An experiment in class field theory

Iwasawa 2023, University of Cambridge

In memory of John Coates

Barry Mazur

In memory of John Coates

In memory of his energy, his generosity of thought, his accomplishments.

This talk is an account of joint work with:

Tony Feng, Michael Harris, and Arpon Raksit

—*lots of it just in progress*.

(See FHM *Derived Class Field Theory* `arXiv:2304.14161`)

Our work is, among other things, an (elementary) exercise

- to give a precise construction of Derived Weight Space,
- to understand the deep work of Tilouine & Urban in Derived Iwasawa Theory,
- and the work of Galatius & Venkatesh in Derived Galois Deformation Theory…
- and to install Jacob Lurie's marvelous (800 page?) treatise humbly titled Higher Algebra in our central nervous system.

Classical class field theory

This theory, working over any number field F as base, makes a neat transition from double quotients related to the multiplicative group

$$\mathbb{G}_m = GL(1)$$

(*taken as group scheme over* $\mathcal{O}_F$, *the ring of integers of* F)

to

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the abelianization of the absolute Galois group of F

$$\Gamma_F := Gal(\bar{F}/F) \rightarrow \Gamma_F^{ab}.$$

Structurally, Class Field Theory gives:

the basic relation between:

- —abelian— *Automorphy* (I.e., double quotients arising from the reductive group $GL(1)$ over $\mathcal{O}_F$.)

to

- —abelian—*Representation theory* (I.e., homomorphisms of the absolute Galois group of F to $GL(1)(R) = R^\times$ for a class of commutative rings R).

The classical set-up

Keep F, our number field, and denote by

- $\mathbb{A}_F := \prod'_{\mathfrak{p} \in |F|} F_{\mathfrak{p}}$

  the ring of adéles of F—i.e., the restricted product of the local fields,

- $\mathbb{A}_F^\times$ the group of idéles (the group of units in $\mathbb{A}_F$),

- $\mathbb{I}_F := F^\times \backslash \mathbb{A}_F^\times$ the idèle class group of F,

The idèle class group $\mathbb{I}_F := F^\times \backslash \mathbb{A}_F^\times$

has the property that for any abelian Galois extension L/F there is a natural *level*—i.e., an open compact subgroup $\mathcal{K}_L \subset \mathbb{A}_F^\times$ attached to L/K allowing for a canonical surjection of the corresponding double quotient $F^\times \backslash \mathbb{A}_F^\times / \mathcal{K}_L$ onto the Galois group of L/F:

$$\mathbb{I}_F/\mathcal{K}_L = F^\times \backslash \mathbb{A}_F^\times / \mathcal{K}_L \xrightarrow{\iota} \mathrm{Gal}(L/K)$$

and where the kernel of ι is the connected component of the identity in that double quotient.

How to 'enhance" Class Field Theory by bringing the (homotopy type of the) kernel into the game?

This “kernel” is the topological heart of the “homogeneous symmetric space” of the reductive group $GL(1)$, so—following this thread— one might also ask analogous questions regarding possible enhancements of the Langlands Correspondence connected to any reductive group G.

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—Questions that bring in the full topology of the corresponding homogeneous symmetric space or Shimura variety—

and, in the spirit of John Coates, one could ask for a similar enhancement of the 'Iwasawa-theoretic" format of these structures.

Connections with derived deformation theory?

Noting that the Idéle Class Group,

$$\mathbb{I}_F := F^\times \setminus \mathbb{A}_F^\times$$

is the locally symmetric space associated to the reductive group $G = \mathrm{GL}(1)$ over F

— *and looking ahead to analogous but perhaps subtler relation between the topology of locally symmetric spaces attached to more general reductive groups and their corresponding Galois representations*—

the derived class field theory exercise that we are undertaking here might serve as a very simple, but—we hope—instructive, example of the general programs of Tilouine-Urban and Galatius-Venkatesh. More details of this will appear in an article that Feng, Harris, Raksit and I are working on.

