can play the roll of anything like a guarantee; while the summand "explanation" is—as I have tried to argue—inseparable from the subject to whom the explanation is meant to explain (whatever it

[13] it can be found on-line on Tim Gowers' homepage

[14] That related "equations" have been speculated about for a long time can be seen from the summary written by Felix Klein of a report given on February 16, 1910 in his Göttingen seminar (*On the Psychological Foundations of Mathematics*) by one of the speakers (Bernstein) who distinguished between thinkers of a *constructive* nature and those of an *observing-combining* nature. In mathematicians of the first kind, the lecturer claimed there is perhaps 3/4 logic and 1/4 imagination and with those of the second kind 1/4 logic and 3/4 imagination. Klein discusses this a bit and then abruptly concludes with the comment: "Only when we see clearly here can one hope to write real biographies of mathematicians." (For this text and other insights I am thankful to Eugene Chislenko who will be publishing a full translation and commentary on this seminar.)

is that it is meant to explain). In the future the machine will (as Apostolos would prefer to see it, in his discussion of Tim's essay) do the boring bits of mathematics, and the mathematicians will then be freer to provide "the more intuitive, imaginary ('ghost-like') leaps of creativity."

But a theme represented in a lively manner in our conference is that some day "proof" may boil down largely to a question of "guarantee," with "explanation" occurring only in trace amounts; some day there may come about computer-generated proofs where the supreme judge might–in fact–not be you or me, but rather an android in Michael Harris's sense, an android who graciously provides the argumentation, and simultaneously provides the ironclad "check" of the validity of this argumentation.

As I read Tim's essay, the fun there is to distinguish the *Android* from the *Andr* in the conversation between those two individuals in Tim's chimerical dialogue designed produce a man/machine co-authored theorem. But, of course, there is really no difficulty in distinguishing between those two conversationalists. To argue that there always will be no such difficulty, let me introduce you to Chloe. *Chloe* is what my wife Gretchen and I call the voice that comes out of the Global Positioning System (GPS) device of our car. Chloe, judging by her chats with us, is gentle, almost alluring, always encouraging, and whenever she wants us to make a right or left turn she gives us two-tenths of a mile warning. If we disobey her instructions she exhibits just the tiniest bit of impatience as she says (after taking a breath—perhaps a stifled sigh) "Recalculating!"

Now despite this apparent partnership, there is a great partition between Chloe and us in our communal enterprise: it is we—and not Chloe—who actually have the *desire* to get from one place to another; it is (often) Chloe who possesses the information of how to get from one place to another. This brings me to the second point I want to make, an obvious one: *desire* pure and simple is often the main motivator for explanation ("I want to know how this works, and why") and until we direct our various studies to things that mathematical androids *desire to know,* the relationship between mathematician and computer will be essentially equivalent to the relationship I have to Chloe; and nothing more.

A third major truth ("truism," in fact) about *explanation* is that it is a relative notion. We only explain things in terms of other things. As a result, at any given time it pays to have at hand a good stock of "already understood things" or, at least, of things that we take provisionally *as understood* to which the explanation we are currently being given can then be linked. This type of structure—somewhat like the concept of *stipulation* in the law—is formally branded onto our mathematical syntax in terms of the *common notions* and *postulates* of Euclid, and the axioms of more modern mathematics. This being true, we are often exhorted to (in effect) take things as understood, even when they aren't, so as to provide convenient posts on which to later hang further explanations. When we accept this we are dealing in *explanation-futures* as in a sort of stock-market of the mind. This is more and more curious the closer one examines it, and it is amazing that we feel as satisfied as we sometimes do, in playing this game. Often these posts—utterly un-understood at the time of their installation in our thoughts—eventually grow roots, and become confusedly tagged as "things we understand," out of mere familiarity and nothing more; though—of course, they explain nothing to us. The success of this seems to suggest that the quip

> "Shut-up," he explained.

occurring in a Ring Lardner novel signals a more common—and perhaps a better—explanation than one might at first think. One very common kind of "explanation-future" is experienced when we learn a new word. Consider, for example, what has been explained to us, and what has not, if we participate (as patient) in the following mini-dialogue:

> "Doctor, why do all my muscles ache?"
> "You have myalgia!"
> "Oh!"

