"What should a professional mathematician know?"

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An interesting and difficult question that Phil Davis and David Mumford posed is:

> What should a professional mathematician now know in order to be considered mathematically "educated" and not merely a brilliant specialist in some sub-sub-field?

1 About the question itself, and its implicit assumption

- It seems to me that grappling with exactly that sort of question is one of the many experiences that might be helpful for one's mathematical education.
- I also would not be entirely willing to immediately write off a "mere brilliant specialist in some sub-sub-field" as being "mathematically uneducated." That is, I'm happy to allow the possibility that a very narrow channel may be all that certain professional mathematicians need in order to achieve a rich mathematical education. I'm cautious about actually naming names here, but I'm thinking of least two individuals whose mathematical minds I admire immensely, and who fit that description. These are people who have attained broad comprehension despite the specificity of their experience.
- When one asks

  > What *should* a professional mathematician now know ...?

  one can think of the *should* from the viewpoint of either the individual mathematician in question (i.e., thinking that he or she "should" know these things to have an enriched professional—or inner—mathematical life) or from the point of view of the community (i.e., it would be good for the community of mathematicians to have lots of people of broadly dispersed specialties who are disposed to engage—mathematically—in a deep way with each other). In either case, the same kind of broad mathematical education is key, and I'm sure that the question is being asked with both *individual good* and *collective good* in mind[1].

[1] In email correspondence, David Mumford also made it clear that one of his and Phil Davis's concerns was this particular benefit to the collective good: broad mathematical education might help ward off the fragmentation of mathematical language into many specialized idiolects.

2 The fruits of education

When we think about the education of any specific person (e.g., Henry Adams, or even T.C. MITS) we probably should focus on the particular set of limitations, foibles, and/or aspects of ignorance that the educational process did ameliorate (if we are viewing it in hindsight) or will ameliorate (if we are viewing it with hopes for the future)[2]. But, no matter where we each start from, when we engage in mathematically educating ourselves it seems to me that we hope to achieve ("in the end") two (somewhat different) sorts of things:

1. We want to have, at the very least, an acquaintance with a relatively large range of mathematical viewpoints: we want to have been exposed to an array of ways of looking and ways of thinking; we want to at least know about the existence of the broad traditions of mathematical experience; and for our core, we want to have a working mastery of a somewhat more select body of mathematics[3].

2. We want to develop in ourselves an innate lively inquisitiveness, and concomitantly an outlook, a disposition, that makes it natural and congenial for us to learn new things when it is appropriate to do so, and that makes *thinking mathematics* a vital activity for us; we want to foster the ability in ourselves to *deeply appreciate* a broad range of mathematics—to get pleasure from this "gift of appreciation;" and we want to be sensitive to the varieties of mathematical taste.

Surely it is the second item above that is the more important: it trumps the first; but it is the first item that we can hope to talk about, so let's talk about that.

3 Fields of acquaintance

Certain fields of mathematics, at certain times, play the role of *lingua franca* in the sense that mathematics from vastly different fields get formulated in the vocabulary, the terminology, or even more strikingly in the conceptual framework of that specific field. *Weierstrass's theory of functions, Cantor's Set Theory* and *Group Theory* have played (and continue to play) such a role; the vocabulary of *Category Theory* has permeated disparate disciplines. These are some of the grand forces in mathematics that shape our way of communicating to one another. Other fields formulate powerful viewpoints, templates, that cross over to distant disciplines–let's call these fields *unifiers*. Algebraic Topology has done service as a unifier, as has large aspects of Algebraic Geometry. Other fields are so ubiquitous that they cast light on all other disciplines of mathematics: measure theory, probability and statistics come to mind; perhaps aspects of combinatorics.

In any epoch there will be the *lingua franca*, the *unifiers* and the *ubiquitous* of that epoch. A young (or an old) mathematician, of no matter what specialty, would do well to be acquainted—at least

[2] the word itself comes, as I understand it, from an amalgam of *ex*- and *ducere*, which hints at a process that leads us out of something and towards something else

[3] Practically speaking it has to be a *much more select* body of mathematics!

a tiny bit acquainted—with the mathematical goings-on in these fields. So which fields are of that sort these days?

4 Critical mass

Before offering a concrete list of good "fields of acquaintance" I want to convey an idea of a friend of mine, who is a student of European History. He tells me that at one point in his career studying European History, he experienced an abrupt *phase shift*. Once you've achieved—says my friend—a certain critical mass of historical information, all of a sudden your view of the entire subject changes. First, your power of simply retaining information increases multifold; but more importantly, your way of thinking about the subject bears no relation to the way you approached things initially. My friend accounted for this surprising moment as a consequence of accumulation, perhaps to overload, of somehow-connected specifics that forced him to involuntarily re-configure—in a more meaningful way— his modes of organization, and contemplation, of the entirety of this corpus of knowledge.