On the Automorphic side: The Idèle Class *Group*

We have the exact sequences of topological groups:

$$\begin{array}{ccccccccc}
 & & & & 1 & \longrightarrow & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (\mathbb{A}_F)^{\times}_{\{0\}} & \longrightarrow & \mathbb{I}_{F,\{0\}} & & \\
 & & & \nearrow & \downarrow \subset & & \downarrow \subset & & \\
1 & \longrightarrow & F^{\times} & \longrightarrow & \mathbb{A}_F^{\times} & \longrightarrow & \mathbb{I}_F & \longrightarrow & 1 \\
 & & & & \downarrow \text{Norm} & & \downarrow \text{Norm} & & \\
 & & & & \mathbb{R}^{\times} & \xrightarrow{=} & \mathbb{R}^{\times} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & \longrightarrow & 1 & &
\end{array}$$

On the Automorphic side: The Idèle Class *Groupoid*

The *Idèle Class Groupoid* (of 'level' $\mathcal{K} := \mathcal{K}_{finite} \times \mathcal{K}_\infty$) is the 'double quotient'

$$\mathbb{I}_{F,\mathcal{K}} := \mathbb{I}_F/\mathcal{K} = [F^\times \backslash \mathbb{A}_F^\times / \mathcal{K}_{finite} \cdot \mathcal{K}_\infty].$$

which can be viewed as *quotient stack*.

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Here the brackets mean that we take the quotient in the sense of groupoids, or in other words we form the *homotopy quotient* rather than the naive quotient group.

Group versus groupoid

That is, the " idèle class group" traditionally considered in class field theory is the (naive) quotient group

$$F^\times \backslash \mathbb{A}_F^\times / \mathcal{K},$$

and modulo the connected component of the identity this can be thought of as the (group of) connected components of the groupoid $\mathbb{I}_\mathcal{K}$:

$$\pi_0(\mathbb{I}_\mathcal{K}).$$

Considering the homotopy type of the groupoid $\mathbb{I}_{\mathcal{K}}$

is how we will be "bringing in kernel."

This homotopy type is fairly simple. E.g., If "level $\mathcal{K}$" is meant to cut out the Hilbert Class Field of F and if F is a totally imaginary number field then the connected components of $\mathbb{I}_{\mathcal{K}}$ have the rational homotopy type of $(S^1)^{r_2 - 1}$, which is non-trivial as long as F is not a quadratic imaginary field.

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We'll get into these groupoids later. But we'll be starting on the Galois side:

Program for this hour

1. The 'Galois side': derived abelianization.
2. Expressing the 'Galois side' in the category of chain complexes—via the setting(s) of Quillen, Lurie, Dold-Kan...
3. The 'Automorphic side': Picard groupoids.
4. Expressing the 'Automorphic side' in the category of chain complexes.
5. Relating the'Galois side' to the 'Automorphic side' via an enhanced version of Finite Flat Duality (i.e., on the level of cochain complexes).
6. Derived Class Field Theory.

Preview for the 'Galois side': Abelianization

Let Γ be a discrete group.

The abelianization of Γ is an abelian group Γ^{ab} for which the projection $\Gamma \longrightarrow \Gamma^{\mathrm{ab}}$ is the universal solution to the problem of morphisms from Γ to any abelian group.

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So, for an abelian group A,

$$Hom_{\mathrm{gps}}(\Gamma, A) = Hom_{\mathrm{gps}}(\Gamma^{\mathrm{ab}}, A).$$

We have

$$\Gamma^{\mathrm{ab}} = \Gamma/[\Gamma, \Gamma] = \mathrm{H}_1(\Gamma, \mathbb{Z}). \tag{0.1}$$

Derived abelianization

Just as Γ^{ab} gives us the one-dimensional homology of the group Γ

$$\Gamma^{\mathrm{ab}} \simeq \mathrm{H}_1(\Gamma, \mathbb{Z})$$

the *derived abelianization* of Γ, denoted $\Gamma^{\mathrm{ab},\bullet}$, is represented by a simplicial abelian group that is constructed canonically in the appropriate category and captures all of $\mathrm{H}_*(\Gamma, \mathbb{Z})$.