For one thing, that our disease has a name is already information; knowing this alone, we know that we are not alone —that there is some recognition of it as an actual disease — that our doctor might possibly have some experience in treating it —that health insurance might pay, etc. But despite the usefulness of all this new information, the response the doctor gave to the patient's question is not an explanation in any reasonable sense; it is, however, a peg on which to hang future real explanations if they ever appear: I can go off, for example, and GOOGLE "myalgia."

It is ridiculously unfair to liken such an "explanation-future" (as X learning that a particular disease—known to X only by its symptoms—has a standard technical Latin name with Greek roots) to a mathematical formula—e.g., such as Gauss's formula, the one cited in the footnote above that expresses a square root as a linear combination of roots of unity. It is unfair because, except for formulas that we label tautologies, any mathematical formula that equates one thing with some other thing is (if correct) valuable and is prima facie explanatory on some level or other. A key to understanding Kronecker's vision is to ask—given, of course, the hindsight won by over a century of further mathematical development—on precisely *what* level is Gauss's formula explanatory?

8 The relative nature of "Explicit"

Just as Gauss's formula offers an explicit expression of square roots in terms of roots of unity, Kronecker's dream is to provide us with some way of explicitly understanding *fields* of *algebraic numbers* that are *abelian* over a given *number field.* As for the italicized technical terms in the previous sentence, let us take them as promissory notes, for I'll discuss them later. For now, it suffices to know that *algebraic numbers* are solutions to polynomial equations of one variable (say X) with coefficients that are ordinary whole numbers or fractions, i.e., are *rational numbers.* So, for example, the (two) solutions to

$$X^2 + X - 1 = 0$$

are both algebraic numbers (the positive one being none other than the "golden mean").

Thus, the general issue in question here is to discuss solutions to certain classes of polynomial equations and to somehow express these solutions *explicitly.*

Now high school algebra prides itself (or at least once did) in offering the famous quadratic formula to its students, so that given any quadratic equation with essentially arbitrary coefficients a, b, c:

$$aX^2 + bX + c = 0$$

high school students can produce explicit solutions to this equation, following the rule:

$$X \;=\; \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

These solutions are then "explicit" in terms of the initial coefficients a, b, c and the operation of extracting a square root. With *explicit,* then, we again have a relative notion: to understand what is meant when someone says that A *is an explicit solution of equation* B*,* we must ask *explicit in terms of what?* We must understand the vocabulary that is allowed. To convince oneself that this is not a trivial point, imagine that you were a traveling judge, trudging through the centuries, judging a contest for the best explicit determination of roots of cubic polynomials in one variable X, and specifically for those polynomial equations that have three real roots[15]. In the 16th century treatise of Bombelli a precise compact little formula is given for "the roots" and it was readily checkable that if you substituted this precise formula for X, it worked; even though the formula could—at least in that century—not lead to even the grossest approximation of its three roots. Would you have awarded Bombelli with his precise formula the prize for producing an "explicit solution" or not? In the subsequent century imagine that Newton entered the contest, sporting Newton's method, which indeed provides usable approximations to the roots, as finely accurate as desired. Would you have awarded Newton the prize? My point, here, is that until you, the judge, decide upon the format and the vocabulary that you will count as explicit, you have no way of gauging who is the victor[16].

[15]Such equations will be discussed by Federica La Nave in this conference.

[16]The wonder of the internet is that the question of what *explicit* means, related to issues such as the one we have raised above has been discussed and even *voted on* in a site called "Yahoo! Answers." The best answer to the questions *what is an explicit solution?* as chosen by voters is the following:

> Numerical solution schemes are often referred to as being explicit or implicit. When a direct computation of the dependent variables can be made in terms of known quantities, the computation is said to be explicit. In contrast, when the dependent variables are defined by coupled sets of equations, and either a matrix or iterative technique is needed to obtain the solution, the numerical method is said to be implicit.

I find some problems and ambiguities in this definition (e.g., the phrase "in terms of" and the gratuitous insistence on "coupled sets of equations") but am intrigued that voters judged that iterative solutions should be classed among the *implicit.* E.g., is

$$X = 1 + \frac{1}{1 + \frac{1}{1+\ldots}}$$

an explicit or implicit solution to the problem of finding a root of the quadratic polynomial cited at the beginning of this section?

Of course, there are mathematical situations when the usage of *explicit* has a perfectly clear interpretation as in an article by Nikos Tzanakis from the University of Crete in Heraklion entitled *Explicit solution of a class of quartic Thue equations*[17] in which, for example it is shown that the equation

$$x^4 - 4x^3y - 3x^2y^2 + 14xy^3 - 4y^4 = -4$$

has only four solutions, these being given explicitly: $(0, \pm 1), (2, 1)$, and $(-2, -1)$.

