Star-like configurations in data related to the computation of L-values.

Murmuration Conference

Barry Mazur

This is joint work with Karl Rubin[1]

[1]For more: see Karl's web-page: https://www.math.uci.edu/~krubin/, or mine: http://people.math.harvard.edu/~mazur/

In computation of special values of L-functions Karl and I have come up with some data that give pictures somewhat surprising to us.

So far our computations aren't substantial enough to make firm conjectures—let alone statements that we can prove—so this is work in the very early stages of . . . 'progress.'

We would be grateful for any advice, and help in accumulating more data.

Dirichlet Characters

Let $d, m \in \mathbb{Z}_{\geq 1}$. Denoting by μ_d the subgroup of $\mathbb{C}$ consisting of d-th roots of unity, let

$$\chi : \mathbb{Z} \to \{0\} \sqcup \mu_d \subset \mathbb{C}$$

be a primitive Dirichlet character of conductor m and order d, cutting out a (real) degree d cyclic field extension $F/\mathbb{Q}$.

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for $F/\mathbb{Q}$ a specific cyclic Galois extension of degree d.
The map χ generates a ring homomorphism of the corresponding group ring:

$$\chi : \mathbb{Z}[\mathrm{Gal}(F/\mathbb{Q})] \to \mathbb{C}.$$

The L-function of E

Fix an elliptic curve E over $\mathbb{Q}$.

Consider its L-function—for choice of Dirichlet character χ (and associated cyclic field extension $F/\mathbb{Q}$):,

$L(E, \chi, s) := \prod_{p \text{ prime}} \left(1 - a_p p^{-s} + \chi(p) p^{k-1-2s}\right)^{-1}$

where the integers a_p have (dually) a local arithmetic meaning related to E over $\mathbb{F}_p$ and also an automorphic meaning since they are also the coefficients (of prime index) of the normalized modular form

$$f_E(q) = \sum_{N \geq 1} a_n q^n$$

uniformizing the elliptic curve E.

Special L- values

The Birch-Swinnerton-Dyer Conjecture gives (important) global arithmetic meaning to the ‘special L-values:’

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Among other things,

$$L(E, \chi, 1) = 0 \quad \rightarrow \quad Rank(E(F)) > Rank(E(\mathbb{Q})).$$

The route we take in computing $L(E, \chi)$:

Modular symbols $\rightarrow$

Theta coefficients $\rightarrow$

Theta elements$\rightarrow$

L-values

The Statistical Picture given by. . .

Modular symbols

Theta coefficients

Theta elements

L-values

are all *different*. . .

We'll

(**a**) Give a rapid description of what's known and conjectured about Modular symbols;

(**b**) discuss data for Theta coefficients, and then

(**c**) present some data for Theta elements

pointing out the curious features this data presents—in hopes of launching a discussion about interpretation, and possibilities for augmenting our computations.

Modular Symbols

The (plus or minus) **modular symbol** attached to E

$$[r]^{\pm}$$

is a certain function—relatively straightforward to compute—on rational numbers $r = \frac{a}{m} \in \mathbb{Q}$:

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$$\boxed{\frac{a}{m} \mapsto [\tfrac{a}{m}]^{\pm}_E = [\tfrac{a}{m}]^{\pm} \in \tfrac{1}{\delta}\mathbb{Z} \in \mathbb{Q}}$$

‘

Here $\delta = \delta_E \in \mathbb{Z}_{\geq 1}$is just a ‘denominator’ that shows up.
Also: for ease of notation in these slides, unless noting otherwise, I’ll suppose we’re consistently dealing with, for example, $[\frac{a}{m}]^{+}$ and drop the “$+$”.

The combinatorics of Modular Symbols

Let N be the conductor of E. For every $r \in \mathbb{Q}$, modular symbols satisfy:

- $\boxed{[\infty] = 0}$ by definition
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- $\boxed{[r]^{\pm} = \pm[-r]^{\pm}}$ by definition