Derived abelianization

Specifically, we'll see that there is a canonical isomorphism

$$\pi_i(\Gamma^{\mathrm{ab},\bullet}) \simeq \mathrm{H}_{i+1}(\Gamma, \mathbb{Z}) \tag{0.2}$$

for $i \geq 0$.

For this, we'll use these classical constructions:

- Classifying Space,
- Infinite symmetric product,
- Loop space.

The Classifying space of a group

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- If Γ is a topological group acting faithfully and nicely on some contractible space E, then $B\Gamma$ has the homotopy type of the quotient space E/Γ.
- If Γ is a simplicial group then $B\Gamma$ can be taken to be the (pointed) simplicial set given by the beautiful combinatorial (classical) bar construction on Γ.
- If Γ is a discrete group, then $B\Gamma$ has the homotopy type of the Eilenberg-Maclane space $K(\Gamma, 1)$—i.e., a space with fundamental group isomorphic to Γ and higher homotopy groups vanishing.

So $H_i(\Gamma; \mathbb{Z}) = H_i(B\Gamma; \mathbb{Z})$.

The Infinite symmetric product of a pointed space

Given any pointed space (X, x), there is the infinite symmetric product.

$$Sym(X, x) := \varinjlim_{n} Sym^n(X)$$

where $Sym^n(X)$ is the quotient of the n-th power of X by the action of the symmetric group on n letters.

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$Sym(X, x)$ is a topological abelian monoid, under concatenation.

The Dold-Thom Theorem: From Homology to Homotopy

Letting $e :=$ the origin, i.e., the image of $(x, , x, x, \dots)$, there is a canonical isomorphism:

$$\pi_i(Sym(X, x); e) \simeq \mathrm{H}_i(X; \mathbb{Z})$$

for $i = 0, 1, 2, \dots$

The Loop Space functor

Given any pointed connected topological space (Y, y) if $\Omega(Y, y)$ is the (topological) space of loops in Y (beginning and ending at the point y)—letting ω be the constant loop based at y we have the canonical isomorphism:

$$\pi_i(\Omega(Y, y); \omega) \simeq \pi_{i+1}(Y, y)$$

The Derived Abelianization $\Gamma^{\mathrm{ab},\bullet}$ of any group Γ

Definition

$$\Gamma^{\mathrm{ab},\bullet} : \text{“} = \text{”} \quad \Omega Sym(B\Gamma, e)$$

where Ω is the (based) loop space functor.

Explain the “ = ”

Categorical Framework

Derived Abelianization versus Abelianization

Note that if Γ is discrete, then:

$$\pi_0(\Gamma^{\mathrm{ab},\bullet}) \;=\; \pi_0(\Omega Sym(B\Gamma, e)) \cong$$

$$\cong \pi_1(Sym(B\Gamma, e)) \cong$$

$$\cong \mathrm{H}_1(B\Gamma; \mathbb{Z}) \cong \mathrm{H}_1(\Gamma; \mathbb{Z}) \cong$$

$$\cong \Gamma^{\mathrm{ab}}.$$

So, (for discrete Γ)

$\Gamma^{\mathrm{ab},\bullet}$ is a group whose underlying group of connected components is Γ^{ab}:

$$\Gamma^{\mathrm{ab},\bullet} \longrightarrow \Gamma^{\mathrm{ab}}$$

and

$$\pi_i(\Gamma^{\mathrm{ab},\bullet}) \cong \pi_{i+1}(Sym(B\Gamma, e)) \cong \mathrm{H}_{i+1}(\Gamma; \mathbb{Z}), \tag{0.3}$$

Derived abelianization of profinite groups

Since the Galois groups we are interested in are profinite groups we need derived abelianization of profinite groups, which we'll still call $\Gamma^{\mathrm{ab},\bullet}$.

If $\Gamma = \lim_\alpha \Gamma_\alpha$ is a profinite group with Γ_α the system of finite quotients defining Γ, then:

$$\Gamma^{\mathrm{ab},\bullet} \cong \lim_\alpha \Gamma_\alpha^{\mathrm{ab},\bullet}.$$

Mention issues related to pro-p-completion.