To seek an explicit solution of some equation is, first, to have a vocabulary in mind in terms of which you wish to phrase your solution explicitly, and only then, to manage to do it. So, if we say that Gauss's equation described in footnote 8 above offers an explicit representation of square roots, we have (implicitly) chosen as our target vocabulary: linear combinations (with coefficients ± 1) of roots of unity. We still need to know why this is a particularly good vocabulary in which to express square roots, and how Kronecker took off from this to achieve his grand vision.

9 How Gauss solves a fifth degree equation

Although the vocabulary of extracting roots is sufficient to offer (in the 16th century) formal solutions, and in later centuries also (approximating) numerical solutions to all polynomial equations in one variable with rational coefficients if they are of degree < 5, the mechanism of root extraction alone is insufficient for general fifth degree equations (and also general higher degree equations). Gauss must have known this, and so he tried his hand on finding roots of what might be considered to be the "smallest" (if this makes any sense) polynomial unsolvable by radicals alone, namely

$$X^5 + X + 1.$$

On a page of his private notebooks in the collection of Göttingen mathematical archives one can find his numerical contemplation of that equation where he finds an approximation to its roots. I want to thank the director of the Göttingen mathematical archives, and also Yuri Tschinkel, Professor of the *Mathematisches Institut* for photographing this page of Gauss's notebooks, and allowing me to reproduce it here; and I thank Marie-France Vigneras for her help in figuring out what Gauss seems to be doing on that page.

Gauss labels his page: *Solution of* $X^5 + X + 1$ *by approximation* and he is swifter at this designated task than any of us would be. For example, he begins by simply figuring -0.754877 as the unique real root, and it seems that he has just done the computation to arrive at this entirely in his head—no scratch paper needed at all. But to get the complex roots he works by expressing them as $z := r\cos\phi + ir\sin\phi$ and jots down the equation that you get between r and ϕ if the evaluation of the polynomial on z has vanishing imaginary part. Making a (trial-and-error) guess for the approximate value of ϕ and knowing log

[17]Acta Arithmetica LXIV.3 (1993) 271-283

tables by heart, it seems, he computes r obtaining thereby a candidate z and then checks whether the real part of the polynomial evaluated on z is respectably small; based on the computation of this real part he adjusts his trial-and-error guess accordingly, hoping for a yet smaller real part, etc.[18]

[18]Felix Klein, in a comment in one of his notebooks (May 11, 1875: p. 161 of Nr. 1 Protokolle 1872-1880) writes down the equation $X^5 - X - k = 0$ and says that Kronecker was the first to pose the problem of studying such equations; Klein refers to Hermite who—he says—showed that the relationship of the five roots could be expressed via elliptic functions.