Invariance under the action of $\Gamma_0(N)$

If

$$T := \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N) \subset \mathrm{SL})2(\mathbb{Z}),$$

so that for $r \in \mathbb{Q} \sqcup \{\infty\}$,

$$T(r) = \frac{ar+b}{cNr+d} \in \mathbb{Q} \sqcup \{\infty\},$$

we have the following relation in modular symbols:

$$\boxed{[r] = [T(r)] - [T(\infty)].}$$

The Atkin-Lehner and Hecke relations

- **Atkin-Lehner relation:** if w is the global root number of E, and $aa'N \equiv 1 \pmod{m}$, then

$$\boxed{[a'/m] = w[a/m]}$$

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- **Hecke relation:** if a prime $\ell \nmid N$ and a_ℓ is the ℓ-th Fourier coefficient of f_E, then

$$\boxed{a_\ell[r] = [\ell r] + \sum_{i=0}^{\ell-1} [(r+i)/\ell]}$$

Modular symbols are connected *(by the Birch-Stephens Theorem)* to 'special L-values' for χ a primitive Dirichlet character cutting out a cyclic field extension $F/\mathbb{Q}$

We will be dealing with sums of modular symbols and sums of Theta coefficients weighted by (say, *even*) Dirichlet characters χ:

$$\sum_{a=1}^{m} \chi(a)[a/m] \in \bar{\mathbb{Q}} \quad \text{(modular symbols)}$$

$$= \quad \chi\Big(\sum_{\alpha \in \mathrm{Gal}(F/\mathbb{Q})} c_\alpha \cdot \alpha \Big)$$

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with

$$c_\alpha \in \mathbb{Q}.$$

Theta coefficients

These c_α are the 'Theta coefficients':

$$c_\alpha := \sum_{a \mapsto \alpha} [\frac{a}{m}].$$

Theta elements

Setting:

$$\Theta_{E,F} := \sum_{\alpha \in \mathrm{Gal}(F/\mathbb{Q})} c_\alpha \alpha \in \mathbb{Q}[\mathrm{Gal}(F/\mathbb{Q})]$$

(these are the 'Theta elements')

$$\chi(\Theta_{E,F}) = \sum_{a=1}^{m} \chi(a)[a/m] = \sum_{\alpha \in \mathrm{Gal}(F/\mathbb{Q})} c_\alpha \chi(\alpha) \in \bar{\mathbb{Q}}$$

Which brings us to L-values:

$$\sum_{a=1}^{m} \chi(a)[a/m]^{+} \quad = \quad \frac{\tau(\chi)L(E,\bar{\chi},1)}{\Omega^{+}} \in \mathbb{C} \quad .$$

The Birch-Stephens Theorem

$$\sum_{a=1}^{m} \chi(a)[a/m]^+$$

$$= \quad \chi\Big(\sum_{\alpha \in \mathrm{Gal}(F/\mathbb{Q})} c_\alpha \cdot \alpha\Big)$$

$$= \quad \chi(\Theta_{E,F})$$

$$= \quad \frac{\tau(\chi) L(E, \bar{\chi}, 1)}{\Omega^+}.$$

E.g.:

If χ is a Dirichlet character of conductor $m \in \mathbb{Z}_{\geq 1}$ and degree d cutting out $F/\mathbb{Q}$ then

$$L(E, \chi, 1) = 0 \iff \sum_{a \in (\mathbb{Z}/m\mathbb{Z})^\times} \chi(a)[a/m] = 0.$$

So the Birch-Stephens Theorem connects the statistical structure of. . .

$$\sum_{a=1}^{m} \chi(a)[a/m]^+ \in \bar{\mathbb{Q}} \quad \text{(modular symbols)}$$

$$\sum_{\alpha \in \mathrm{Gal}(F/\mathbb{Q})} c_\alpha \cdot \alpha \in \bar{\mathbb{Q}}[\mathrm{Gal}(F/\mathbb{Q})] \quad \text{(Theta coefficients)}$$

$$\Theta_{E,F} \in \bar{\mathbb{Q}}[\mathrm{Gal}(F/\mathbb{Q})] \quad \text{(Theta elements)}$$

with

$$\frac{\tau(\chi)L(E,\bar{\chi},1)}{\Omega^+} \in \mathbb{C} \quad (L-\text{values}).$$

Discussion of Data

We'll briefly discuss data for

- modular symbols and
- Theta coefficients,

  but then get on to consider the curious data for

- Theta Elements.