The Categories Involved (an example)

Let S be any finite set of (nonarchimedean) primes of F and $\mathcal{K}_S$ the level structure that cuts out the maximal abelian Galois extension L_S/F unramified outside S. Denoting by $\Gamma^{\mathrm{ab},\bullet}_{F,S}$ the derived abelianization of $\Gamma_{F,S} := \mathrm{Gal}(L_S/F)$ The 'object'

$$\Omega Sym(B\Gamma_{F,S}, e) \qquad (\text{``} = \Gamma^{\mathrm{ab},\bullet}_{F,S}\text{''})$$

can be elementarily constructed and realized as a projective limit of simplicial abelian groups.

But we want to view it in Quillen's theory of *homotopical algebra*.

—Discuss. . . —

Intuitively speaking, the derived abelianization should be a kind of "derived functor of abelianization". However, the process of "deriving" the abelianization functor cannot be approached as in classical homological algebra, since the category of (not necessarily abelian) groups is far from being the sort of abelian category to which the classical theory of derived functors applies.

The Categories Involved

The Quillen formalism of derived functors has as its equipment not only simplicial objects, but also of appropriate generalizations of “quasi-isomorphisms” and “projective resolutions”.

. . . and with designated classes of morphisms called *fibrations* and *cofibrations*.

We will also be in the framework of ∞-categories[1].

[1] By the way, take a look at Emily Riehl’s great Scientific American article *Infinity Category Theory Offers a Bird’s-Eye View of Mathematics*

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With all this in place, one constructs derived functors by a procedure (roughly) analogous to the traditional calculus in derived categories, using "projective resolutions," viewing our simplicial abelian group as *equivalent* to a certain chain complex, in this setting.

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Derived Iwasawa-type Theories

:

Climbing the rings of a cyclotomic tower, taking the Derived abelianizations of the Galois groups of the maximal unramified extension of the rings of integers of those rungs, one gets an *explicit* natural prosystem of (pro-p) simplical abelian groups—which underlies Derived Classical Iwasawa Theory.

Doing something similar with the knot group of a knot $K \subset S^3$ as one climbs the tower of abelian extensions of the three-dimensional sphere S^3 ramified over K one gets a similar natural pro-system that encodes—and some way enhances—the Alexander polynomial of K.

Moving toward the Automorphic side (with the example)

Recalling that $\Gamma_{F,S}^{\mathrm{ab},\bullet}$ is the derived abelianization of $\Gamma_{F,S} := \mathrm{Gal}(L_S/F)$ we have:

$$
\begin{array}{ccc}
\text{Automorphic side} & & \text{Galois side} \\
\downarrow & & \downarrow \\
\text{“}\mathbb{I}_{F,S} = F^{\times} \backslash \mathbb{A}_F^{\times}/\mathcal{K}_S\text{”} & \overset{?}{\dashrightarrow} & \Gamma_{F,S}^{\mathrm{ab},\bullet} \\
\downarrow ? & & \downarrow \pi_0 \\
\dfrac{\mathbb{I}_{F,S}}{\mathit{connected\ component}} & \overset{\simeq}{\dashrightarrow} & \Gamma_{F,S}^{\mathrm{ab}}
\end{array}
$$

Passing to Chain Complexes

The mechanism is given by the Dold-Kan theorem.

Theorem

There is an explicit 'normalization functor' N that is an equivalence of categories:

$$\begin{array}{c} \{\text{simplicial abelian groups}\} \\ \Big\downarrow N \\ \{\text{chain complexes in non} - \text{negative degrees}\}. \end{array}$$

.

Moreover, under the equivalence, the nth homology group of a chain complex is the nth homotopy group of the corresponding simplicial abelian group, and a chain homotopy corresponds to a simplicial homotopy.

(We'll be making use of Jacob Lurie's enhanced version of the Dold-Kan Theorem working in an ambient 'stable ∞-category.')

There is also an explicit 'inverse normalization functor' (inverse up to equivalence—call it "N^{-1}") that goes backwards from non-positively graded cochain complexes to simplicial abelian groups that sends cochains of degree $-i$ in the cochain complex to simplices of degree i.