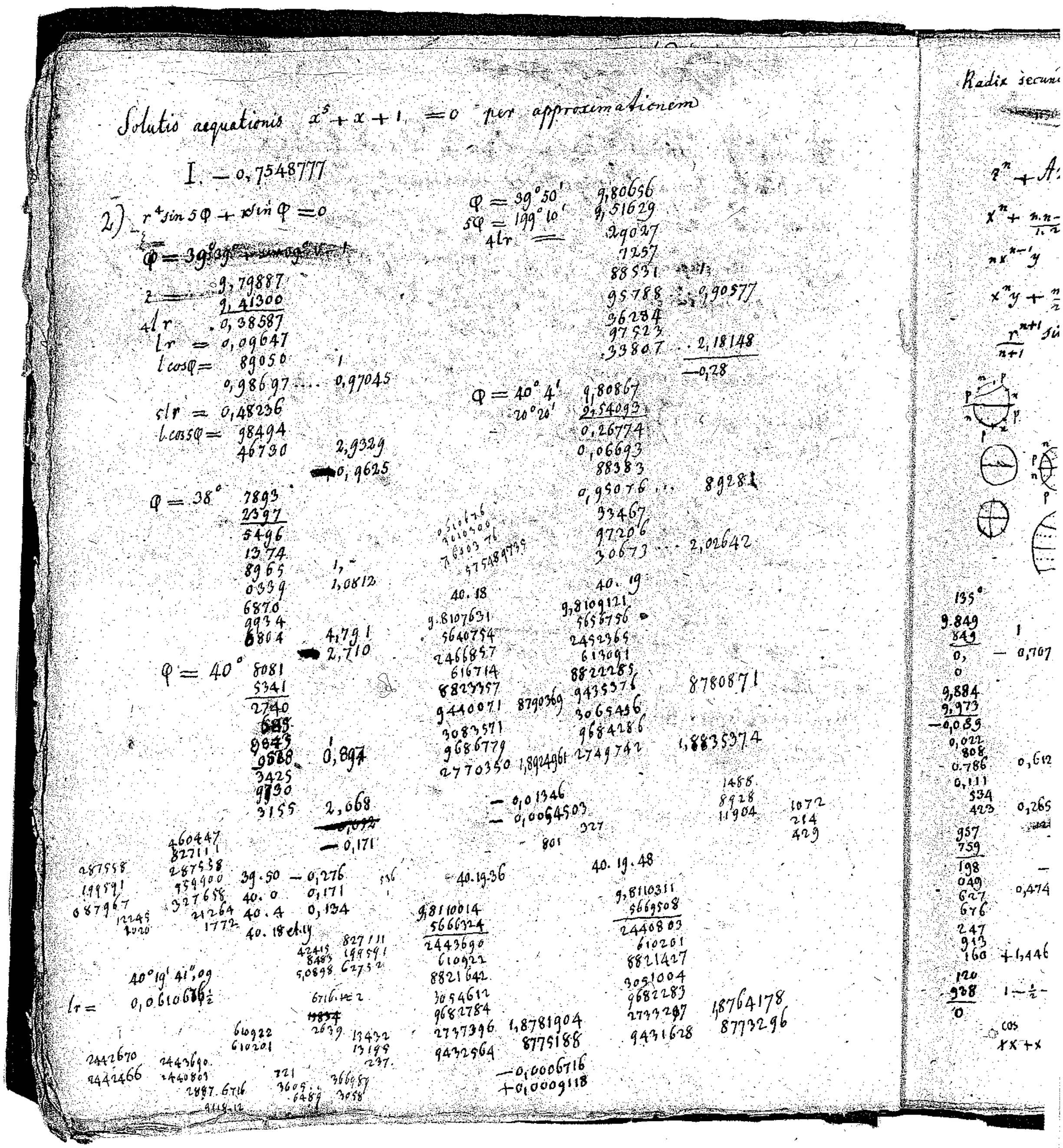
Solutio aequationis $x^5 + x + 1 = 0$ per approximationem
I. $-0,7548777$
2) $r^4 \sin 5\varphi + r \sin \varphi = 0$
$\varphi = 39°50'$
$5\varphi = 199°10'$
$\varphi = 38°$
$\varphi = 40°$
$\varphi = 40°4'$
$40°19'41'',09$
Radix secun

Figure 2:

10 Gradus ad Parnassum

The collection of rational numbers forms what is referred to as a *field* which signifies that it is a collection (of "numbers") in which we can add, subtract, and multiply any two of them to get a well-defined result (a "number") and where the usual rules of addition, subtraction, and multiplication hold; and we can divide, except, of course, by zero. The *rational field of numbers* is a basic object of study, and a key to its hidden properties is to examine the question of which polynomials (in one variable with coefficients that are rational numbers) have rational numbers as roots; and in the event that they don't have rational numbers as roots, how to express their (then necessarily irrational) roots. So $X^2 - 3X + 2$ presents us with no conundrum about "how to express its roots" for the roots are simply $X = 1$ and $X = 2$, while $X^5 - X + 1$ might provide us with a greater challenge to "explicitly" present its roots (especially with the warning of the previous sections in this article, that—strictly speaking–we don't really know what the word "explicitly" means without some agree upon vocabulary in which the sought-for solution is the be explicitly given).

Moreover, just as the word "explicit" is a relative notion, after Galois' famous treatise, *Algebra* itself has come be understood as a relative notion. So if you wish to study any *field of "numbers"*–not only the field of rational numbers—the analogous key to its hidden properties is to examine the question of which polynomials (in one variable with coefficients in that field) have their roots in that field; and in the event that they don't have roots in that field, to cope somehow. If, for example, we adjoin to the rational field some irrational number (say $\sqrt{2}$) to get a larger field (which also means, of course, throwing into this larger field numbers like $\sqrt{2} + 5$, etc. so that we can add, multiply, and appropriately divide within this larger field) we get an example of a *number field* one of the italicized words in section 8 that I had promised to discuss.