The modular symbol data

For $m, X > 1$, let S_m denote the multiset

$$S_m := \left\{ \frac{[a/m]}{\sqrt{\log(m)}} : a \in (\mathbb{Z}/m\mathbb{Z})^\times \right\}$$

and put:

$$\Sigma_X := \bigcup_{m<X} S_m.$$

Distribution of modular symbols

Theorem (Petridis-Risager)

There is an explicit $V_E \in \mathbb{R}$ *such that as* $X \to \infty$*, the distribution of the* Σ_X *converges to a normal distribution with mean zero and variance* V_E*.*

The variance V_E is essentially $L(\mathrm{Sym}^2(E), 1)$.

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The variance V_E is essentially $L(\mathrm{Sym}^2(E), 1)$.

Conjecture

As $m \to \infty$, the distribution of the individual S_m converges to a normal distribution with mean zero and variance V_E.

Summary: Distribution of Theta-coefficients (say, when $d = p$, a prime)

- Each c_α is a sum of $\varphi(m)/p$ modular symbols $[a/m]$,
- As $X \to \infty$ the multiset of all $[a/m]/\sqrt{\log(m)}$; $m < X$ converges to a normal distribution with variance V_E.
- Conjecturally, the finer multiset of $[a/m]/\sqrt{\log(m)}$ also converges to a normal distribution with variance V_E as $m \to \infty$.

so one might expect that as $m \to \infty$ the collection of all (appropriately normalized) Theta coefficients; i.e.,

$$c_\alpha \sqrt{\frac{p}{V_E \log(m)\varphi(m)}}$$

satisfies a normal distribution with variance 1.

Distribution of the c_α

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But it looks not too far off.

Without going into the specifics of how we normalize and collect $\Lambda_{E,d}(t)$, the data of normalized Theta coefficients for elliptic curves E and cyclic Galois extensions of $\mathbb{Q}$ of degree d here are some pictures:

- For each of the three elliptic curves 11A1, 37A1, and 32A1 (in the notation of Cremona's tables, and
- five (prime) values of d,
- we computed the first (approximately) 50,000 generic[2] Theta Coefficients $c_{F,\alpha}$ as F runs through cyclic extensions of degree d and smallest conductor (prime to d and to the conductor of E).

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The resulting approximations to $\Lambda_{E,d}$ are shown in Figures 1 through 3. In each plot the dashed line is the normal distribution.

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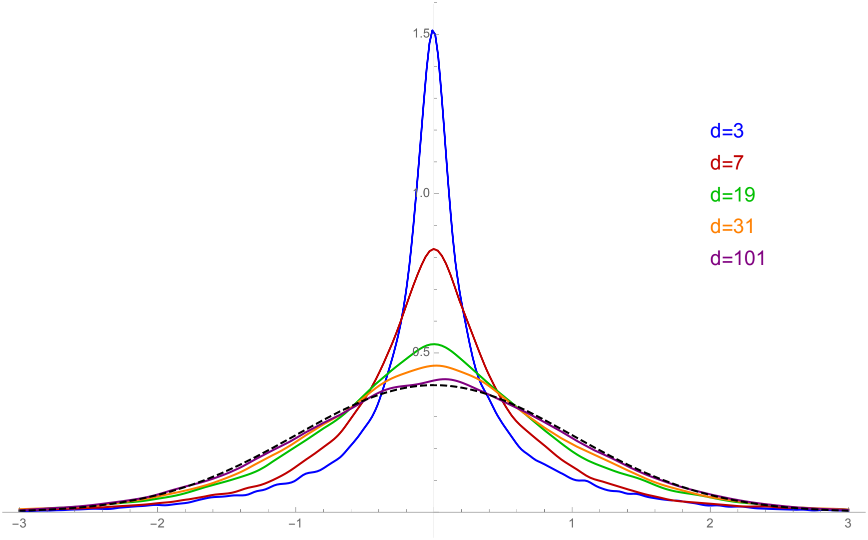

Figure: Approximations to $\Lambda_{E,d}$ for $E = 11A1$.