Also, a good source, posted last month:
An introduction to derived (algebraic) geometry J.Eugster and J.P.Pridham `https://arxiv.org/pdf/2109.14594.pdf`

Canonical truncations of chain complexes and 'connective covers'

Given a Chain complex

$$C_* : \quad \dots C_m \overset{d_m}{\rightarrow} C_{m-1} \overset{d_{m-1}}{\rightarrow} \dots \overset{d_{n+1}}{\rightarrow} C_n \overset{d_n}{\rightarrow} C_{n-1} \overset{d_{n-1}}{\rightarrow} \dots$$

- we can turn it into a cochain complex by redefining degrees ($n \mapsto -n$);
- we can take its canonical truncation in degree ≥ 0 to get the 'connective cover' of C_*:

$$\bar{\mathrm{C}}_* : \quad [\dots C_m \overset{d_m}{\rightarrow} \dots \overset{d_1}{\rightarrow} ker(d_0) \rightarrow 0 \overset{d_{-1}}{\rightarrow} 0 \quad \dots]$$

The Dold-Kan equivalence gives that:

$$\Gamma_{F,S}^{\mathrm{ab},\bullet}$$

corresponds to a connective chain complex quasi-isomorphic to the (reduced, then shifted) homology chain complex

$$\overline{\mathrm{C}}_*(X_{F,S},\hat{\mathbb{Z}})[-1]$$

where $X_{F,S} := \mathit{Spec}(\mathcal{O}_F[1/S])$.

The Chain Complex attached to a commutative (flat) group scheme over a scheme X

This is canonically given as the Cech cohomology complex with sections of the group scheme over the relevant hypercoverings of X in the flat (fppf) topology.

For the rest of this talk

to explain the basic ideas, we will

- suppose that S is empty, so we're dealing with "level 1"

$$X_F := X_{F,\emptyset} = Spec(\mathcal{O}_F).$$

- fix a prime p and replace $\Gamma_F^{\mathrm{ab},\bullet}$ by its p-completion —which then corresponds to the chain complex

$$\overline{C}_*(X_F, \mathbb{Z}_p)[-1],$$

- and assume that either F is totally complex or $p > 2$.

The automorphic side: *Picard groupoids*

For X any scheme consider the *Picard groupoid* $\underline{Pic}(X)$: the category whose objects are line bundles on X and morphisms are isomorphisms of line bundles.

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One can also put level structures on them. E.g., if $X = Spec(\mathcal{O}_F)$ we can consider locally free rank one $\mathcal{O}_F$-modules L such that for every real embedding $v : \mathcal{O}_F \hookrightarrow \mathbb{R}$ we have fixed an isomorphism of $\mathbb{R}$-vector spaces

$$L \otimes_v \mathbb{R} \simeq \mathbb{R}$$

—with a similar level structure for every complex archimedean place of F.

From Picard groupoids to simplicial sets

Picard groupoids may also be viewed as simplicial sets via the simplicial nerve construction. As we hinted, we follow the approach to higher category theory offered by Jacob Lurie's work where spaces and categories are treated in a unified way under the umbrella of simplicial sets.

The étale—(also fppf)—cohomology of $\mathbb{G}_m$

Letting $X := Spec(\mathcal{O}_F)$, we have:

$$H^0(X_{ét}, \mathbb{G}_m) = \mathcal{O}_F^\times$$

$$H^1(X_{ét}, \mathbb{G}_m) = Pic(X) = \mathcal{C}l(\mathcal{O}_F)$$

$$H^3(X_{ét}, \mathbb{G}_m) = \mathrm{Q}/\mathbb{Z}$$

and $H^r(X_{ét}, \mathbb{G}_m)$ vanishes for $r = 2$ and $r > 3$.

From Picard groupoids to simplicial abelian groups

The tensor product of line bundles induces a symmetric monoidal structure on the category $Pic(X)$, and this translates into an abelian group structure on the corresponding simplicial set.

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The tensor product of line bundles induces a symmetric monoidal structure on the category $\underline{Pic}(X)$, and this translates into an abelian group structure on the corresponding simplicial set.