One lesson hammered home in Descartes' treatise "Rules for the Direction of the Natural Intelligence" is that one should assiduously proceed—in any intellectual work—*by degrees,* i.e., that there is (often, always?) a natural succession of steps of thought to go through (*Step* 1, *Step* 2, . . .) and that you miss a step at your peril. I assume that Descartes was thinking, in analogy, of the degrees that occur in polynomial equations that cut out curves in "his" Cartesian plane[19]. There are linear equations, quadratic equations, and so on.

Often it makes good sense to go step-by-step, and particularly for the problem of finding roots of polynomials (say in one variable) with rational coefficients. We can deal with linear equations, as could the ancient Babylonians; we can deal with quadratic equations thanks to the "quadratic formula;" with cubic and fourth degree equations, thanks to the work of the 16th century Italian mathematicians. The issue of quintic equations and beyond is the mainstay of modern algebraic techniques, beginning with the work of Abel and Galois.

So Descartes' dictum seem to have been somewhat borne out in this study of equations: we proceed by steps of increasing difficulty, never skipping any of them.

[19]For a concise, yet detailed, description of Descartes' contribution, see page 463 of Robin Hartshorne's *Teaching Geometry According to Euclid* published in the NOTICES of the AMS **47**, April (2000) 460-465.

But there are several ways of cutting steps into this mountain, and in recent times, following Galois, we have an alternate way of gauging the "difficulty" of an irreducible polynomial equation which I will hint at in this footnote[20], and the *abelian* number fields mentioned in section 8 are simply the "step 1" number field extensions—the easiest ones from this alternate way of grading "difficulty."

11 The exponential function as spark

Ernst Kummer knew (and this was before Kronecker had his dream) that if you allowed in your vocabulary a gadget that can extract n-th roots (for any $n = 2, 3, \dots$) you have all the vocabulary you need to describe any algebraic number contained in an abelian extension of a given number field. So when Kronecker dreamt his dream, he was seeking more than this. He–and Weber[21]—had significantly sharpened what would have been Kummer's take on abelian exensions of the rational number field by proving one of the glorious results of that epoch; namely, that the maximal abelian extension field of the rational numbers is the field generated by the collection of all roots of unity; i.e., by the values of the complex analytic function

$$e^{2\pi i z}$$

as z ranges through all rational numbers or equivalently, through all reciprocal positive integers.

Now it is the "i.e." of the previous sentence that reflects more accurately the framework for Kronecker's vision. Here we have one of the most basic (transcendental) analytic functions in mathematics, the exponential function, a lone function, one that will take on transcendental values when z is an algebraic number that is *not* rational, and will take on algebraic values if z is rational. And, surprise, it is precisely those algebraic values, the roots of unity, that turn out to generate precisely the maximal abelian extension of the rational number field—explicating the modern "Step 1" algebraic extensions of the rational number field!

Just as in Gauss's formula, where any square root of a integer can be expressed as a linear combination of roots of unity with ± 1 as coefficients, after Kronecker-Weber we know that any abelian algebraic number can be expressed as a finite linear combination (with rational numbers as coefficients) of roots of unity, or equivalently: of the *values of*

[20]Here the grading of difficulty comes by considering the structure of the group of symmetries of its roots (the *Galois group* of the equation). One marker is the possible *degrees* of linear representations of this group. For this other—more modern—notion of *degree,* Descartes' operating principle also seems to apply: we have an immense literature on the degree 1 (i.e., one-dimensional) linear representations of Galois groups of polynomial equations that have coefficients in the rational field of numbers, and of more general number fields; we have an impressive, but still less complete, literature on degree 2 linear representations of these Galois groups, and for degrees ≥ 3 a very interesting literature, but still less complete.

Given a number field K, an *abelian* number field extension L of K is an extended number field obtained by adjoining all the roots of a polynomial equation with coefficients in K that has a Galois group that can be cut out by degree 1 linear representations.

[21]See Schappacher's account of the intricacy of the history of the Kronecker-Weber Theorem

the transcendental function $e^{2\pi iz}$ *as* z *runs through all rational numbers*[22]. To gauge how mathematicians, on occasion, expressed exuberance over this, count the "wunderbars" in the following (word-for-word) transcription of a piece of a lecture David Hilbert gave in his course (*Vorlesung über die Theorie der Algebraischen Zahlen*) in 1926[23].