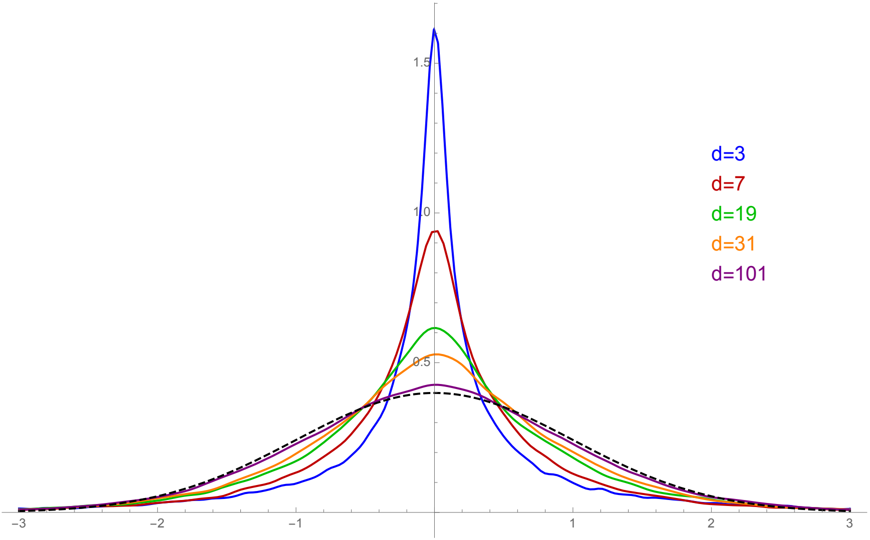

Figure: Approximations to $\Lambda_{E,d}$ for $E = 37A1$.

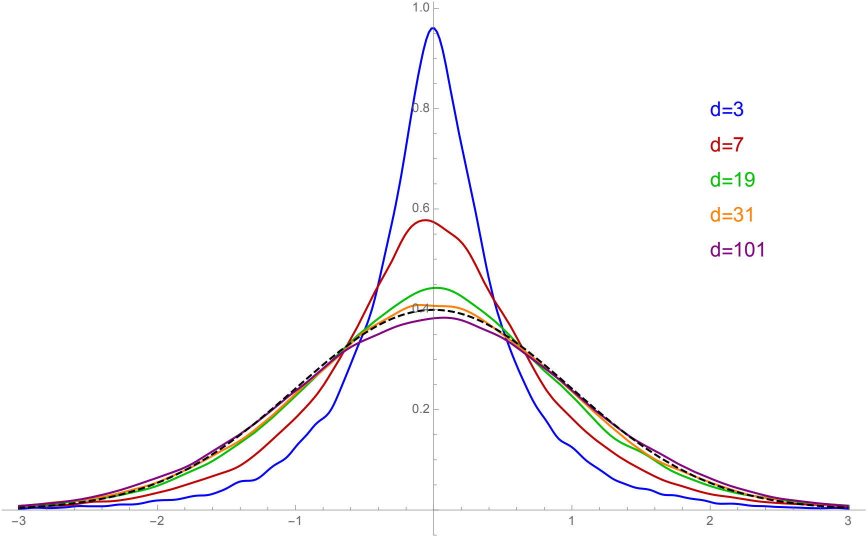

Figure: Approximations to $\Lambda_{E,d}$ for $E = 32A1$.

Question

What can we say about these $\Lambda_{E,d}(t)$*?*

Correlation of Theta-coefficients in a given Theta-element

Letting $d = p$ be prime, and χ a Dirichlet character of order p cutting out $F/\mathbb{Q}$, recall that

$$L(E, \chi, 1) = 0 \quad \leftrightarrow \quad \sum_{\alpha \in \mathrm{Gal}(F/\mathbb{Q})} c_{E,F;\alpha}\chi(\alpha) = 0 \quad \leftrightarrow$$

—the $c_{E,F;\alpha}$ are all equal for $\alpha \in \mathrm{Gal}(F/\mathbb{Q})$ —

I.e., to get vanishing you must have lots of "correlation" in the Theta coefficients corresponding to the same Theta element.

Setting up the ambient space

This is the ambient space in which we want to visualize data related to Theta Elements

(corresponding to characters χ of order $p \geq 3$ a prime and conductor m).