We can take the Picard groupoid $\underline{Pic}(X)$ (interchangeably) as a symmetric monoidal category or as a simplicial abelian group.

The homotopy groups of $\underline{Pic}(X)$

We have:

$$\pi_0(\underline{Pic}(X)) = \mathrm{H}^1(Spec(\mathcal{O}_F), \mathbb{G}_m) = \mathcal{C}l(\mathcal{O}_F) \tag{0.4}$$

I.e., its group of connected components, the group of equivalence classes of line bundles on $X = Spec(\mathcal{O}_F)$ is $\mathcal{C}l(\mathcal{O}_F)$, the ideal class group of $\mathcal{O}_F$.

Also

$$\pi_1(\underline{Pic}(X)) = \mathrm{H}^0(Spec(\mathcal{O}_F), \mathbb{G}_m) = \mathcal{O}_F^\times, \tag{0.5}$$

and $\pi_i(\underline{Pic}(X))$ vanishes for $i > 0$.

The Cochain Complex associated to the Picard Groupoid

For any scheme X the group of equivalence classes of line bundles over X, $\mathrm{H}^1(X, \mathbb{G}_m)$, is the 0th cohomology group of the (shifted) cohomology complex

$$\mathrm{C}^\bullet(X, \mathbb{G}_m[1]),$$

an object well-defined in the homotopy category of complexes.

More specifically, making use of the appropriate truncation functor that extracts the connective cover of a complex, $\mathcal{C}^\bullet - - > \bar{\mathcal{C}}^\bullet$, form

$$\bar{\mathrm{C}}^\bullet(X, \mathbb{G}_m[1]),$$

and we have:

By Dold-Kan

$$\text{The simplicial abelian group : } \underline{Pic}(X)$$

$$\updownarrow \simeq$$

$$\text{The cochain complex : } \bar{\mathrm{C}}^\bullet(X, \mathbb{G}_m[1])$$

So,

$$\bar{\mathrm{C}}^\bullet(X, \mathbb{G}_m[1])$$

is the connective cover of the cohomology complex

$$\mathrm{C}^\bullet(X, \mathbb{G}_m[1]),$$

and taking the 0th cohomology group of $\bar{\mathrm{C}}^\bullet(X, \mathbb{G}_m[1])$ also recovers the ideal class group.

This truncation doesn't do that much except turn it into a complex that directly compares with its associated simplical abelian group.

The Picard groupoid and the idèle class groupoid

If F is the function field of a smooth projective curve C over a finite field, then Weil's construction identifies the idèle class group $I_{F,\mathcal{K}_{max}}$ with the groupoid of line bundles on C.

Here $\mathcal{K}_{max}$ is the product over all places v of F of the maximal compact subgroups of $F_v^\times$.

The Picard groupoid and the idèle class groupoid

A similar construction can be applied when F is a (totally complex) number field. Under this assumption we get a natural homotopy equivalence

$$\underline{Pic}(\mathrm{Spec}\ (\mathcal{O}_F)) \simeq I_{F,\mathcal{K}_{max}}$$

giving us a 'Cochain Complex' description of the idèle class group:

$$\bar{\mathrm{C}}^\bullet(\mathrm{Spec}\ (\mathcal{O}_F), \mathbb{G}_m[1]) \simeq \underline{Pic}(\mathrm{Spec}\ (\mathcal{O}_F)) \simeq I_{F,\mathcal{K}_{max}}. \quad (0.6)$$

The profinite completion of the Picard groupoid and flat cohomology

Consider the Kummer exact sequence

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 1 \tag{0.7}$$

which we can write as an isomorphism in the derived category of sheaves on the flat topology of $\mathrm{Spec}\,(\mathcal{O}_F)$:

$$[\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m] \simeq \mu_n[1]; \tag{0.8}$$

$$(\mathbb{G}_m)\hat{} := \varprojlim_{n\to\infty} \text{“}\mathbb{G}_m/n\text{”} \simeq \varprojlim_{n\to\infty} \mu_n[1]. \tag{0.9}$$

(Denote by $T\mu$ the pro-finite flat group scheme $\lim_{n\to\infty} \mu_n$)

For $\mathcal{F}$ an fppf sheaf over $X := \mathrm{Spec}\,(\mathcal{O}_F)$, denote by $C^*(X, \mathcal{F})$ the (fppf) cohomology complex of $\mathcal{F}$.