> Das ist etwas ganz Eigenartiges. Wir besitzen eine analytische Funktion $e^{2\pi iz}$ mit der wunderbaren Eigenschaft, dass sie für rationale Argumentwert immer algebraische Werte liefert und dass man durch sie alle Abelschen Körper und nur diese erhält. Diese zweite Eigenschaft ist ja der Inhalt des grossen Kroneckerschen Satzes, dass alle Abelschen Körper Kreiskörper sind. Dass ist nun in der Tat eine ganz wunderbare Eigenschaft. Schon allein dass eine transzendente Function algebraiche Werte liefert, wenn man das Argument $z = a/b$ setzt! Dass es so etwas überhaupt gibt! Das Seltsame ist nun dabei, dass man nur die Funktion $e^{2\pi iz}$ zu besitzen braucht und dass dann alles andere sich ganz von selbst, förmlich ohne unser Hinsutun sich einstellt! Dass gilt also für das Problem, alle Abelschen Körper über den Körper der rationalen Zahlen aufzustellen. Unser neues Problem heisst nun, alle Abelschen Körper über dem imaginär quadratischen Körper $k(\sqrt{m})$ $m < 0$ zu erhalten. Für die Erledigung des ersten Problems stellte such uns de Funktion $e^{2\pi iz}$ zur Verfügung, dieser wunderbare Geschenk des Himmels. Werden wir nun auch für den zweiten Fall etwas Aehnliches erhalten? Das ist de Frage, die wir auch gar nicht umgehen können.

But what economy! And what a template! A single analytic function—the exponential function—a major player in Calculus, and analysis, giving us the explanatory key to a major piece of number theory, all abelian extensions of the rational field. In sum, a concept in one field (analysis) explaining with striking economy an important structure in quite a different field (number theory). And so the quest is on.

- Can we use, or modify, this template to similarly explicate the modern "Step 1" algebraic extensions of other number fields?

  Kronecker dreamt specifically of the following kind of ground number field over which to consider abelian extensions; namely *quadratic imaginary fields* which are fields that are generated over the rational field by the adjunction of some imaginary quadratic

[22] The fact that we have stated to equivalent formulations parallels the statement, in section 6 that Kronecker's dream uses *Algebraic Geometry* and/or *Analytic Geometry* to explain whatever algebraic number fields (i.e., the ones abelian over the rational field) that it explains: the vocabulary of "roots of unity" provides the algebraic geometric formulation, which in further aspects of the dream would be replaced by torsion points in commutative algebraic groups, while the vocabulary of "values of the transcendental function $e^{2\pi iz}$ for rational z" gives the analytic geometric formulation where one unravels the algebraic group analytically and expresses things in terms of values of some uniformizing analytic function.

[23] I'm thankful to Yuri Tschinkel and the Göttingen library for permission to quote from these notes.

quantity, i.e., $\sqrt{-1}$ or $\sqrt{-2}$ or $\sqrt{-3}$, etc. and wished to extend, and/or replace, the exponential function and its values for rational z—by appropriate values of the elliptic modular function, and of its companion functions (these being certain Weierstrass $\mathcal{P}$-functions). The elliptic modular function, also colloquially referred to as the "j-function," has a famous Fourier expansion

$$j(z) = e^{-2\pi iz} + 744 + 1986884 \cdot e^{2\pi iz} + 21493760 \cdot e^{4\pi iz} + 864299970 \cdot e^{6\pi iz} + \dots$$

where each of its coefficients tells its own story (these coefficients link to what is known in the trade as "monstrous moonshine").

An example is now necessary:

12 How Cardano, Gauss, Newton, and Kronecker might solve a cubic equation

For our example I want to take a random equation like the polynomial of degree three

$$X^3 + 3491750X^2 - 5151296875X + 12771880859375.$$

(Well, it isn't quite random ... it *is* rigged, a bit.) How might the mathematicians in the title of this section deal with the problem of finding roots of this polynomial?