But here we'll only deal with elliptic curves E over $\mathbb{Q}$ (and with conductors m)

- such that E has root number $w = -1$
- and—we restrict m to have the property that its conductor, N viewed as an element in $(\mathbb{Z}/m\mathbb{Z})^*$ is in the kernel of
$$(\mathbb{Z}/m\mathbb{Z})^* \to \mathrm{Gal}(F/\mathbb{Q}).$$

Symmetry

In such a case we have clean symmetry:

$$c_\alpha = -c_{\{\alpha^{-1}\}}.$$

The ambient space (for the data given by Theta Elements)

Letting $\mathbb{R}[\mathbb{Z}/p\mathbb{Z}]$ be the group ring of the cyclic group $\mathbb{Z}/p\mathbb{Z}$—with $\mathbb{R}$ as coefficient ring, define:

$$W := (\mathbb{R}[\mathbb{Z}/p\mathbb{Z}])^{-} \subset \mathbb{R}[\mathbb{Z}/p\mathbb{Z}];$$

i.e., W is the $\mathbb{R}$-vector space of dimension $\frac{p-1}{2}$ on which

- $(\mathbb{Z}/p\mathbb{Z})^*$ in its action as a group of automorphisms of $\mathbb{Z}/p\mathbb{Z}$ induces an action on the $\mathbb{R}$-vector space W,

  and where

- $-1 \in (\mathbb{Z}/p\mathbb{Z})^*$ acts on W as scalar multiplication by -1.

A basis for the $\mathbb{R}$-vector space W in $\mathbb{R}[\mathbb{Z}/p\mathbb{Z}]$:

Elements

$$z = \sum_{i=0}^{p-1} z_i \langle i \rangle \in \mathbb{R}[\mathbb{Z}/p\mathbb{Z}]$$

$$(\text{for } z_i \in \mathbb{R} \text{ and } \langle i \rangle \in \mathbb{Z}/p\mathbb{Z})$$

that are in W have the property that

- $z_0 = 0$ and
- for $1 \leq a \leq p-1$ we have: $z_{(p-a)} = -z_a$.

So a basis for W is given by:

$$\{w_i := \langle i \rangle - \langle p - i \rangle \in \mathbb{R}[\mathbb{Z}/p\mathbb{Z}] \;\; \text{for} \;\; 1 \leq i \leq (p-1)/2\}$$

The spines

The data we get appear to organize themselves around certain lines in W that we call **the spines.**

For $1 \leq a \leq (p-1)/2$ put:

$$s_a := \sum_{k=1}^{(p-1)/2} \sin(\frac{2\pi ka}{p}) w_a \in W \subset \mathbb{R}[\mathbb{Z}/p\mathbb{Z}]$$

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The "spines" are the lines (in W)—i.e. one-dimensional $\mathbb{R}$-vector spaces—generated by the $\mathrm{s_a}$.

There are $(p-1)/2$ such spines forming a single orbit under the action of $(\mathbb{Z}/p\mathbb{Z})^*$ on W.

Each spine is stabilized by the action of $\langle -1 \rangle \in (\mathbb{Z}/p\mathbb{Z})^*$ where this action induces the involution 'multiplication by -1' on each spine.

These $(p-1)/2$ spines are linearly independent in the $(p-1)/2$-dimensional $\mathbb{R}$-vector space W.

The $(\mathbb{Z}/p\mathbb{Z})^*$ orbit in W determined by a Theta element $\Theta_{E,F} = \sum_\alpha c_\alpha \cdot \alpha$

Since

$$c_\alpha = -c_{\{\alpha^{-1}\}}$$

is the symmetry in the coefficients of $\Theta_{E,F}$, if we choose any isomorphism

$$\phi : Gal(F/\mathbb{Q}) \xrightarrow{\simeq} \mathbb{Z}/p\mathbb{Z}$$

We get:

$$\begin{array}{ccc} \phi : Gal(F/\mathbb{Q}) & \xrightarrow{\simeq} & \mathbb{Z}/p\mathbb{Z} \\ \downarrow \subset & & \downarrow = \\ \mathbb{Z}[Gal(F/\mathbb{Q})] & \xrightarrow{\phi} & \mathbb{Z}[\mathbb{Z}/p\mathbb{Z}] \end{array}$$

and the image of $\Theta_{E,F} \in \mathbb{Q}[Gal(F/\mathbb{Q})]$ in $\mathbb{R}[\mathbb{Z}/p\mathbb{Z}]$ lands in the minus subspace,

$$W := \mathbb{R}[\mathbb{Z}/p\mathbb{Z}]^-,$$

i.e., the ambient space for our data.

The images of $\Theta_{E,F}$ under the various choices of isomorphisms ϕ taken together produce a single $(\mathbb{Z}/p\mathbb{Z})^*$ orbit of points in W.