The profinite completion of $C^*(X, \mathbb{G}_m)$ can be seen to have the form

$$C^*(X, \mathbb{G}_m)\hat{} := \varprojlim_n C^*(X, \mathbb{G}_m)/n \xrightarrow{\sim} \tag{0.10}$$

$$\xrightarrow{\sim} C^*(X, \varprojlim_n \text{“}\mathbb{G}_m/n\text{”}) \cong C^*(X, T\mu[1]) \tag{0.11}$$

So:

$$\mathrm{C}^*(X, \mathbb{G}_m)\hat{} \quad \simeq \quad \mathrm{C}^*(X, T\mu[1]) \tag{0.12}$$

and:

Moving to simplicial sets and simplicial abelian groups

we can put all this together to get:

$$\{I_{F,\mathcal{K}_{max}}\}\hat{} \xrightarrow{\sim} Pic(X)\hat{} \xrightarrow{\sim} \mathrm{C}^*(X,\mathbb{G}_m[1])\hat{} \xrightarrow{\sim} \mathrm{C}^*(X, T\mu[2]).$$

The Passage from the Automorphic side to The Galois side via Finite Flat Duality

For a finite flat fppf commutative group scheme $\mathcal{F}$ over a Zariski-open subscheme

$$X \hookrightarrow Spec(\mathcal{O}_F),$$

denoting its Cartier dual by $\mathcal{F}^D := \underline{\mathrm{H}om}(\mathcal{F}, \mathbb{G}_m)$, there are isomorphisms

$$\mathrm{H}_i(X, \mathcal{F}) \cong \mathrm{H}_c^{3-i}(X, \mathcal{F}^D).$$

E.g.,

$$\mathrm{H}_i(X, \mathbb{Z}/p^n\mathbb{Z}) \cong \mathrm{H}_c^{3-i}(X, \mu_{p^n}).$$

Enhanced finite flat duality

The compactly supported cohomology groups $\mathrm{H}_c^{3-i}(X, \mathcal{F}^D)$ are a bit more involved to define in general but under our simplifying assumptions (S empty and F totally complex) we will have $\mathrm{H}_c^{3-i}(\mathcal{O}_F, \mathcal{F}^D) = \mathrm{H}^{3-i}(\mathcal{O}_F, \mathcal{F}^D)$

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and its enhancement as an equivalence of (fppf) chain (and cochain) complexes:

$$C_*(X, \mathcal{F}) \cong C^*(X, \mathcal{F}^D[3]).$$

Taking $\mathcal{F} = \mathbb{Z}/n\mathbb{Z}$

(and taking limits 'over n') we get the natural equivalence of chain complexes

$$C_*(X, \widehat{\mathbb{Z}}) \cong C^*_{flat}(X, T\mu[3]).$$

Combining this with the Kummer sequence we get:

$$Pic(X)\hat{} \xrightarrow{\sim} \overline{C}^*_{flat}(X, T\mu[2]) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} \overline{C}_*(X, \widehat{\mathbb{Z}}[-1])$$

Derived class field theory

And putting all this together we get:

Theorem

Suppose that F is a totally imaginary number field. Then there is a natural isomorphism

$$(\pi_1(\mathrm{Spec}\ (X))^{\mathrm{ab},\bullet})\hat{} \simeq (I_{F,\mathcal{K}})\hat{}$$

where $\mathcal{K} = \prod_v \mathcal{O}_{F,v}^\times$ is the maximal compact subgroup of the finite idèles, such that taking π_0 recovers the usual isomorphism of class field theory

$$\pi_1(\mathrm{Spec}\ (\mathcal{O}_F))^{\mathrm{ab}} \cong \pi_0(I_{F,\mathcal{K}}).$$