Cardano: He would use the famous formula giving the solutions of the general cubic equation in terms of square roots and cube roots of simple expressions involving the coefficients of the polynomial, and since this cubic polynomial has a *single* real root, this will work fine. Having found a real root θ he can then divide the polynomial by $X - \theta$ to get a quadratic polynomial, whose two roots he will get by the classic quadratic formula.
Gauss: He could, of course, rely on Cardano's formula. But I imagine that he'd also just figure in his head an astoundingly good approximation to the real root, and proceed from there.
Newton: His famous iterative method (*Newton Approximation*) will do the job of approximating a real root.
Kronecker: If someone gave him the hint that the polynomial was rigged for his benefit, he would surely put his cherished j-function to use, and—given the methods available to him—he could readily see that $j\left(\frac{1+\sqrt{-23}}{2}\right)$ is a real root of that displayed polynomial.

If you adjoin all the roots of this polynomial to the rational number field you get what is called the *Hilbert Class Field* of the quadratic imaginary field generated by $\sqrt{-23}$ that is, the largest abelian field extension of this quadratic imaginary field that is *everywhere unramified*[24]. It is, in particular an abelian extension of the quadratic imaginary field and therefore is subject to Kronecker's very general theory which guarantees that it—-the field—can be generated by special values of the very special functions like the j-function that Kronecker works with. In this instance—Kronecker's Jugendtraum tells us that the field extension can be generated by the single value $j\left(\frac{1+\sqrt{-23}}{2}\right)$. Kronecker would know this intimately, since this much is part of his general framework. What is a particular piece of luck for Kronecker, is my choice of the polynomial to be the minimal irreducible polynomial over the rationals that this algebraic number $j\left(\frac{1+\sqrt{-23}}{2}\right)$ satisfies[25].

This "example" is a pitifully small instance of what Kronecker has given us, for in the context of quadratic imaginary fields generally, Kronecker's dream has come to a strikingly explicit realization!

But this has only sparked the search for ways of extending Kronecker's template to the range of other number fields—where the template sometimes fits (so-called CM-fields) and where the template tantalizes more than it fits (non-CM-fields). Sometimes, though, it is better for an idea to "tantalize" rather than to "snugly fit," for then it stands a chance of being expanded to act as a guide for even grander goals.

- Can we use, or modify, this template to similarly explicate the modern "Step 2" algebraic extensions of the rational number field or of other number fields?

And here we can only say that the template has been stretched and reshaped to now embrace the goals of a large part of modern number theory, including the Langlands program. The template, in short, is a grand moving frame as—I believe–all great mathematical visions end up being, in their maturity. But what kind of moving frame?

13 The story of a mathematical vision

The person referred to as "I" in René Descartes' *Discourse on Method* claims to have studied a bit of philosophy, logic, and among types of mathematics, geometrical analysis and algebra in his youth, and—finding them each deficient for his purposes (*logic* being infused with harmful elements, and the various branches of *mathematics* being generally confusing and arcane arts)—he sought another "method" which would have the benefits of the virtues of the separate disciplines, but would be free of their cloistered defects. He goes on to offer four economical principles that should guide thought, and then much more.

[24]For an abelian cubic field extension such as this one *everywhere unramified* means that every prime ideal in the ring of integers of the quadratic imaginary field either generates a prime ideal in the ring of integers of the extension field, or else it generates an ideal that splits into the product of three prime ideals.

[25]Kronecker is not the only person enamored of the j-function for its ability to solve polynomial equations. Felix Klein's book *The Icosahedron* is devoted to the use of the j-function to express the roots of the general quintic polynomial. And that some such thing was a possibility had also been noted in the writings of Hermite, and Jacobi.

I imagine that Descartes' famous discoveries in mathematics (the union of geometry and algebra) provided some impetus for this more general view of a *method of thought.*

And assuming this interpretation, I want to take *Discourse on Method* as a ready-at-hand prototype for an entire genre of intellectual voyage where the structure of some initial fundamental and arresting discovery—by virtue of its energy and explanatory force—inspires the formation of a template designed to organize,unify, and explicitly survey a much more general range of ideas; a template that doesn't quite fit but inspires all the more for that.

Many great mathematical visions have this trajectory and Kronecker's Jugendtraum is very much of this form, where what you take as the initial template depends upon how far back in the story you wish to go, but the Kronecker-Weber Theorem offers a strong template—tantalizingly related to, but not perfectly fitting the fully general context of algebraic number theory, and yet suffusing into grand unifying principles; principles that promise mathematics of the future capable of surveying a wide range of material—all-in-one-shot—and explicitly, where the very ground-rules of this explicitness is modeled on the initial template.

EXPLICITUS[26]

[26] "Explicitus" was written at the end of medieval books, originally short for *explicitus est liber* "the book is unrolled."