Equivalently,

choosing any generator $g \in Gal(F/\mathbb{Q})$ and defining

$$\Theta_{E,F}^{\{g\}} := \sum_{j=0}^{p-1} c_{g^j} \cdot \langle j \rangle \in W$$

note that the coefficients of $\Theta_{E,F}^{\{g\}}$ in terms of the basis we gave for W are just the '*first*' $(p-1)/2$ *Theta coefficients*, i.e.,

$$\Theta_{E,F}^{\{g\}} := \sum_{j=1}^{(p-1)/2} c_{g^j} \cdot (\langle j \rangle - \langle -j \rangle) \in W$$

What we're plotting:

For fixed prime p and fixed elliptic curve E and for varying Galois cyclic field extensions $F/\mathbb{Q}$ of degree p we'll plot those $(\mathbb{Z}/p\mathbb{Z})^*$ orbits of points in W that are the images of $\Theta_{E,F}$.

We find that the data forms a star-like structure about the spines. We see no clear reason why that should be so, and hope to gather enough data to be able to make a quantitative conjecture that accounts for the pictures.

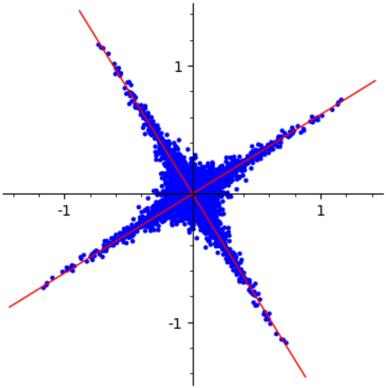
The starlike structure in W of the 'distribution' given by the $(\mathbb{Z}/p\mathbb{Z})^*$ orbits of images of $\Theta_{E,F}$

Here they are for the prime $p = 5$ and elliptic curves

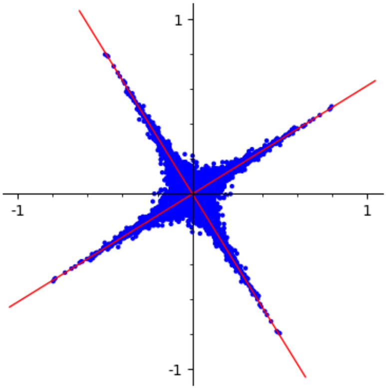
$$E = 37a1, 43a1, 53a1, 1001a1, \text{ and } 10001a1$$

where the data was tabulated for 10,381 fields F (essentially all cyclic Galois extensions of degree 5 with prime conductor up to 500,000);

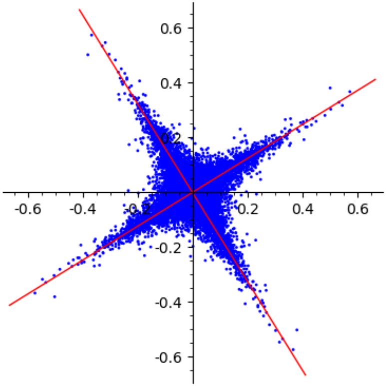
$E = 37a1$



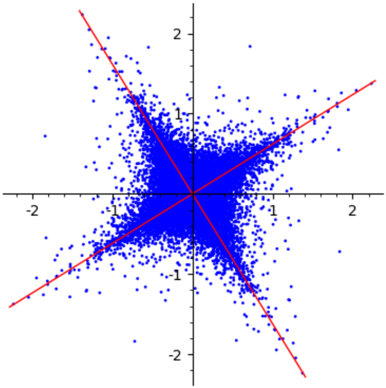
$E = 43a1$



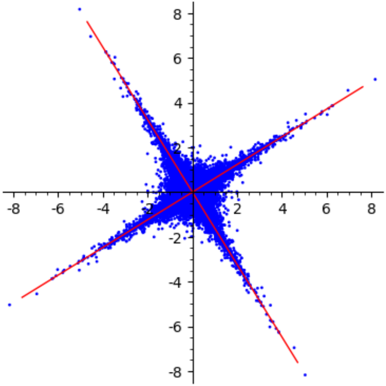
$E = 53a1$



$E = 1001a1$



$E = 10001a1$



The 3-d video: $E = 37a1$ with $p = 7$

—if I can figure out how to present it—