Well, it would be interesting if we could put our finger on the critical moments, the phase shifts—if they exist—in our mathematical education. If they do exist they may depend less on our having devoured any *specific* collection of mathematical ideas, and more on our having exposed ourselves to some un-specifiable critical mass. With this in mind, I'll end, below, with a list that nowadays, in my opinion, stands for "fields of acquaintance" any critical mass of them being a good choice for a good mathematical education[4].

5 Broad types of mathematical intuitions

Our undergraduate education[5] would have done well for us if it left us with an abiding taste for what are, to my mind, the most classical four (very overlapping!) aspects of mathematical thought: *Geometry, Algebra, Computation*, and *The mathematical intuition(s) derived from Physics*. I say that these are "overlapping,' but in reality, what makes mathematics *one subject* is that nothing that we do is entirely contained in one of these categories: they seem to stand for distinct intuitions, and have given rise to distinct realms of thought, and yet they are inseparably welded together. These four categories have been fused together so substantially in recent times that it may even be misleading to keep bringing them up. For there are subjects of great importance such as the theory of modular (or automorphic) forms that defy this categorization completely, since they stretch their substance, their tools, and the intuition they rely on over all four of these items.

Nevertheless, *Geometry, Algebra, Computation*, and *Mathematical Physics* represent a recognizable, if wobbly, partition of mathematical sensibilities. Let us consider them as motivating intuitions

[4] I realized, when first making up this list, how little of it I felt adequately comfortable with—i.e., how much of a wish-list for my future education it represents!

[5] I am assuming that, on background, we have a mathematical education that would normally come from a good undergraduate curriculum, including—of course—standard ODE, Riemann surfaces, complex analysis more generally, and real analysis including measure theory and the theory of Fourier transforms. And the standard algebra/Galois theory/commutative algebra sequence. I also omit mention of number theory for the moment.

rather than fixed repositories of knowledge; i.e., as fundamental types of *highly developed senses* that some mathematicians enjoy: there are people with strong *geometric* intuitions; there are those with strong *algebraic intuitions*; there are those who are very sensitive to various aspects of *computation* and *estimation* and then there are the lucky people who also bring into the mix some basic *physical intuition*. I also think that we all understand what it means for some mathematician to have any one of these gifts—whether or not we ourselves possess it.

So, a broad mathematical education should, perhaps, aim to help a person achieve (at least somewhat) a predilection for each of these ways of thinking. I would like to view each of these intuitions as "core" rather than any particular conglomeration of subject matter as core.

In response to an earlier draft, David Mumford asked me where I would put the traditional, and grand, subject of Analysis in this classification, Analysis being a subject dealing with intuitions as fundamental as *time, change,* and *continuity*; my feeling is that the subject is so many-dimensional that it derives its inspiration and intuitions from every, or any, direction: when you say that someone is a strong analyst, you might mean that this person has a keen sense of a priori estimates in PDE, which I would tag as computation and estimation; or you might be talking of a good complex analyst dealing with dynamical systems which I would tag as geometry, etc. I would probably want to spread Analysis over all four categories.

6 A specific, but very tentative, list

I've tried to make a tentative and necessarily partial list of typical problems dealt with in each category such that some–at least—of these items would be good to acquaint ourselves with, but I discover that my list changes radically from day to day. Here is today's:

- *Geometry-in the broad sense:* This can be experienced in so many different ways that any one person's "critical mass" will be disjoint from another person's. Our sense of geometry might go from knot theory to differential geometry (and the related analysis; e.g., the spectrum of the Laplacian) to classification of n-dimensional manifolds to symplectic geometry to dynamical systems to sphere-packing to fixed point theorems to K-theory to systems of elliptic PDE's in the large to the Index Theorem to Bott periodicity to the homotopy groups of spheres... and here we would be moving into the more algebraic realms of algebraic topology, which nowadays is also commingling with algebraic geometry.

- *Algebra-in the broad sense.* This includes the ubiquitous notion of groups (say; finite, finitely presented, Lie, algebraic, arithmetic, and adelic) and their linear and projective (finite- and infinite-dimensional) representations. It includes elementary aspects of any of the subjects with "algebraic" as an adjective in their name: algebraic topology, algebraic geometry, and algebraic number theory. It includes the entire basic vocabulary of very general languages such as *category theory* and more specific languages of use in traditional functional analysis such as the theory of Hilbert and Banach spaces.

- *Computation–in the broad sense.* This stretches from machine computation, algorithms, numerical analysis, estimation, and statistics, to combinatorics, analysis and probability; i.e., ways of dealing with data, practically and/or theoretically.

- *Mathematics related to Physics.* Newtonian Mechanics, Optics, Maxwell's Equations, Relativity, some Quantum Mechanics and some physics related to field theories and string theory; and—of course—the mathematics that connects to this.